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A NOTE ON THE SOLVABILITY OF A FINITE GROUP IN WHICH EVERY NON-NILPOTENT MAXIMAL SUBGROUP IS NORMAL

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Abstract. We provide a new and simple proof to show that a finite group in which every non-nilpotent maximal subgroup is normal is solvable.

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1. Introduction

In this paper all groups are assumed to be finite. It is known that every maximal subgroup of a group G is normal if and only if G is a nilpotent group. As a generalization, Li and Shi [1] gave a proof to show that the following result holds.

Theorem 1.1. [1, Theorem 1.1] *A group G with all non-nilpotent maximal subgroups being normal is solvable.*

Moreover, based on the solvability of the group G in [1, Theorem 1.1], Shi [3, Theorem 5] proved that such a group G has a Sylow tower.

In this paper, our main goal is to provide a new and simpler proof of [1, Theorem 1.1], see Section 2.

2. New proof of [1, Theorem 1.1]

Proof. We first claim that G has a normal subgroup of prime-power order.

Suppose not. We divide the following discussions into three cases.

Case 1: Assume that every maximal subgroup of G is nilpotent. It follows that G is either a nilpotent group or a minimal non-nilpotent group. Then one can easily get that G has a normal Sylow subgroup that has prime-power order by the structure of minimal non-nilpotent group [2, Theorem 9.1.9], a contradiction.

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Case 2: Assume that every maximal subgroup of G is non-nilpotent. It follows that every maximal subgroup of G is normal by the hypothesis and then G is nilpotent, this contradicts that every maximal subgroup of G is non-nilpotent.

Case 3: Assume that G not only has nilpotent maximal subgroups but also has non-nilpotent maximal subgroups. Since G has no normal subgroup of prime-power order, every Sylow p-subgroup P of G is not normal in G for any prime divisor p of $|G|$, that is $N_G(P) < G$. Then there exists a maximal subgroup M of G such that $N_G(P) \leq M$. Note that every non-nilpotent maximal subgroup of G is normal and P is not normal in G, one has that M is nilpotent by the Frattini-argument. Therefore, every Sylow subgroup of G is contained in some nilpotent maximal subgroup of G.

For any nilpotent maximal subgroup M of G, if there exists a prime divisor q of $|M|$ such that the Sylow q-subgroup Q_M of M is not a Sylow q-subgroup of G, then $N_G(Q_M) > M$ as M being nilpotent. It follows that Q_M is a normal subgroup of G of prime-power order since M is maximal in G, a contradiction.

Next assume that every Sylow subgroup of M is also a Sylow subgroup of G for any nilpotent maximal subgroup M of G.

For the case when G has exactly one nilpotent maximal subgroup M, then M is normal in G, which implies that G has a normal Sylow subgroup, a contradiction.

For another case when G has at least two nilpotent maximal subgroups. Let M_1 and M_2 be any two distinct nilpotent maximal subgroups of G.

(i) Suppose $(|M_1|, |M_2|) = 1$. Let N be a non-nilpotent maximal subgroup of G, then $G = M_1N = M_2N$. One has $|G| = \frac{|M_1||N|}{|M_1\cap N|} = \frac{|M_2||N|}{|M_2\cap N|}$ and then $\frac{|M_1|}{|M_1\cap N|} = \frac{|M_2|}{|M_2\cap N|}$. Note that $(\frac{|M_1|}{|M_1\cap N|}, \frac{|M_2|}{|M_2\cap N|}) = 1$ by the hypothesis. It follows that $|M_1| = |M_1 \cap N|$ and then $M_1 \leq N$. One has $M_1 = N$ since M_1 is maximal in G, a contradiction.

(ii) Suppose $(|M_1|, |M_2|) > 1$ and $|M_1| \neq |M_2|$. Then there exists a prime r such that $r \mid (|M_1|, |M_2|)$. Let R_1 be a Sylow r-subgroup of M_1 and R_2 be a Sylow r-subgroup of M_2. Since both R_1 and R_2 are also Sylow r-subgroups of G, there exists an $x \in G$ such that $R_2 = {R_1}^x$. That is ${R_1}^x \in \mathrm{Syl}_r(M_2)$. It follows that $R_1 \in \mathrm{Syl}_r({M_2}^{x^{-1}})$. Since $|M_1| \neq |M_2|$, one has $M_1 \neq {M_2}^{x^{-1}}$. Then $N_G(R_1) \geq \langle M_1, {M_2}^{x^{-1}}\rangle > M_1$. Thus $N_G(R_1) = G$ since M_1 is maximal in G, which implies that R_1 is a normal Sylow subgroup of G, a contradiction.

(iii) Suppose that all nilpotent maximal subgroups of G have the same order. Let M be any nilpotent maximal subgroup of G. Since every Sylow subgroup of G is contained in some nilpotent maximal subgroup of G and all nilpotent maximal subgroups of G have the same order, one has $|G| = |M|$, a contradiction.

All above arguments imply that our assumption is not true. Hence G has a normal subgroup of prime-power order.

In the following let G_1 be a normal subgroup of G of prime-power order. Consider the quotient group G/G_1. It is clear that every non-nilpotent maximal subgroup of G/G_1 is also normal, arguing as above, one has that G/G_1 has a normal subgroup G_2/G_1 of prime-power order. We go on considering the quotient group G/G_2, one by one, we can obtain a normal subgroups series: $1 = G_0 \lhd G_1 \lhd G_2 \lhd \cdots \lhd G_i \lhd \cdots \lhd G_{s-1} \lhd G_s = G$, where $s > 1$ and every quotient group G_i/G_{i-1} has prime-power order for each $1 \leq i \leq s$. Therefore, one has that G is solvable. □

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