International Electronic Journal of Algebra
Published Online: November 2023
DOI: 10.24330/ieja.1385180

DIMENSION OF UNCOUNTABLY GENERATED SUBMODULES

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Received: 28 July 2023; Accepted: 21 August 2023
Communicated by Alberto Facchini

Abstract. In this article we introduce and study the concepts of uncountably generated Krull dimension and uncountably generated Noetherian dimension of an R-module, where R is an arbitrary associative ring. These dimensions are ordinal numbers and extend the notion of Krull dimension and Noetherian dimension. They respectively rely on the behavior of descending and ascending chains of uncountably generated submodules. It is proved that a quotient finite dimensional module M has uncountably generated Krull dimension if and only if it has Krull dimension, but the values of these dimensions might differ. Similarly, a quotient finite dimensional module M has uncountably generated Noetherian dimension if and only if it has Noetherian dimension. We also show that the Noetherian dimension of a quotient finite dimensional module M with uncountably generated Noetherian dimension β is less than or equal to $\omega_1 + \beta$, where ω_1 is the first uncountable ordinal number.

Mathematics Subject Classification (2020): 16P60, 16P20, 16P40
Keywords: Chain condition on uncountably generated submodules, Noetherian dimension, Krull dimension, uncountably generated Krull dimension, uncountably generated Noetherian dimension

1. Introduction

Throughout this paper let R be an arbitrary associative ring. Unless stated otherwise, R-modules are always understood to be right R-modules. Lemonnier [19] has introduced the concepts of deviation and codeviation of an arbitrary poset. In particular, if we apply these two concepts to the lattice of all submodules of an R-module M we obtain the concepts of Krull dimension (in the sense of Rentschler and Gabriel) and dual Krull dimension of M, respectively, see [12,18,21,22] and [1,2,3,4,13,14,15,17]. The dual Krull dimension of M in the latter references is called Noetherian dimension of M. The Noetherian dimension of an R-module M is denoted by $n\text{-}\dim M$ and by $k\text{-}\dim M$ we denote the Krull dimension of M. The above two dimensions and some other Krull type (respectively, Noetherian

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The author is grateful to the Research Council of Shahid Chamran University of Ahvaz for financial support (SCU.MM1401.192).

type) dimensions for the poset of certain submodules of an R-module M have been studied by many authors who obtained some useful results in the literature, see [1,5,6,7]. The concept of α-short modules is introduced in [9]. We recall that an R-module M is called α-short, if for each submodule N of M, either $n\text{-}\dim N \leq \alpha$ or $n\text{-}\dim \frac{M}{N} \leq \alpha$ and α is the least ordinal number with this property. We recall that the Goldie dimension of an R-module M, denoted by $G\text{-}\dim M$ is the supremum λ of the cardinals k such that M contains a direct sum of k nonzero submodules, Given a cardinal number k, we say k is attained in M if M contains a direct sum of k nonzero submodules, see [10]. An R-module M is called countably generated if there exists a countable subset A of M such that $< A >= M$; otherwise M is uncountably generated. The author [8] extensively studied modules with chain condition on uncountably generated submodules.

In this paper we introduce and study the concepts of uncountably generated Krull dimension and uncountably generated Noetherian dimension of an R-module M, denoted by $uck\text{-}\dim M$ and $ucn\text{-}\dim M$ respectively. These dimensions are ordinal numbers and reflect the behavior of descending and ascending chains of uncountably generated submodules. Our main aim of studying these dimensions is to compare them with Krull dimension and Noetherian dimension of modules.

In Section 2 of this paper we investigate some basic properties of uncountably generated Krull dimension. In essence, this dimension measures the deviation of the poset of the uncountably generated submodules of M from being Artinian. We briefly study this dimension and trivially observe that if M has Krull dimension then $uck\text{-}\dim M \leq k\text{-}\dim M$. We also observe that some basic facts for modules with this dimension are similar to the basic properties of modules with Krull dimension. We show that if an R-module M has uncountably generated Krull dimension, then the Goldie dimension of M is less than or equal to c, where c is the first uncountable cardinal number. We also observe that if $G\text{-}\dim M = c$, then c is not attained in M.

Section 3 is devoted to the study of the concept of uncountably generated Noetherian dimension, which is the dual of the concept of uncountably generated Krull dimension. It is proved that a quotient finite dimensional R-module M has uncountably generated Noetherian dimension if and only if it has Noetherian dimension. We show that if M is a quotient finite dimensional R-module and $ucn\text{-}\dim M = \beta$, then M has Noetherian dimension and $n\text{-}\dim M \leq \beta + \omega_1$, where ω_1 is the first uncountable ordinal number. We also observe that a quotient finite dimensional R-module M has uncountably generated Krull dimension if and only if it has Krull dimension, but the values of these dimensions might differ. We recall that if an R-module M has Noetherian dimension and α is an ordinal number, then M is called α-atomic if $n\text{-}\dim M = \alpha$ and $n\text{-}\dim N < \alpha$, for all proper submodules N of M. An R-module M is called atomic if it is α-atomic for some ordinal α (note, atomic modules are also called conotable, dual critical and N-critical in some other articles; see for example [2], [4] and [20]).

The notation $N \subseteq M$ (resp. $N \subset M$) means that N is a submodule (resp. proper submodule) of M. For all concepts and basic properties of rings and modules, which are not defined in this paper, we refer the reader to [12], [15], [20], and [21].

We should remind the reader that by a quotient finite dimensional module M, we mean for each submodule N of M, $\frac{M}{N}$ has finite Goldie dimension.

Let us close this introduction with the following well-known and important result, see [20, Theorem 2.4] and [1, Proposition 2.2].

Proposition 1.1. *The following statements are equivalent for any R-module M and any ordinal $\alpha \geq 0$:*

(1) $n\text{-}\dim M \leq \alpha$*;*
(2) *M is quotient finite dimensional and for any chain of submodules $N \subset P \subseteq M$, there exists an R-module X with $N \subseteq X \subset P$ and $n\text{-}\dim \frac{P}{X} \leq \alpha$.*

2. Uncountably generated Krull dimension

In this section we introduce and study the concept of uncountably generated Krull dimension of an R-module M, which is a Krull-like dimension extension of the concept of descending chain condition on uncountably generated submodules of M. In other word, it is the deviation of the poset of the uncountably generated submodules of M.

Next, we give our definition of uncountably generated Krull dimension.

Definition 2.1. Let M be an R-module. The uncountably generated Krull dimension of M (briefly, uc-Krull dimension), denoted by $uck\text{-}\dim M$ is defined by transfinite recursion as follows: If M does not have any uncountably generated submodule, then $uck\text{-}\dim M = -1$. If α is an ordinal number and $uck\text{-}\dim M \not< \alpha$, then $uck\text{-}\dim M = \alpha$ provided that there is no infinite descending chain $M_0 \supseteq M_1 \supseteq M_2 \supseteq \cdots$ of uncountably generated submodules of M such that $uck\text{-}\dim \frac{M_i}{M_{i+1}} \not< \alpha$ for each $i \geq 1$. In otherwise $uck\text{-}\dim M = \alpha$, if $uck\text{-}\dim \not< \alpha$ and for each infinite descending chain of uncountably generated submodules of M such as $M_0 \supseteq M_1 \supseteq M_2 \supseteq \cdots$ there exists an integer t, such that for each $i \geq t$, we have

$uck\text{-}\dim \frac{M_i}{M_{i+1}} < \alpha$. A ring R has uncountably generated Krull dimension, if as an R-module has uncountably generated Krull dimension. It is possible that there is no ordinal α such that $uck\text{-}\dim M = \alpha$, in this case we say M does not have uncountably generated Krull dimension.

If $uck\text{-}\dim M > \alpha$, there exists an infinite descending chain $M_0 \supsetneq M_1 \supsetneq M_2 \supsetneq \cdots$ of uncountably generated submodules of M such that $uck\text{-}\dim \frac{M_i}{M_{i+1}} \geq \alpha$, for all i.

Clearly $uck\text{-}\dim M \leq 0$ if and only if M satisfies the descending chain condition on its uncountably generated submodules, see [8].

The following evident lemma is needed.

Lemma 2.2. *Let an R-module M have Krull dimension. Then M has uc-Krull dimension and $uck\text{-}\dim M \leq k\text{-}\dim M$.*

Let M be an R-module. In [16], it is shown that every submodule of modules with countable Noetherian dimension is countably generated. Hence, if M has countable Noetherian dimension or is ω_1-atomic, then $uck\text{-}\dim M \leq 0$. There are Noetherian V-domains (called Cozzen's domains) which are neither left nor right Artinian, see [11, Theorem 7.45]. This shows that the values of Krull dimension and uc-Krull dimension for an R-module might differ.

Lemma 2.3. *Let the R-module M have uc-Krull dimension. Then, for each submodule A of M, A has uc-Krull dimension and $uck\text{-}\dim A \leq uck\text{-}\dim M$.*

Proof. It is clear that every chain of uncountably generated submodules of A is a chain of uncountably generated submodules of M and we are done. $\square$

Theorem 2.4. *Let the R-module M have uc-Krull dimension. Then, for each submodule A of M, $\frac{M}{A}$ has uc-Krull dimension and $uck\text{-}\dim \frac{M}{A} \leq uck\text{-}\dim M$.*

Proof. Let $uck\text{-}\dim M = \alpha$ and let $\frac{M_1}{A} \supsetneq \frac{M_2}{A} \supsetneq \cdots$ be a chain of uncountably generated submodules of $\frac{M}{A}$. It is clear that $M_1 \supsetneq M_2 \supsetneq \cdots$ is a chain of uncountably generated submodules of M. Hence there exists an integer number t such that for each $i \geq t$ we get $uck\text{-}\dim \frac{M_i/A}{M_{i+1}/A} = uck\text{-}\dim \frac{M_i}{M_{i+1}} < \alpha$. Thus $uck\text{-}\dim \frac{M}{A}$ exists and it is less than or equal to α. $\square$

In view of previous theorem and Lemma 2.3 we have the following result.

Corollary 2.5. *Let the R-module M have uc-Krull dimension. If A is a submodule of M, then*

$$\sup\{uck\text{-}\dim A, uck\text{-}\dim \frac{M}{A}\} \leq uck\text{-}\dim M.$$

For any R-module M we shall denote by $UC(M)$ the poset of all uncountably generated submodules of M.

Lemma 2.6. *Assume that each $A \in UC(M) \setminus \{M\}$ has uc-Krull dimension. Then M has uc-Krull dimension and $uck\text{-}\dim M = \sup\{uck\text{-}\dim A : A \in UC(M) \setminus \{M\}\}$.*

Proof. Put $\sup\{uck\text{-}\dim A : A \in UC(M) \setminus \{M\}\} = \alpha$. Let $M_1 \supseteq M_2 \supseteq \cdots$ be any chain of uncountably generated submodules of M, hence there exists an integer number t such that for each $i \geq t$, we have $uck\text{-}\dim \frac{M_i}{M_{i+1}} < uck\text{-}\dim M_1 \leq \alpha$. This implies that M has uc-Krull dimension and $uck\text{-}\dim M \leq \alpha$. Now, in view of Lemma 2.3, we infer that $uck\text{-}\dim M = \alpha$. □

Lemma 2.7. *Assume that $\frac{M}{A}$ has uc-Krull dimension for all $A \in UC(M)$. Then M has uc-Krull dimension and*

$$uck\text{-}\dim M \leq \sup\{uck\text{-}\dim (\frac{M}{A}) + 1 : A \in UC(M)\}.$$

Proof. Let $M_1 \supseteq M_2 \supseteq \cdots$ be an infinite chain of uncountably generated submodules of M. In view of Lemma 2.3, we infer that $\frac{M_i}{M_{i+1}} \subseteq \frac{M}{M_{i+1}}$ has uc-Krull dimension and $uck\text{-}\dim \frac{M_i}{M_{i+1}} \leq uck\text{-}\dim \frac{M}{M_{i+1}} < uck\text{-}\dim \frac{M}{M_{i+1}} + 1$. Hence $uck\text{-}\dim M \leq \sup\{uck\text{-}\dim (\frac{M}{A}) + 1 : A \in UC(M)\}$ and we are done. □

Lemma 2.8. *Assume that for each $A \in UC(M)$ either A or $\frac{M}{A}$ has uc-Krull dimension. Then, M has uc-Krull dimension, too.*

Proof. If there exists some i such that M_i has uc-Krull dimension, then each $\frac{M_k}{M_{k+1}}$ has uc-Krull dimension for $k \geq i$. Otherwise $\frac{M}{M_i}$ has uc-Krull dimension for each i. Thus there exists some integer k such that in any case $\frac{M_i}{M_{i+1}}$ has uc-Krull dimension for each $i \geq k$. Consequently, M has uc-Krull dimension. □

The following proposition shows that there exists an R-module M which does not have uc-Krull dimension.

Proposition 2.9. *Let the R-module M have uc-Krull dimension. Then $G\text{-}\dim M \leq c$, where c the first uncountable cardinal number; moreover if $G\text{-}\dim M = c$, then c is not attained in M.*

Proof. The proof is by induction on $uck\text{-}\dim M$. If $uck\text{-}\dim M \leq 0$, then M satisfies the descending chain condition on uncountably generated submodules, hence by [8, Lemma 4.6], we get $G\text{-}\dim M \leq c$ and in this case if $G\text{-}\dim M = c$, then c is not attained in M. Now let α be an ordinal number and let $uck\text{-}\dim M = \alpha$. Let $\mathbb{R}$ be the set of real numbers. We know that $card(\mathbb{R}) = c$. We assume that $\oplus_{i \in \mathbb{R}} M_i \subseteq M$, with $card(\mathbb{R}) = c$ and $M_i \neq 0$ for all $i \in \mathbb{R}$ and seek a contradiction. Let $I_0 = \mathbb{R}$, and $I_1 = \mathbb{R} - (0,1)$, $I_2 = \mathbb{R} - (0,2)$ and $I_n = \mathbb{R} - (0,n)$, where n is an integer number. Put $N_i = \oplus_{k \in I_n} M_k$ for all $i \geq 0$. Thus we have the following chain

$$N_0 \supset N_1 \supset N_2 \supset \cdots$$

of uncountably generated submodules of M. Inasmuch as $uck\text{-}\dim M = \alpha$, we infer that there exists an integer k, such that for each $i \geq k$, $uck\text{-}\dim \frac{N_i}{N_{i+1}} < \alpha$. But it is clear that $G\text{-}\dim \frac{N_i}{N_{i+1}} \geq c$ for each i; moreover if $G\text{-}\dim \frac{N_i}{N_{i+1}} = c$, then c is attained in $\frac{N_i}{N_{i+1}}$. Hence by induction hypothesis $\frac{N_i}{N_{i+1}}$ does not have uc-Krull dimension for each i, which is a contradiction. □

In view of the previous proposition and Theorem 2.4, the following corollary is now evident.

Corollary 2.10. *Let the R-module M have uc-Krull dimension. Then for each submodule N of M, we get $G\text{-}\dim \frac{M}{N} \leq c$; moreover if $G\text{-}\dim \frac{M}{N} = c$, then c is not attained in $\frac{M}{N}$.*

We recall that an R-module M is called α-critical if $k\text{-}\dim M = \alpha$ and $k\text{-}\dim \frac{M}{N} < \alpha$, for each nonzero submodule N of M. M is called critical if it is α-critical for some ordinal number α. In the following definition we consider a related concept.

Definition 2.11. Let M be an R-module. We say that M is α-uc-critical, if $uck\text{-}\dim M = \alpha$ and for each uncountably generated submodule N of M we have $uck\text{-}\dim \frac{M}{N} < \alpha$. M is called uc-critical if it is α-uc-critical for some ordinal number α.

The proof of the following result is similar to [12, Theorem 2.1], and thus is omitted.

Lemma 2.12. *Let M be a nonzero module with uc-Krull dimension. Then M has a uc-critical submodule.*

Finally, we conclude this section with the following result.

Theorem 2.13. *Let the R-module M satisfy the ascending chain condition on uncountably generated submodules. If $uck\text{-}\dim M = \alpha$, then for each $\beta \leq \alpha$, there exists a submodule N of M such that $\frac{M}{N}$ is a β-uc-critical module.*

Proof. Fix an ordinal $\beta \leq \alpha$ and put

$$F = \{N \in UC(M) : uck\text{-}\dim \frac{M}{N} \geq \beta\}.$$

If $F = \emptyset$, then M is α-uc-critical. Otherwise, since M satisfies the ascending chain condition on uncountably generated submodules and in view of [8, Lemma 3.4], we infer that F has a maximal element, N say. It suffices to show that uck-dim $\frac{M}{N} = \beta$. If uck-dim $\frac{M}{N} > \beta$, then there exists a chain

$$\frac{N_1}{N} \supseteq \frac{N_2}{N} \supseteq \cdots$$

of uncountably generated submodules of $\frac{M}{N}$ such that for each i, uck-dim $\frac{N_i}{N_{i+1}} \geq \beta$ and this is a contradiction. Note that N is a maximal element of F; thus for each i, uck-dim $\frac{N_i}{N_{i+1}} \leq uck$-dim $\frac{M}{N_{i+1}} < \beta$. □

3. Uncountably generated Noetherian dimension

In this section we introduce and study the concept of uncountably generated Noetherian dimension, which is the dual of the concept of uncountably generated Krull dimension. In fact this dimension is a Krull-like dimension extension of the concept of ascending chain condition on uncountably generated submodules of M. In other word, it is the codeviation of the poset of uncountably generated submodules of M.

Next, we give our definition of uncountably generated Noetherian dimension.

Definition 3.1. Let M be an R-module. The uncountably generated Noetherian dimension of M (briefly, uc-Noetherian dimension), denoted by ucn-dim M is defined by transfinite recursion as follows: If M does not have any uncountably generated submodule, then ucn-dim $M = -1$. If α is an ordinal number and uck-dim $M \not< \alpha$, then uck-dim $M = \alpha$ provided that there is no infinite ascending chain $M_0 \subseteq M_1 \subseteq M_2 \subseteq \cdots$ of uncountably generated submodules of M such that ucn-dim $\frac{M_i}{M_{i+1}} \not< \alpha$ for each $i \geq 1$. In otherwise ucn-dim $M = \alpha$, if ucn-dim $\not< \alpha$ and for each infinite ascending chain of uncountably generated submodules of M such as $M_0 \subseteq M_1 \subseteq M_2 \subseteq \cdots$ there exists an integer t, such that for each $i \geq t$, we have ucn-dim $\frac{M_{i+1}}{M_i} < \alpha$. A ring R has uncountably generated Noetherian dimension, if as an R-module has uncountably generated Noetherian dimension. It is possible that there is no ordinal α such that ucn-dim $M = \alpha$, in this case we say M does not have uncountably generated Noetherian dimension.

If ucn-dim $M > \alpha$, there exists an infinite ascending chain $M_0 \subseteq M_1 \subseteq M_2 \subseteq \cdots$ of uncountably generated submodules of M such that ucn-dim $\frac{M_{i+1}}{M_i} \geq \alpha$, for all i.

In [16], it is shown that every submodule of modules with countable Noetherian dimension is countably generated. Hence, if M has countable Noetherian dimension or ω_1-atomic, then ucn-dim $M \leq 0$.

The following evident remarks are needed.

Remark 3.2. Let an R-module M have Noetherian dimension. Then M has uc-Noetherian dimension and $ucn\text{-}\dim M \leq n\text{-}\dim M$.

Remark 3.3. Let M be a nonzero R-module, then $ucn\text{-}\dim M \leq 0$ if and only if M satisfies the ascending chain condition on uncountably generated submodules.

The proof of the following six results are similar to the proof of its dual results in Section 2, and hence they are omitted.

Lemma 3.4. *Let the R-module M have uc-Noetherian dimension. Then, each submodule A of M has uc-Noetherian dimension and $ucn\text{-}\dim A \leq ucn\text{-}\dim M$.*

Lemma 3.5. *Let the R-module M have uc-Noetherian dimension. Then, for each submodule A of M, $\frac{M}{A}$ has uc-Noetherian dimension and $ucn\text{-}\dim \frac{M}{A} \leq ucn\text{-}\dim M$.*

Corollary 3.6. *Let the R-module M have uc-Noetherian dimension. If N is a submodule of M, then*

$$\sup\{ucn\text{-}\dim N, ucn\text{-}\dim \frac{M}{N}\} \leq ucn\text{-}\dim M.$$

Lemma 3.7. *Assume that each $A \in UC(M)\backslash\{M\}$ has uc-Noetherian dimension. Then M has uc-Noetherian dimension and*

$$ucn\text{-}\dim M \leq \sup\{ucn\text{-}\dim A | A \in UC(M)\backslash\{M\}\} + 1.$$

Lemma 3.8. *Assume that $\frac{M}{A}$ has uc-Noetherian dimension for all $A \in UC(M)$. Then, M has uc-Noetherian dimension and*

$$ucn\text{-}\dim M \leq \sup\{ucn\text{-}\dim (\frac{M}{A}) : A \in UC(M)\} + 1.$$

Lemma 3.9. *Assume that for each $A \in UC(M)$ either A or $\frac{M}{A}$ has uc-Noetherian dimension. Then M also has uc-Noetherian dimension.*

In view of Corollary 2.10, and the previous comment, we have the following result.

Corollary 3.10. *Let M be a module with uc-Noetherian dimension. If N is a submodule of M, then $G\text{-}\dim \frac{M}{N} \leq c$; moreover if $G\text{-}\dim \frac{M}{N} = c$, then c is not attained in $\frac{M}{N}$.*

Next, we present our main result of this section, which gives an upper bound for the value of Noetherian dimension of a quotient finite dimensional R-module M with uc-Noetherian dimension.

Theorem 3.11. *Let M be a quotient finite dimensional R-module with ucn-$\dim M = \beta$, then M has Noetherian dimension and n-$\dim M \leq \omega_1 + \beta$.*

Proof. Let N be a proper submodule of M. Put $\frac{M}{N} = Q$, by Lemma 3.5, we infer that ucn-$\dim Q \leq \beta$. In view of Proposition 1.1, it is sufficient to show that Q has a nonzero factor module with Noetherian dimension less than or equal to $\omega_1 + \beta$. We proceed by induction on β. For $\beta = 0$, by [8, Proposition 3.10], n-$\dim Q \leq \omega_1$ and we are done. Now, let us assume that ucn-$\dim Q = \beta$. Suppose that for each proper uncountably generated submodule N of Q, ucn-$\dim N < \beta$, hence by induction hypothesis n-$\dim N < \omega_1+\beta$. Now let P be a proper countably generated submodule of Q. If there exists a proper uncountably generated submodule N of Q such that $P \subset N$, then by our assumption n-$\dim P < \omega_1 + \beta$. Otherwise ucn-$\dim \frac{Q}{P} = 0$ and in view of [8, Proposition 3.10] we have n-$\dim \frac{Q}{P} \leq \omega_1$ and therefore n-$\dim \frac{Q}{P} < \omega_1 + \beta$. It follows that for each proper submodule K of Q, either n-$\dim K < \omega_1 + \beta$ or n-$\dim \frac{Q}{K} < \omega_1 + \beta$, i.e., Q is γ-short module for some $\gamma < \omega_1 + \beta$. This immediately implies that n-$\dim Q \leq \omega_1 + \beta$, see [9, Proposition 1.12]. Now, assume that there is a proper uncountably generated submodule N_1 of Q such that ucn-$\dim N_1 = \beta$. If ucn-$\dim \frac{N}{N_1} < \beta$ for each proper uncountably generated submodule $\frac{N}{N_1}$ of $\frac{Q}{N_1}$, then by the argument just given n-$\dim \frac{Q}{N_1} \leq \omega_1+\beta$ and we are done. Otherwise there exists a proper uncountably generated submodule $\frac{N_2}{N_1}$ of $\frac{Q}{N_1}$ such that ucn-$\dim \frac{N_2}{N_1} = \beta$. By continuing with this method, we either find a proper submodule N of Q such that n-$\dim \frac{Q}{N} \leq \omega_1+\beta$, or else we get a chain

$$N_1 \subset N_2 \subset N_3 \subset \cdots$$

of uncountably generated submodules of Q such that ucn-$\dim \frac{N_{i+1}}{N_i} = \beta$ for each i. But this is a contradiction. □

Lemonnier [19], has interestingly shown that any poset E has codeviation if and only if it has deviation. Consequently for an R-module M, uck-$\dim M$ (resp. k-$\dim M$) exists if and only if ucn-$\dim M$ (resp. n-$\dim M$) exists. Now, in view of Theorem 3.11 and the comment which follows Lemma 2.2, the following proposition is evident.

Proposition 3.12. *Let M be a quotient finite dimensional R-module. Then M has Krull dimension if and only if it has uncountably generated Krull dimension, but the values of these dimensions might differ.*

Acknowledgement. The authors would like to thank the referee for the valuable suggestions and comments.

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