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THE FURTHER RESULTS ON SEP ELEMENTS IN A RING WITH INVOLUTION

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Abstract. In this paper, we further study many new characterizations of SEP elements in a ring with involution. Firstly, combining Moore-Penrose invertible element, group invertible element, we find some PE elements to characterize SEP elements and then further discover some equivalent conditions for SEP elements especially around the element aa^*a^+a. Mainly, by constructing some equations in a given set including $a^+, a^*, (a^\#)^*, a^+a, aa^+$, we obtain a lot of new characterizations of SEP elements. Next, we study the expression forms of related bivariate equations to depict SEP elements. Finally, we use nil-cleanity of the element aa^*a^+a to link SEP elements with PE elements.

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1. Introduction

The study of generalized inverse problems has expanded from linear operators to operator algebras, C^*-algebras, and semigroups and rings, the produced conceptions like EP elements, PI elements, PE elements and so on play key roles in the development of generalized inverse in a ring. Numerous challenges in systems and control theory necessitate the addressing of associated equations. The paper aims to discuss the solution of some related equations to further characterize SEP elements.

An involution $a \longmapsto a^*$ in a ring R is an anti-isomorphism of degree 2, that is, $(a^*)^* = a$; $(a+b)^* = a^* + b^*$; and $(ab)^* = b^*a^*$. In this case, R is called a $*$-ring.

We know that $a \in R$ satisfying $a^2 = a$ is called an idempotent element. The set of all idempotent elements will be denoted by $E(R)$.

An element $a \in R$ is called Hermitian if $a^* = a$ [14], and a is called a projection if $a^2 = a = a^*$. We denote the set of all projections of R by $PE(R)$.

An element $a \in R$ is called Moore-Penrose invertible if there exists $x \in R$ satisfying the following equations:

$$axa = a,\ xax = x,\ (ax)^* = ax,\ (xa)^* = xa;$$

such an x is the uniquely determined Moore-Penrose inverse (or MP-inverse) of a [3,4], denoted by $x = a^+$. The set of all Moore-Penrose invertible elements of R will be denoted by R^+.

Let $a \in R$. Then a is called group invertible if there exists $x \in R$ satisfying

$$axa = a,\ xax = x,\ ax = xa;$$

such an x is uniquely determined group inverse of a (see [5,8,9]), written $x = a^\#$. Denote by $R^\#$ the set of all group invertible elements of R.

An element $a \in R$ satisfying $a = aa^*a$ is called partial isometry of R [3,4]. Let R^{PI} denote the set of all partial isometries of R. Obviously, we have that $a \in R^+$ is a partial isometry if and only if $a^* = a^+$.

Let $a \in R^\# \cap R^+$. If $a^\# = a^+$, then a is called an EP element. We denote the set of all EP elements in R by R^{EP}. On the studies of EP, the readers can refer to [1,4,6,7,10,14,15,16,17].

If $a \in R^{EP} \cap R^{PI}$, then a is said to be a strong EP element of R [3,4,14,19,20]. Let R^{SEP} denote the set of all SEP elements of R.

In [14,17], Mosic and Djordjevic give many characterizations of SEP elements, we have learned some equivalent conditions for SEP elements. In [18], many new characterizations of strongly EP elements have been presented. Then many researchers characterize SEP elements by constructing related equations. In [19], it is known that $a \in R^\# \cap R^+$ is SEP if and only if the equation $yxa^* = yxa^\#$ has at least one solution in a given set $\chi_a^2 =: \{(x, y)|x, y \in \chi_a\}$. In [11], it is shown the basic solution formula of the bivariate equation $xa^+(a^\#)^* = aa^+y$. In [2], Guan uses core invertible elements to characterize and discuss the solution of $xa^* = a^{\textcircled{\#}}x$. In [12,13], the forms of solutions of parametric equations in a certain given set is proved.

Motivated by these results, this paper mainly study the further characterizations of SEP elements by PE elements.

2. Characterizing SEP elements by projections

Theorem 2.1. *Let $a \in R^\# \cap R^+$. Then $a \in R^{SEP}$ if and only if $aa^*a^+a \in PE(R)$.*

Proof. " $\Longrightarrow$ " If $a \in R^{SEP}$, then $a^+ = a^* = a^\#$, this gives that

$$aa^*a^+a = aa^\#a^+a = a^\#a = a^+a \in PE(R).$$

" $\Longleftarrow$ " From the assumption, we have

$$aa^*a^+a = aa^*a^+a^2a^*a^+a,$$

and

$$aa^*a^+a = (aa^*a^+a)^* = a^+a^2a^* = a^+aa^+a^2a^* = a^+a^2a^*a^+a.$$

Multiplying the last equality on the right by $(a^\#)^*a^+a$, one has $a = a^+a^2$. Hence $a \in R^{EP}$ and

$$a = aa^+a = a(a^*a^+a(a^\#)^*a^+)a = aa^*a^+a^2a^*a^+a(a^\#)^*a^+a = aa^*a^+a^2 = aa^*a.$$

It follows that $a \in R^{PI}$. Thus $a \in R^{SEP}$. □

Noting that $a \in PE(R)$ if and only if $a^* \in PE(R)$. Hence, Theorem 2.1 implies the following corollary.

Corollary 2.2. *Let $a \in R^\# \cap R^+$. Then $a \in R^{SEP}$ if and only if $a^+a^2a^* \in PE(R)$.*

Since $e \in R$ is a projection if and only if $e = ee^*$, this induces

Corollary 2.3. *Let $a \in R^\# \cap R^+$. Then $a \in R^{SEP}$ if and only if $a^+a^2a^* = a^+a^2a^*aa^*a^+a$.*

Lemma 2.4. *Let $e \in R$. Then $e \in R^{Her}$ if and only if $e - ee^* \in R^{Her}$.*

Proof. " $\Longrightarrow$ " It is evident because $e = e^*$.
" $\Longleftarrow$ " Assume that $e - ee^* \in R^{Her}$. Then

$$e - ee^* = (e - ee^*)^* = e^* - ee^*.$$

This gives $e = e^*$ and so $e \in R^{Her}$. □

Lemma 2.4 and Corollary 2.2 imply the following theorem.

Theorem 2.5. *Let $a \in R^\# \cap R^+$. Then $a \in R^{SEP}$ if and only if $a^+a^2a^* \in E(R)$ and $a^+a^2a^* - a^+a^2a^*aa^*a^+a \in R^{Her}$.*

Theorem 2.6. *Let $a \in R^\# \cap R^+$. Then $a \in R^{SEP}$ if and only if $aa^*a^\#a \in PE(R)$.*

Proof. " $\Longrightarrow$ " Since $a \in R^{SEP}$, $a^+ = a^\#$ and $aa^*a^+a \in PE(R)$ by Theorem 2.1. Hence, $aa^*a^\#a \in PE(R)$.
" $\Longleftarrow$ " Using the hypothesis, one gets

$$aa^*a^\#a = (aa^*a^\#a)^* = (aa^\#)^*aa^* = ((aa^\#)^*aa^*)aa^+ = (aa^*a^\#a)aa^+.$$

Multiplying the equality on the left by $a^{\#}(a^+)^*a^+$, one yields $a^{\#} = a^{\#}aa^+$. Hence $a \in R^{EP}$ by [14, Theorem 1.2.1], this infers $aa^*a^+a = aa^*a^{\#}a \in PE(R)$. By Theorem 2.1, $a \in R^{SEP}$. □

Since $aa^*a^+a = aa^*a^*(a^+)^*$, it follows that for $a \in R^{PI}$, one gets $aa^*a^+a = aa^+a^+(a^+)^*$. Thus, Theorem 2.1 implies the following result.

Theorem 2.7. *Let $a \in R^{\#} \cap R^+$. Then $a \in R^{SEP}$ if and only if $aa^+a^+(a^+)^* \in PE(R)$.*

Proof. We only show " $\Longleftarrow$ ". From the hypothesis, we have

$$aa^+a^+(a^+)^* = aa^+a^+(a^+)^*(aa^+a^+(a^+)^*)^*.$$

Multiplying the equality on the left by $aa^*a(aa^{\#})^*$, we get

$$a = (a^+)^*aa^+.$$

Hence, $a^* = aa^+a^+$ by [14, Theorem 1.5.3], $a \in R^{SEP}$. □

According to [14, Theorem 2.1.1], $a \in R^{EP}$ if and only if $aa^+a^+ = a^{\#}$. Thus we have

Corollary 2.8. *Let $a \in R^{\#} \cap R^+$. Then $a \in R^{SEP}$ if and only if $a^{\#}(a^+)^* \in PE(R)$.*

Theorem 2.9. *Let $a \in R^{\#} \cap R^+$. Then $a \in R^{SEP}$ if and only if $a^{\#}(a^+)^* \in E(R)$ and $a^{\#}(a^+)^* - a^{\#}(a^+)^*a^+(a^{\#})^* \in R^{Her}$.*

Proof. " $\Longrightarrow$ " Assume that $a \in R^{SEP}$. Then $a^{\#}(a^+)^* \in PE(R)$ by Corollary 2.8, this implies

$$a^{\#}(a^+)^* \in E(R) \cap R^{Her}.$$

By Lemma 2.4, one gets $a^{\#}(a^+)^* - a^{\#}(a^+)^*a^+(a^{\#})^* \in R^{Her}$.

" $\Longleftarrow$ " The condition $a^{\#}(a^+)^* - a^{\#}(a^+)^*a^+(a^{\#})^* \in R^{Her}$ gives

$$a^{\#}(a^+)^* - a^{\#}(a^+)^*a^+(a^{\#})^* = (a^{\#}(a^+)^* - a^{\#}(a^+)^*a^+(a^{\#})^*)^*,$$

this induces

$$a^{\#}(a^+)^* = (a^{\#}(a^+)^*)^*.$$

Noting that $a^{\#}(a^+)^* \in E(R)$. Then $a^{\#}(a^+)^* \in PE(R)$ and so $a \in R^{SEP}$ by Corollary 2.8. □

3. Some equivalent conditions for SEP elements

Lemma 3.1. *Let $a \in R^{\#} \cap R^{+}$. Then*

(1) $(aa^*a^+a)^{\#} = aa^{\#}(a^{\#})^*a^+aa^{\#}$;
(2) $(aa^*a^+a)^{+} = (a^{\#})^*a^+$;
(3) $aa^*a^+a \in R^{EP}$ *if and only if* $a \in R^{EP}$;
(4) $aa^*a^+a \in R^{SEP}$ *if and only if* $aa^* = (a^{\#})^*a^+$.

Proof. (1) and (2) can be verified routinely.

(3) " $\Longrightarrow$ " Since $aa^*a^+a \in R^{EP}$, $(aa^*a^+a)^{\#} = (aa^*a^+a)^{+}$. By (1) and (2), one obtains

$$aa^{\#}(a^{\#})^*a^+aa^{\#} = (a^{\#})^*a^+.$$

Multiplying the equality on the left by a^*a^+a, one gets $a^+aa^{\#} = a^+$. Hence, $a \in R^{EP}$ by [14, Theorem 1.2.1].

" $\Longleftarrow$ " Assume that $a \in R^{EP}$. Then $a^+ = a^{\#}$, it follows that

$$aa^{\#}(a^{\#})^*a^+aa^{\#} = aa^{\#}(a^+)^*a^+aa^+ = (a^+)^*a^+ = (a^{\#})^*a^+.$$

By (1) and (2), $aa^*a^+a \in R^{EP}$.

(4) " $\Longrightarrow$ " Since $aa^*a^+a \in R^{SEP}$ if and only if $aa^*a^+a \in R^{EP}$ and $(aa^*a^+a)^* = (aa^*a^+a)^+$, it follows that

$$aa^* = (aa^*)^* = (aa^*a^+a)^* = (aa^*a^+a)^+ = (a^{\#})^*a^+.$$

" $\Longleftarrow$ " The condition $aa^* = (a^{\#})^*a^+$ leads to

$$a^+a^2a^* = a^+a(a^{\#})^*a^+ = (a^{\#})^*a^+ = aa^*$$

and

$$a = aa^*(a^+)^* = a^+a^2a^*(a^+)^* = a^+a^2.$$

Hence, $a \in R^{EP}$, one has

$$(aa^*a^+a)^* = (aa^*)^* = aa^* = (a^{\#})^*a^+ = aa^{\#}(a^{\#})^*a^+aa^{\#} = (aa^*a^+a)^{\#}.$$

Thus $aa^*a^+a \in R^{SEP}$. □

Corollary 3.2. *Let $a \in R^{\#} \cap R^{+}$. Then $aa^*a^+a \in R^{SEP}$ if and only if $aa^*a = (a^{\#})^*$.*

Proof. " $\Longrightarrow$ " Since $aa^*a^+a \in R^{SEP}$, $a \in R^{EP}$ and $aa^* = (a^{\#})^*a^+$ by Lemma 3.1. Hence,

$$aa^*a = (a^{\#})^*a^+a = (a^+)^*a^+a = (a^+)^* = (a^{\#})^*.$$

" $\Longleftarrow$ " Using the assumption, one gets

$$(a^{\#})^* = aa^*a = (aa^*a)a^+a = (a^{\#})^*a^+a,$$

and

$$a^{\#} = a^+aa^{\#}.$$

Hence, $a \in R^{EP}$, it follows that $aa^* = aa^*aa^+ = (a^{\#})^*a^+$. By Lemma 3.1, $aa^*a^+a \in R^{SEP}$. □

Corollary 3.3. *Let $a \in R^{\#} \cap R^+$. Then $a \in R^{SEP}$ if and only if $a \in R^{PI}$ and $aa^*a^+a \in R^{SEP}$.*

Proof. " $\Longrightarrow$ " Since $a \in R^{SEP}$, $a \in R^{PI}$ and $a^+ = a^{\#} = a^*$, this gives

$$(a^{\#})^* = a = aa^*a.$$

By Corollary 3.2, $aa^*a^+a \in R^{SEP}$.

" $\Longleftarrow$ " Since $a \in R^{PI}$, $aa^*a^+a \in R^{SEP}$, by Corollary 3.2, one gets

$$a = aa^+a = aa^*a = (a^{\#})^*.$$

Hence, $a \in R^{SEP}$. □

Theorem 3.4. *Let $a \in R^{\#} \cap R^+$. Then the following are equivalent:*

(1) $a \in R^{SEP}$;
(2) $aa^{\#}(a^{\#})^*a^+aa^{\#} = aa^+$;
(3) $aa^{\#}(a^{\#})^*a^+aa^{\#} = a^+a$;
(4) $a^+aa^{\#}(a^{\#})^*a^+aa^{\#} = a^+$;
(5) $(a^{\#})^*a^+aa^{\#} = aa^+$;
(6) $a^+aa^{\#} = a^*$.

Proof. (1) $\Longrightarrow$ (2) Since $a \in R^{SEP}$, $aa^*a^+a \in R^{SEP}$ by Corollary 3.3. Using Lemma 3.1, one gets

$$aa^{\#}(a^{\#})^*a^+aa^{\#} = (aa^*a^+a)^{\#} = (aa^*a^+a)^+ = (a^{\#})^*a^+ = aa^+.$$

(2) $\Longrightarrow$ (3) From $aa^{\#}(a^{\#})^*a^+aa^{\#} = aa^+$, one obtains

$$aa^+ = (aa^{\#}(a^{\#})^*a^+aa^{\#})a^+a = aa^+a^+a.$$

Hence, $a \in R^{EP}$, this gives

$$aa^{\#}(a^{\#})^*a^+aa^{\#} = aa^+ = a^+a.$$

(3) $\Longrightarrow$ (4) The condition $aa^{\#}(a^{\#})^*a^+aa^{\#} = a^+a$ gives

$$a^+a = aa^{\#}a^+a = aa^{\#}.$$

Hence, $a \in R^{EP}$, which infers $a^+ = a^+a^+a$. Thus,

$$a^+aa^\#(a^\#)^*a^+aa^\# = a^+a^+a = a^+.$$

(4) $\Longrightarrow$ (5) From $a^+ = a^+aa^\#(a^\#)^*a^+aa^\#$, one yields

$$a^+ = a^+aa^\#,$$

so $a \in R^{EP}$ and

$$a^+ = a^+aa^\#(a^\#)^*a^+aa^\# = a^+(a^\#)^*a^+aa^\#,$$

$$aa^+ = aa^+(a^\#)^*a^+aa^\# = a^+a(a^\#)^*a^+aa^\# = (a^\#)^*a^+aa^\#.$$

(5) $\Longrightarrow$ (6) Since $aa^+ = (a^\#)^*a^+aa^\#$, we get

$$a^* = a^*aa^+ = a^*(a^\#)^*a^+aa^\# = a^+aa^\#.$$

(6) $\Longrightarrow$ (1) From the assumption $a^+aa^\# = a^*$, one has

$$aa^* = aa^+aaa^\# = aa^\#.$$

Hence, $a \in R^{SEP}$ by [14, Theorem 1.5.3]. □

Theorem 3.5. *Let $a \in R^\# \cap R^+$. Then $a \in R^{SEP}$ if and only if $(a^\#a)^*aa^\# = aa^*$.*

Proof. " $\Longrightarrow$ " Assume that $a \in R^{SEP}$. Then $(a^\#)^*a^+aa^\# = aa^+$ by Theorem 3.4. Noting that $a^+ = a^*$. Then

$$(aa^\#)^*aa^\# = (a^\#)^*a^*aa^\# = (a^\#)^*a^+a^\# = aa^+ = aa^*.$$

" $\Longleftarrow$ " From the equality $(aa^\#)^*aa^\# = aa^*$, one has

$$a^* = a^+aa^* = a^+(aa^\#)^*aa^\# = a^+aa^\#.$$

Hence, $a \in R^{SEP}$ by Theorem 3.4. □

Lemma 3.6. *Let $a \in R^\# \cap R^+$. Then $(a^\#a)^*aa^\# \in R^{EP}$ with $((a^\#a)^*aa^\#)^+ = a^+a^2a^+a^+a$.*

Proof. It is a routine to verify. □

Theorem 3.7. *Let $a \in R^\# \cap R^+$. Then $a \in R^{SEP}$ if and only if $((a^\#a)^*aa^\#)^+ = aa^*a^+a$.*

Proof. "$\Longrightarrow$" Since $a \in R^{SEP}$, $a \in R^{EP}$ and $aa^{\#} = aa^{+} = aa^{*}$. Hence,

$$((a^{\#}a)^{*}aa^{\#})^{+} = (a^{\#}aaa^{\#})^{+} = (aa^{\#})^{+} = (aa^{+})^{+} = aa^{+} = aa^{*} = aa^{*}a^{+}a.$$

"$\Longleftarrow$" From the equality $((a^{\#}a)^{*}aa^{\#})^{+} = aa^{*}a^{+}a$, we get

$$(a^{\#}a)^{*}aa^{\#} = (aa^{*}a^{+}a)^{+} = (a^{\#})^{*}a^{+}.$$

Multiplying the equality on the left by a^{*}, we obtain

$$a^{*}aa^{\#} = a^{+}.$$

Hence, $a \in R^{SEP}$ by [14, Theorem 1.5.3]. □

4. Characterizing SEP elements by the solution of equations in a given set

Lemma 4.1. *Let $a \in R^{\#} \cap R^{+}$. Then $a \in R^{SEP}$ if and only if $(a^{+})^{*}a^{+}a^{+} = a^{+}$.*

Proof. "$\Longrightarrow$" Since $a \in R^{SEP}$, $(a^{+})^{*} = a, a^{+} = a^{\#}$. Hence

$$(a^{+})^{*}a^{+}a^{+} = aa^{+}a^{\#} = a^{\#} = a^{+}.$$

"$\Longleftarrow$" From $(a^{+})^{*}a^{+}a^{+} = a^{+}$, we have

$$a^{*} = a^{+}aa^{*} = (a^{+})^{*}a^{+}a^{+}aa^{*} = (a^{+})^{*}a^{+}a^{*},$$

and

$$(aa^{\#})^{*} = (a^{+})^{*}a^{+}a^{*}(a^{\#})^{*} = (a^{+})^{*}a^{+}.$$

Hence, $a \in R^{EP}$ by [14, Theorem 1.1.3] and $a^{*} = a^{*}(aa^{\#})^{*} = a^{*}(a^{+})^{*}a^{+} = a^{+}$. Thus $a \in R^{SEP}$. □

Observing Theorem 3.4, we can construct the following equation:

$$aa^{\#}(a^{\#})^{*}xa^{\#} = x. \tag{4.1}$$

Theorem 4.2. *Let $a \in R^{\#} \cap R^{+}$. Then $a \in R^{SEP}$ if and only if Eq.(4.1) has at least one solution in $\Omega_a = \{a^{+}, a^{*}, (a^{\#})^{*}, a^{+}a, aa^{+}\}$.*

Proof. "$\Longrightarrow$" Since $a \in R^{SEP}$, $a^{+} = a^{*} = a^{\#}$. Obviously, Eq.(4.1) has at least one solution in $\Omega_a = \{a^{+}, a^{*}, (a^{\#})^{*}, a^{+}a, aa^{+}\}$.

"$\Longleftarrow$" (1) If $x = a^{+}$ is the solution of Eq.(4.1), then

$$aa^{\#}(a^{\#})^{*}a^{+}a^{\#} = a^{+}.$$

Multiplying the equality on the right by $a^{+}a$, we obtain

$$aa^{\#}(a^{\#})^{*}a^{+}a^{\#}a^{+}a = a^{+}a^{+}a.$$

Noting that $a^{\#}a^{+}a = a^{\#}$, so

$$a^{+}a^{+}a = a^{+}.$$

Then, $a = a^{+}aa$, multiplying the equality on the right by $a^{\#}a^{\#}$, we obtain

$$a^{\#} = a^{+}aa^{\#}.$$

Hence, $a \in R^{EP}$ by [14, Theorem 1.2.1]. It follows that $aa^{\#}(a^{\#})^* = aa^{\#}(a^{+})^* = (a^{+})^*$. From $aa^{\#}(a^{\#})^*a^{+}a^{\#} = a^{+}$, we get

$$(a^{+})^*a^{+}a^{+} = a^{+}.$$

Hence, $a \in R^{SEP}$ by Lemma 4.1.

(2) If $x = a^*$ is the solution of Eq.(4.1), then

$$aa^{\#}(a^{\#})^*a^*a^{\#} = a^*.$$

Multiplying the equality on the right by $a^{+}a$, one obtains $a^* = a^*a^{+}a$. We apply the involution to the equality, and then we conclude $a = a^{+}a^2$. Hence, $a \in R^{EP}$ by the proof of Theorem 4.2(1). This gives $a^{\#} = a^t$ and $a^* = aa^{\#}(a^{\#})^*a^*a^{\#}$ imply

$$a^* = aa^{\#}(a^{\#})^*a^*a^{\#} = (aa^t)(aa^t)^*a^{\#} = (aa^t)(aa^t)^*a^{\#} = (aa^{\#})(aa^{\#})a^{\#}.$$

It yields $a^* = a^{\#}$. Thus $a \in R^{SEP}$.

(3) If $x = (a^{\#})^*$ is the solution of Eq.(4.1), then

$$aa^{\#}(a^{\#})^*(a^{\#})^*a^{\#} = (a^{\#})^*.$$

Multiplying the equality on the right by $a^{+}a$, we obtain $(a^{\#})^* = (a^{\#})^*a^{+}a$. Applying the involution on the equality, one yields $a^{\#} = a^{+}aa^{\#}$. Hence, $a \in R^{EP}$ by [14, Theorem 1.2.1]. From $aa^{\#}(a^{\#})^*(a^{\#})^*a^{\#} = (a^{\#})^*$, we get $(a^{\#})^*(a^{\#})^*a^{+} = (a^{\#})^*$ and

$$a^* = a^*a^*(a^{\#})^* = a^*a^*(a^{\#})^*(a^{\#})^*a^{+} = a^{+}.$$

Hence, $a \in R^{SEP}$.

(4) If $x = a^{+}a$ is the solution of Eq.(4.1), then $a \in R^{SEP}$ by Theorem 3.4.

(5) If $x = aa^{+}$ is the solution of Eq.(4.1), then $a \in R^{SEP}$ from Theorem 3.4. □

Revising Eq.(4.1) as follows

$$aa^{\#}(a^{\#})^*xa^{+} = x. \tag{4.2}$$

Theorem 4.3. *Let $a \in R^{\#} \cap R^{+}$. Then $a \in R^{SEP}$ if and only if Eq.(4.2) has at least one solution in $\chi_a \cup \{a^{+}a\} = \{a, a^{\#}, a^{+}, a^*, (a^{+})^*, (a^{\#})^*, a^{+}a\}$.*

Proof. " $\Longrightarrow$ " Since $a \in R^{SEP}$, $a^+ = a^* = a^\#$. Clearly, $x = a^*$ is a solution to Eq.(4.2).

" $\Longleftarrow$ " (1) If $x = a$, then

$$aa^\#(a^\#)^*aa^+ = a.$$

Multiplying the equality on the right by aa^+, we obtain $a = a^2a^+$. Hence, $a \in R^{EP}$ by [14, Theorem 1.2.1]. From $aa^\#(a^\#)^*aa^+ = a$, we get

$$a = aa^\#(a^\#)^*aa^+ = aa^\#(a^+)^*aa^+ = (a^+)^*aa^+ = (a^\#)^*aa^+ = (a^\#)^*.$$

Hence, $a \in R^{SEP}$.

(2) If $x = a^\#$, then

$$aa^\#(a^\#)^*a^\#a^+ = a^\#.$$

Multiplying the equality on the right by aa^+, we obtain $a^\# = a^\#aa^+$. Hence, $a \in R^{EP}$ by [14, Theorem 1.2.1]. From $aa^\#(a^\#)^*a^\#a^+ = a^\#$, we get $a^+ = (a^+)^*a^+a^+$. Hence, $a \in R^{SEP}$ by Lemma 4.1.

(3) If $x = a^+$, then

$$aa^\#(a^\#)^*a^+a^+ = a^+.$$

Multiplying the equality on the left by $aa^\#$, we obtain $aa^\#a^+ = a^+$. Hence, $a \in R^{EP}$ by [14, Theorem 1.2.1]. Hence, $x = a^+ = a^\#$, by (2), $a \in R^{SEP}$.

(4) If $x = a^*$, then $aa^\#(a^\#)^*a^*a^+ = a^*$, e.g.,

$$aa^\#a^+ = a^*.$$

Hence, $a \in R^{EP}$ by [14, Theorem 1.5.3].

(5) If $x = (a^+)^*$, then

$$aa^\#(a^\#)^*(a^+)^*a^+ = (a^+)^*.$$

Multiplying the equality on the right by aa^+, we obtain $(a^+)^* = (a^+)^*aa^+$. We take $*$ to the last equality, one yields $a^+ = aa^+a^+$. Hence, $a \in R^{EP}$ by [14, Theorem 1.2.1]. From $aa^\#(a^\#)^*(a^+)^*a^+ = (a^+)^*$, we get $(a^+)^*(a^+)^*a^+ = (a^+)^*$. Applying the involution and Lemma 4.1, one yields $a \in R^{SEP}$.

(6) If $x = (a^\#)^*$, then

$$aa^\#(a^\#)^*(a^\#)^*a^+ = (a^\#)^*.$$

Multiplying the equality on the left by $aa^\#$, we obtain

$$aa^\#(a^\#)^* = (a^\#)^*.$$

By [14, Theorem 1.1.3], $a \in R^{EP}$. Hence, $x = (a^\#)^* = (a^+)^*$, by (5), $a \in R^{SEP}$.

(7) If $x = a^+a$, then $aa^\#(a^\#)^*a^+aa^+ = a^+a$, e.g.,

$$aa^\#(a^\#)^*a^+ = a^+a.$$

Multiplying the equality on the right by aa^+, we obtain $aa^+ = a^+a^2a^+$. Hence, $a \in R^{EP}$ by [14, Theorem 1.2.1]. From $aa^\#(a^\#)^*a^+aa^+ = a^+a$, we get $(a^+)^*a^+ = a^+a$. This gives

$$aa^* = ((a^+)^*a^+)^+ = (a^+a)^+ = a^+a.$$

Hence, $a \in R^{SEP}$ by [14, Theorem 1.5.3]. □

Multiplying Eq.(4.1) on the left by $a^\#$, we get

$$a^\#(a^\#)^*xa^\# = a^\#x. \tag{4.3}$$

Theorem 4.4. *Let $a \in R^\# \cap R^+$. Then $a \in R^{SEP}$ if and only if Eq.(4.3) has at least one solution in $\Omega_a \backslash \{a^+a\}$.*

Proof. " $\Longrightarrow$ " Since $a \in R^{SEP}$, $a^+ = a^* = a^\#$. Obviously, $x = a^+$ is a solution to Eq.(4.3).

" $\Longleftarrow$ " (1) If $x = a^+$, then

$$a^\#(a^\#)^*a^+a^\# = a^\#a^+.$$

Multiplying the equality on the right by a^+a, we obtain

$$a^\#a^+a^+a = a^\#a^+.$$

Multiplying the last equality on the left by a^+a^2, we get $a^+a^+a = a^+$. Hence, $a \in R^{EP}$. From $a^\#(a^\#)^*a^+a^\# = a^\#a^+$, we get

$$a^+(a^+)^*a^+a^+ = a^+a^+.$$

This gives

$$(a^+)^*a^+a^+ = aa^+(a^+)^*a^+a^+ = aa^+a^+ = a^+.$$

Hence, $a \in R^{SEP}$ by Lemma 4.1.

(2) If $x = a^*$, then

$$a^\#(a^\#)^*a^*a^\# = a^\#a^*.$$

Multiplying the equality on the right by a^+a, we obtain

$$a^\#a^*a^+a = a^\#a^*.$$

Multiplying the equality on the left by a^+a^2, we obtain

$$a^*a^+a = a^+a^2(a^\#a^*a^+a) = a^+a^2a^\#a^* = a^*.$$

This infers $a = a^+a^2$. Hence, $a \in R^{EP}$ by [14, Theorem 1.2.1]. From $a^\#(a^\#)^*a^*a^\# = a^\#a^*$, we get

$$a^+(a^+)^*a^*a^+ = a^+a^*,$$

e.g., $a^+a^+ = a^+a^* = a^\#a^*$. Hence, $a \in R^{SEP}$ by [14, Theorem 1.5.3].

(3) If $x = (a^\#)^*$, then

$$a^\#(a^\#)^*(a^\#)^*a^\# = a^\#(a^\#)^*.$$

Multiplying the equality on the right by a^+a, we obtain

$$a^\#(a^\#)^* = a^\#(a^\#)^*a^+a.$$

Multiplying the last equality on the left by a^+a^2, we obtain

$$(a^\#)^* = (a^\#)^*a^+a.$$

This implies that $a^\# = a^+aa^\#$. Hence, $a \in R^{EP}$ by [14, Theorem 1.2.1]. From $a^\#(a^\#)^*(a^\#)^*a^\# = a^\#(a^\#)^*$, we get

$$a^+(a^+)^*(a^+)^*a^+ = a^+(a^+)^*,$$

and

$$(a^+)^*(a^+)^*a^+ = aa^+(a^+)^*(a^+)^*a^+ = aa^+(a^+)^* = (a^+)^*.$$

Applying the involution, one gets $(a^+)^*a^+a^+ = a^+$. Hence, $a \in R^{SEP}$ by Lemma 4.1.

(4) If $x = aa^+$, then

$$a^\#(a^\#)^*aa^+a^\# = a^\#aa^+.$$

Multiplying the equality on the right by a^+a, we get

$$a^\#aa^+ = a^\#aa^+a^+a.$$

Multiplying the last equality on the left by a^+a, we obtain $a^+a^+a = a^+$. Hence, $a \in R^{EP}$. From $a^\#(a^\#)^*aa^+a^\# = a^\#aa^+$, we get

$$a^+(a^+)^*a^+ = a^+.$$

This gives

$$(a^+)* = aa^+(a^+)^*a^+a = aa^+a = a.$$

Hence, $a \in R^{SEP}$. □

$$a^\#(a^\#)^*x = x. \tag{4.4}$$

Theorem 4.5. *Let $a \in R^\# \cap R^+$. Then $a \in R^{SEP}$ if and only if Eq.(4.4) has at least one solution in $\Omega_a \setminus \{aa^+\}$.*

Proof. " $\Longrightarrow$ " Since $a \in R^{SEP}$, $x = a^+ = a^* = a^{\#}$ is a solution to Eq.(4.4).

" $\Longleftarrow$ " (1) If $x = a^+$, then

$$a^{\#}(a^{\#})^* a^+ = a^+.$$

Multiplying the equality on the right by aa^*a^+a, we get $a^{\#}a^+ = a^*a^+$. Hence, $a \in R^{SEP}$ by [14, Theorem 1.5.3].

(2) If $x = a^*$, then

$$a^{\#}(a^{\#})^* a^* = a^*.$$

This gives

$$a^{\#}a^+ = a^{\#}(a^{\#})^* a^* a^+ = a^* a^+.$$

Hence, $a \in R^{SEP}$.

(3) If $x = (a^{\#})^*$, then

$$a^{\#}(a^{\#})^*(a^{\#})^* = (a^{\#})^*.$$

It follows that

$$a^{\#}(a^{\#})^* a^* = a^{\#}(a^{\#})^*(a^{\#})^* a^* a^* = (a^{\#})^* a^* a^* = a^*.$$

Hence, $a \in R^{SEP}$ by (2).

(4) If $x = a^+a$, then

$$a^{\#}(a^{\#})^* a^+ a = a^+ a.$$

One gets

$$a^{\#}(a^{\#})^* a^* = a^{\#}(a^{\#})^* a^+ a a^* = a^+ a a^* = a^*.$$

Hence, $a \in R^{SEP}$ by (2). $\square$

$$(a^{\#})^* x a^{\#} = x. \tag{4.5}$$

Theorem 4.6. *Let $a \in R^{\#} \cap R^+$. Then $a \in R^{SEP}$ if and only if Eq.(4.5) has at least one solution in $\chi_a \cup \{aa^+\} = \{a, a^{\#}, a^+, a^*, (a^+)^*, (a^{\#})^*, aa^+\}$.*

Proof. " $\Longrightarrow$ " Since $a \in R^{SEP}$, $a = (a^{\#})^*$. Obviously, Eq.(4.5) has at least one solution $x = a$ in χ_a.

" $\Longleftarrow$ " (1) If $x = a$, then

$$(a^{\#})^* a a^{\#} = a.$$

It follows that

$$a^* = (aa^{\#})^* a^{\#}.$$

By multiplying the equality on the right by a^+a, we arrive at $a^*a^+a = a^*$. Applying the involution to both sides again leads to $a^+aa = a$. Hence, $a \in R^{EP}$ by [14, Theorem 1.2.1]. From $(a^{\#})^*aa^{\#} = a$, we get $(a^+)^* = (a^+)^*aa^{\#} = (a^{\#})^*aa^{\#} = a$, then $a \in R^{PI}$ by [14, Theorem 1.5.1]. Hence, $a \in R^{SEP}$ by [14, Theorem 1.5.3].

(2) If $x = a^{\#}$, then

$$(a^{\#})^*a^{\#}a^{\#} = a^{\#}.$$

This gives

$$a = a^{\#}a^2 = (a^{\#})^*a^{\#}a^{\#}a^2 = (a^{\#})^*aa^{\#}.$$

By (1), $a \in R^{SEP}$.

(3) If $x = a^+$, then

$$(a^{\#})^*a^+a^{\#} = a^+.$$

Multiplying the equality on the right by a^+a, we get $a^+a^+a = a^+$. Hence, $a \in R^{EP}$ by [14, Theorem 1.2.1]. Hence $x = a^+ = a^{\#}$, by (2), $a \in R^{SEP}$.

(4) If $x = a^*$, then

$$(a^{\#})^*a^*a^{\#} = a^*.$$

By multiplying the equality on the right by a^+a, we obtain

$$a^*a^+a = a^*.$$

Therefore, we conclude that $a \in R^{EP}$. By [14, Theorem 1.1.3],

$$a^* = (a^{\#})^*a^*a^{\#} = aa^{\#}a^{\#} = a^{\#}.$$

Hence, $a \in R^{SEP}$.

(5) If $x = (a^+)^*$, then

$$(a^{\#})^*(a^+)^*a^{\#} = (a^+)^*.$$

Applying the involution, one gets

$$(a^{\#})^*a^+a^{\#} = a^+.$$

By (3), $a \in R^{SEP}$.

(6) If $x = (a^{\#})^*$, then

$$(a^{\#})^*(a^{\#})^*a^{\#} = (a^{\#})^*.$$

Applying the involution, one obtains

$$(a^{\#})^*a^{\#}a^{\#} = a^{\#}.$$

Hence $a \in R^{SEP}$ by (2).

(7) If $x = aa^+$, then $(a^{\#})^*aa^+a^{\#} = aa^+$. Noting that $(a^{\#})^*aa^+ = (a^{\#})^*$ and $a^{\#} = aa^+a^{\#}$. Then

$$a^{\#} = aa^+a^{\#} = (a^{\#})^*aa^+a^{\#}a^{\#} = (a^{\#})^*a^{\#}a^{\#}.$$

Hence $a \in R^{SEP}$ by (2). □

5. Characterize SEP elements by the solution of bivariate equations

From Eq.(4.5), we construct the following bivariate equations.

$$(a^{\#})^* x a^{\#} = y. \tag{5.1}$$

Theorem 5.1. *Let $a \in R^{\#} \cap R^{+}$. Then the general solution to Eq.(5.1) is given by*

$$\begin{cases} x = p + u - aa^{+}uaa^{+} \\ y = (a^{\#})^* p a^{\#} \end{cases}, \textit{where } p,\ u \in R. \tag{5.2}$$

Proof. First the formula (5.2) is the solution to Eq.(5.1). In fact,

$$(a^{\#})^* x a^{\#} = (a^{\#})^*(p + u - aa^{+}uaa^{+})a^{\#} = (a^{\#})^* p a^{\#} = y.$$

Next, let

$$\begin{cases} x = x_0 \\ y = y_0 \end{cases} \tag{5.3}$$

be any solution to Eq.(5.1). Then

$$(a^{\#})^* x_0 a^{\#} = y_0.$$

Choose $p = aa^{+}a^* y_0 a^2 a^{+}$, and $u = x_0$. Then

$$aa^{+}uaa^{+} = aa^{+}x_0 aa^{+} = aa^{+}a^*((a^{\#})^* x_0 a^{\#})a^2 a^{+} = aa^{+}a^* y_0 a^2 a^{+} = p,$$

it follows that

$$x_0 = p + x_0 - aa^{+}uaa^{+} = p + u - aa^{+}uaa^{+}.$$

Also,

$$(a^{\#})^* p a^{\#} = (a^{\#})^* aa^{+}a^* y_0 a^2 a^{+} a^{\#} = (aa^{\#})^* y_0 aa^{\#} = (aa^{\#})^*((a^{\#})^* x_0 a^{\#})aa^{\#} = (a^{\#})^* x_0 a^{\#} = y_0.$$

Hence, the general solution to Eq.(5.1) is provided by the formula (5.2). □

Theorem 5.2. *Let $a \in R^{\#} \cap R^{+}$. Then $a \in R^{SEP}$ if and only if the general solution to Eq.(5.1) is given by*

$$\begin{cases} x = p + u - aa^{+}uaa^{+} \\ y = (a^{\#})^* p a^* \end{cases}, \textit{where } p,\ u \in R. \tag{5.4}$$

Proof. " $\Longrightarrow$ " Since $a \in R^{SEP}$, we have $a^* = a^{\#}$. Obviously, the formula (5.2) and the formula (5.4) are consistent. By Theorem 5.1, we are done.

" $\Longleftarrow$ " For the condition, one gets

$$(a^{\#})^*(p + u - aa^{+}uaa^{+})a^{\#} = (a^{\#})^* p a^*,$$

i.e.,

$$(a^{\#})^* pa^{\#} = (a^{\#})^* pa^*$$

for any $p \in R$. Choosing $p = a^*$, one gets

$$(aa^{\#})^* a^{\#} = a^*.$$

Applying the involution on the equality, one has

$$(a^{\#})^* aa^{\#} = a.$$

By Theorem 4.6, $a \in R^{SEP}$. □

Theorem 5.3. *Let $a \in R^{\#} \cap R^{+}$. Then $a \in R^{SEP}$ if and only if the general solution to Eq.(5.1) is given by*

$$\begin{cases} x = p + u - aa^{+}uaa^{+} \\ y = apa^{\#} \end{cases}, \textit{where } p,\ u \in R. \tag{5.5}$$

Proof. " $\Longrightarrow$ " Given that $a \in R^{SEP}$, we have $a = (a^{\#})^*$. Clearly, the formula (5.2) can be expressed as the formula (5.5), as desired by Theorem 5.1.

" $\Longleftarrow$ " From the condition, one obtains

$$(a^{\#})^*(p + u - aa^{+}uaa^{+})a^{\#} = apa^{\#},$$

e.g.,

$$(a^{\#})^* pa^{\#} = apa^{\#}$$

for each $p \in R$. Choosing $p = a$. Then $(a^{\#})^* aa^{\#} = a$. By Theorem 4.6, $a \in R^{SEP}$. □

We change Eq.(5.1) as follows.

$$a(aa^{\#})^* xa^{\#} = y. \tag{5.6}$$

Theorem 5.4. *Let $a \in R^{\#} \cap R^{+}$. Then the general solution to Eq.(5.6) is given by*

$$\begin{cases} x = p + u - aa^{+}uaa^{+} \\ y = apa^{\#} \end{cases}, \textit{where } p,\ u \in R \textit{ with } a^{+}p = a^{+}a^{+}ap. \tag{5.7}$$

Proof. Since

$$a(aa^{\#})^* xa^{\#} = a(aa^{\#})^*(p + u - aa^{+}uaa^{+})a^{\#} = a(aa^{\#})^* pa^{\#}$$

$$= a(aa^{\#})^* aa^{+}pa^{\#} = a(aa^{\#})^* aa^{+}a^{+}apa^{\#} = a(aa^{\#})^* a^{+}apa^{\#} = apa^{\#} = y,$$

the formula (5.7) is the solution to Eq.(5.6).

Now let

$$\begin{cases} x = x_0 \\ y = y_0 \end{cases} \tag{5.8}$$

be any solution to Eq.(5.6). It follows $a(aa^{\#})^* x_0 a^{\#} = y_0$. Choose $p = a^+ y_0 a$, and $u = x_0 - p$. This gives

$$aa^+ uaa^+ = aa^+(x_0 - p)aa^+ = aa^+ x_0 aa^+ - aa^+ a^+ y_0 aaa^+$$

$$= aa^+(aa^{\#})^* x_0 a^{\#} aaa^+ - aa^+ a^+ y_0 aaa^+ = aa^+ a^+ a(aa^{\#})^* x_0 a^{\#} aaa^+ - aa^+ a^+ y_0 aaa^+$$

$$= aa^+ a^+ y_0 aaa^+ - aa^+ a^+ y_0 aaa^+ = 0.$$

Hence $x = p + (x_0 - p) = p + u = p + u - aa^+ uaa^+$. At the same time,

$$apa^{\#} = a(a^+ y_0 a)a^{\#} = a(a^+ a(aa^{\#})^* x_0 a^{\#})aa^{\#} = a(aa^{\#})^* x_0 a^{\#} = y_0.$$

Consequently, the formula (5.7) offers the general solution to Eq.(5.6). □

Theorem 5.5. *Let $a \in R^{\#} \cap R^+$. Then $a \in R^{SEP}$ if and only if Eq.(5.1) has the same solution as Eq.(5.6).*

Proof. It is evident from Theorem 5.3 and Theorem 5.4. □

6. The solution to non-homogeneous bivariate equations

$$(a^{\#})^* xa^{\#} - y = a^+. \tag{6.1}$$

Theorem 6.1. *Let $a \in R^{\#} \cap R^+$. Then the general solution to Eq.(6.1) is given by*

$$\begin{cases} x = p + u - aa^+ uaa^+ \\ y = (a^{\#})^* pa^{\#} - a^+ \end{cases}, \textit{where } p,\ u \in R. \tag{6.2}$$

Proof. Clearly, the formula (6.2) is the solution to Eq.(6.1).

Now assuming that

$$\begin{cases} x = x_0 \\ y = y_0 \end{cases} \tag{6.3}$$

represents any solution to Eq.(6.1), then we have

$$(a^{\#})^* x_0 a^{\#} - y_0 = a^+.$$

Choose $p = a^*(a^+ + y_0)a$, and $u = x_0 - p$. Then we have

$$aa^+ x_0 aa^+ = aa^+ a^*((a^{\#})^* x_0 a^{\#})a^2 a^+ = aa^+ a^*(y_0 + a^+)a^2 a^+ = aa^+ paa^+.$$

It follows that $aa^+uaa^+ = 0$. Hence

$$x_0 = p + (x_0 - p) = p + u = p + u - aa^+uaa^+.$$

Also we have

$$(a^\#)^*pa^\# = (a^\#)^*a^*(a^+ + y_0)aa^\# = a^+aa^\# + (aa^\#)^*y_0aa^\#$$

$$= a^+aa^\# + (aa^\#)^*((a^\#)^*x_0a^\# - a^+)aa^\# = (a^\#)^*x_0a^\# = y_0 + a^+. \qquad \square$$

This induces

$$y_0 = (a^\#)^*pa^\# - a^+.$$

Hence the general solution to Eq.(6.1) is given by (6.2).

Theorem 6.2. *Let $a \in R^\# \cap R^+$. Then $a \in R^{SEP}$ if and only if the general solution to Eq.(6.1) is given by*

$$\begin{cases} x = p + u - aa^+uaa^+ \\ y = (a^\#)^*pa^+ - a^* \end{cases}, \text{where } p,\ u \in R. \tag{6.4}$$

Proof. " $\Longrightarrow$ " Given that $a \in R^{SEP}$, we have $a^+ = a^* = a^\#$. It's evident that the formula (6.2) can be expressed as the formula (6.4), as desired by Theorem 6.1.

" $\Longleftarrow$ " The condition implies

$$(a^\#)^*(p + u - aa^+uaa^+)a^\# - ((a^\#)^*pa^+ - a^*) = a^+$$

for $p \in R$. Choose $p = 0$. Then one gets $a^+ = a^*$ which follows that $a \in R^{PI}$. Choose $p = a^\#$. Then the equation becomes $(a^\#)^*a^\#a^\# = (a^\#)^*a^\#a^+$, and

$$aa^\# = a^3a^+a^*(a^\#)^*a^\#a^\# = a^3a^+a^*(a^\#)^*a^\#a^+ = aa^+.$$

Thus $a \in R^{EP}$ by [14, Theorem 1.2.1]. Therefore we deduce $a \in R^{SEP}$. $\square$

Revised Eq.(6.1) as follows:

$$(a^\#)^*xa^+ - y = a^*. \tag{6.5}$$

Similar to the proof of Theorem 6.1, we have the following theorem.

Theorem 6.3. *Let $a \in R^\# \cap R^+$. Then the general solution to Eq.(6.5) is given by*

$$\begin{cases} x = p + u - aa^+ua^+a \\ y = (a^\#)^*pa^+ - a^* \end{cases}, \text{where } p,\ u \in R. \tag{6.6}$$

Theorem 6.4. *Let $a \in R^\# \cap R^+$. Then $a \in R^{SEP}$ if and only if Eq.(6.5) has the same solution as Eq.(6.1).*

Proof. " $\Longrightarrow$ " Suppose that $a \in R^{SEP}$. Then the formula (6.4) and the formula (6.6) are the same, by Theorem 6.2 and Theorem 6.3, we are done.

" $\Longleftarrow$ " Indeed,

$$\begin{cases} x = a \\ y = (a^{\#})^* - a^* \end{cases} \tag{6.7}$$

clearly represents a solution to Eq.(6.5).

By the assumption, it is also a solution to Eq.(6.1). Hence we obtain

$$(a^{\#})^* a a^{\#} - (a^{\#})^* + a^* = a^+.$$

Multiplying the equality on the right by aa^+, one gets

$$(a^{\#})^* a a^{\#} = (a^{\#})^*.$$

This gives $a \in R^{EP}$ by [14, Theorem 1.1.3] and

$$a^+ = (a^{\#})^* a a^{\#} - (a^{\#})^* + a^* = a^*.$$

Therefore $a \in R^{SEP}$. □

7. Using nil-cleanity of aa^*a^+a to characterize SEP elements

Theorem 7.1. *Let $a \in R^{\#} \cap R^+$. Then $a \in R^{SEP}$ if and only if $aa^*a^+a = u + p$, where $u^2 = 0$, $p \in PE(R)$, and $pu = up = 0$.*

Proof. " $\Longrightarrow$ " Given that $a \in R^{SEP}$, we know that $aa^*a^+a \in PE(R)$ by Theorem 2.1. Let's choose $u = 0$ and $p = aa^*a^+a$. Then, we have $aa^*a^+a = u + p$, where $u^2 = 0$ and $pu = up = 0$.

" $\Longleftarrow$ " From the assumption, we have

$$aa^*a^+ap = (u+p)p = up + p^2 = p,$$

$$paa^*a^+a = p(u+p) = pu + p^2 = p.$$

Furthermore, we find

$$0 = u^2 = (aa^*a^+a - p)^2 = (aa^*a^+a)^2 - aa^*a^+ap - paa^*a^+a + p = aa^*a^+a^2a^*a^+a - p.$$

Thus, we conclude that $aa^*a^+a^2a^*a^+a = p = paa^*a^+a$. Applying the involution to the last equality, one gets

$$a^+a^2a^*a^+a^2a^* = p = aa^*a^+a^2a^*a^+a.$$

Then, we have

$$a^+a^2a^*a^+a^2a^* = a^+a^2a^*a^+a^2a^*a^+a.$$

Multiplying $a^{\#}a$ on the left of both sides, we get

$$aa^*a^+a^2a^* = aa^*a^+a^2a^*a^+a.$$

To proceed, multiply the equality by $(a^{\#})^*a^+$, we obtain $a^+a^2a^* = a^+a^2a^*a^+a$. Finally, multiplying $a^+a^{\#}a$ on the left, one gets $a^* = a^*a^+a$, thus, $a^+aa = a$. Consequently, $a \in R^{EP}$. We have

$$aa^*aa^* = aa^*a^+a^2a^*a^+a = p = aa^*a^+ap = aa^*p,$$

$$aa^*aa^*aa^* = aa^*p = p = aa^*aa^*,$$

$$a^*aa^*aa^* = a^+aa^*aa^*aa^* = a^+aa^*aa^* = a^*aa^*,$$

it follows by applying the involution to the last equality, one yields

$$aa^*aa^*a = aa^*a,$$

$$a^*aa^* = a^+(aa^*aa^*a)a^+ = a^+(aa^*a)a^+ = a^*.$$

Hence $a \in R^{PI}$, and so $a \in R^{SEP}$. □

Theorem 7.2. *Let $a \in R^{\#} \cap R^+$. Then $a \in R^{SEP}$ if and only if $aa^*a^+a = u + p$, where $u^2 = 1$, $up = pu = -p$ and $1 - p = aa^*$, $p \in PE(R)$.*

Proof. "$\Longrightarrow$" Since $a \in R^{SEP}$, we have $aa^*a^+a = aa^+$ by Theorem 2.1. Choose $u = 2aa^+ - 1$, $p = 1 - aa^+ \in PE(R)$. Then

$$u^2 = 1,\ up = aa^+ - 1 = -p,\ pu = aa^+ - 1 = -p,\ 1 - p = aa^+ = aa^*.$$

Clearly, $aa^*a^+a = u + p$.

"$\Longleftarrow$" From the conditions, we have

$$aa^*a^+ap = (u + p)p = up + p = -p + p = 0,$$

$$paa^*a^+a = p(u + p) = p + pu = p - p = 0,$$

and

$$1 = u^2 = (aa^*a^+a - p)^2 = aa^*a^+a^2a^*a^+a - aa^*a^+ap - paa^*a^+a + p = aa^*a^+a^2a^*a^+a + p.$$

After applying the involution to the last equality, we obtain

$$aa^*a^+a^2a^*a^+a = a^+a^2a^*a^+a^2a^*.$$

Similar to the proof of Theorem 7.1, we get $a \in R^{EP}$. This gives

$$aa^* = aa^*a^+a = u + p,$$

and

$$aa^*aa^* = (u + p)^2 = u^2 + up + pu + p = 1 - p - p + p = 1 - p = aa^*.$$

This leads to

$$a = aa^*(a^+)^* = aa^*aa^*(a^+)^* = aa^*a.$$

In summary, $a \in R^{SEP}$. □

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References

[1] D. Drivaliaris, S. Karanasios and D. Pappas, *Factorizations of EP operators*, Linear Algebra Appl., 429(7) (2008), 1555-1567.

[2] M. G. Guan, X. R. Wang and J. C. Wei, *Characterizations of SEP elements in a ring with involution*, Filomat, 38(5) (2024), 1703-1710.

[3] R. E. Harte and M. Mbekhta, *On generalized inverses in C^*-algebras*, Studia Math., 103(1) (1992), 71-77.

[4] R. E. Hartwig, *Block generalized inverses*, Arch. Rational Mech. Anal., 61(3) (1976), 197-251.

[5] R. E. Hartwig, *Generalized inverses, EP elements and associates*, Rev. Roumaine Math. Pures Appl., 23(1) (1978), 57-60.

[6] R. E. Hartwig and I. J. Katz, *Products of EP elements in reflexive semigroups*, Linear Algebra Appl., 14(1) (1976), 11-19.

[7] S. Karanasios, *EP elements in rings and semigroup with involution and C^*-algebras*, Serdica Math. J., 41(1) (2015), 83-116.

[8] J. J. Koliha, *The Drazin and Moore-Penrose inverse in C^*-algebras*, Math. Proc. R. Ir. Acad., 99A(1) (1999), 17-27.

[9] J. J. Koliha, D. S. Djordjevic and D. Cvetkovic, *Moore-Penrose inverse in rings with involution*, Linear Algebra Appl., 426(2-3) (2007), 371-381.

[10] J. J. Koliha and P. Patricio, *Elements of rings with equal spectral idempotents*, J. Aust. Math. Soc., 72(1) (2002), 137-152.

[11] A. Q. Li, M. G. Guan and J. C. Wei, *SEP elements and solutions of related equations in a ring with involution*, Filomat, 37(24) (2023), 8213-8228.

[12] A. Q. Li and J. C. Wei, *The influence of the expression form of solutions to related equations on SEP elements in a ring with involution*, J. Algebra Appl., 23(6) (2024), 2450113 (16 pp).

[13] A. Q. Li and J. C. Wei, *The equation characterizations of generalized inverses in a ring with involution*, J. Algebra Appl., 24(8) (2025), 2550206 (25 pp).

[14] D. Mosic, Generalized Inverses, Faculty of Sciences and Mathematics, University of Nis, Nis, Serbia, 2018.

[15] D. Mosic and D. S. Djordjevic, *Further results on partial isometries and EP elements in rings with involution*, Math. Comput. Modelling, 54(1-2) (2011), 460-465.

[16] D. Mosic and D. S. Djordjevic, *New characterizations of EP, generalized normal and generalized Hermitian elements in rings*, Appl. Math. Comput., 218(12) (2012), 6702-6710.

[17] D. Mosic, D. S. Djordjevic and J. J. Koliha, *EP elements in rings*, Linear Algebra Appl., 431(5-7) (2009), 527-535.

[18] Y. Sui, Y. Huang and J. C. Wei, *Strongly EP elements and the solutions of equations in rings*, Int. Electron. J. Algebra, 32 (2022), 129-140.

[19] Z. C. Xu, R. Tang and J. C. Wei, *Strongly EP elements in a ring with involution*, Filomat, 34(6) (2020), 2101-2107.

[20] D. D. Zhao and J. C. Wei, *Strongly EP elements in rings with involution*, J. Algebra Appl., 21(5) (2022), 2250088 (10 pp).

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