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BAER SUBMODULES OF MODULES OVER COMMUTATIVE RINGS

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Abstract. Let R be a commutative ring and M be an R-module. A submodule N of M is called a d-submodule (resp., an fd-submodule) if $\mathrm{Ann}_R(m) \subseteq \mathrm{Ann}_R(m')$ (resp., $\mathrm{Ann}_R(F) \subseteq \mathrm{Ann}_R(m')$) for some $m \in N$ (resp., finite subset $F \subseteq N$) and $m' \in M$ implies that $m' \in N$. Many examples, characterizations, and properties of these submodules are given. Moreover, we use them to characterize modules satisfying Property T, reduced modules, and von Neumann regular modules.

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1. Introduction

Throughout this paper, all rings are assumed to be commutative with nonzero identity and all modules are nonzero unitary. Let R be a ring and M be an R-module. If N is a submodule of M and K is a nonempty subset of M, then the ideal $\{x \in R \mid xK \subseteq N\}$ of R is denoted by $(N :_R K)$ and the annihilator of N is the ideal $\mathrm{Ann}_R(N) := (0 :_R N)$. For any ideal I of R and any submodule N of M, the submodule $\{m \in M \mid Im \subseteq N\}$ is denoted by $(N :_M I)$ and the annihilator of I in M is the submodule $\mathrm{Ann}_M(I) := (0 :_M I)$. An R-module M is said to be faithful if $\mathrm{Ann}_R(M)$ is the zero ideal of R. A proper submodule N of M is said to be a maximal submodule if it is not properly contained in any other proper submodule of M. A proper submodule N of M is a prime submodule if for any $x \in R$ and $m \in M$, $xm \in N$ implies either $m \in N$ or $x \in (N :_R M)$. For any submodule N of M, the (prime) radical of N in M, denoted by $\mathrm{rad}_M(N)$, is defined to be the intersection of all prime submodules of M containing N. If there is no prime submodule containing N, we define $\mathrm{rad}_M(N) = M$. A submodule N of M is called radical if $N = \mathrm{rad}_M(N)$. We shall need the notion of the envelope of a submodule

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introduced by McCasland and Moore in [20]. For a submodule N of M, the envelope of N in M, denoted by $E_M(N)$, is defined to be the collection $E_M(N) := \{rm \mid r \in R \text{ and } m \in M \text{ such that } r^k m \in N \text{ for some positive integer } k\}$. Note that, in general, $E_M(N)$ is not an R-module. We use $\langle E_M(N)\rangle$ to denote the submodule of M generated by $E_M(N)$. Obviously $(N :_R M)M \subseteq N \subseteq E_M(N) \subseteq \mathrm{rad}_M(N)$ for every submodule N of M, where the equalities do not hold in general. A submodule N of M satisfies *the radical formula* in M if $\langle E_M(N)\rangle = \mathrm{rad}_M(N)$. Moreover, we say that an R-module M *satisfies the radical formula* if every submodule of M satisfies the radical formula in M (for details, see [15], [18], [20], [22] and [24]). We say that an R-module M is a *multiplication module* [4] if every submodule of M has the form IM for some ideal I of R, i.e., $N = (N :_R M)M$. It is well known that maximal submodules and prime submodules exist in multiplication modules (for details, see [7]). Recall from [1], an R-module M satisfies *Property T* (resp., *strong Property T*) if for every finitely generated submodule N (resp., finite subset F) of M with $N \subseteq T_R(M)$ (resp., $F \subseteq T_R(M)$), $\mathrm{Ann}_R(N) \neq 0$ (resp., $\mathrm{Ann}_R(F) \neq 0$), where $T_R(M) := \{m \in M \mid rm = 0 \text{ for some nonzero } r \in R\}$. According to [17], an R-module M is called *reduced* if for any $m \in M$ and $x \in R$, $xm = 0$ implies $Rm \cap xM = 0$. An R-module M is called *von Neumann regular* [12] if for each $m \in M$, there exists an element $r \in R$ such that $rM = r^2M$ and $Rm = rM$. It is well known that a finitely generated R-module M is von Neumann regular if and only if for any $m \in M$, there exists a weak idempotent $e \in R$ of M (i.e., $e - e^2 \in \mathrm{Ann}_R(M)$) such that $Rm = eM$ (see [12, Lemma 5]).

The notion of d-ideals in a commutative ring was introduced by Speed [25] who called them Baer ideals. These ideals were also put to good use in 1972 by Evans [8] when characterizing commutative rings that are finite direct sums of integral domains. In [11], Jayaram introduced fd-ideals (as strongly Baer ideals) and 0-ideals in reduced rings and characterize quasi regular and von Neumann regular rings. In [16], Khabazian, Vedadi and Safaeeyan extended the concept of d-ideals to the category of modules and investigated the modules for which their submodules are d-submodules (see [17, Theorem 2.1]). In [23], Safaeeyan and Taherifar studied d-ideals and fd-ideals in general rings, and not just the reduced ones. In [2], the authors studied rings in which every ideal contained in the set of zero-divisors is a d-ideal. The authors of [13] recently extended the concepts of d-ideals and 0-ideals to submodules and they called them Baer submodules and $*$-submodules, respectively. They used them to characterize quasi regular modules and weak quasi regular modules.

Our objective in this study is to investigate the concepts of d-submodules, fd-submodules, and 0-submodules of a module over a commutative ring. In Section 2, we establish some examples of these submodules (see Example 2.2). Also we characterize modules which have an element with zero annihilator and satisfies the Property T (see Proposition 2.9). This result can be applied to produce examples of modules in which all maximal submodules are fd-submodules. We use these concepts to characterize reduced modules (see Proposition 2.14). Moreover, we observe that in any faithful reduced p.p. R-module, the sum of two d-submodules is a d-submodule (see Theorem 2.16), where an R-module M is called *p.p.* if for any $m \in M$, $\mathrm{Ann}_R(m) = Re$ for some idempotent $e \in R$ [17]. We also characterize finitely generated von Neumann regular modules in terms of d-submodules (see Theorem 2.25).

2. Main results

Let M be an R-module and let S be a multiplicative subset of R, i.e., S satisfies: $1 \in S$ and if $s_1, s_2 \in S$, then $s_1 s_2 \in S$. If N is a submodule of M, we set $N_S(M) := \{x \in M \mid sx \in N \text{ for some } s \in S\}$. Then $N_S(M)$ is a submodule of M containing N, which is called the *component of N determined by S*, or simply the *S-component of N*. In particular, the S-component of the zero submodule 0 is $\ker(\phi)$, where $\phi : M \to S^{-1}M$, the localization of M by S, is a canonical map. Note that $N_S(M) = M$ if $(N :_R M) \cap S \neq \emptyset$. We shall begin with the following definitions:

Definition 2.1. Let M be an R-module and N be a submodule of M.

(1) N is said to be a *d-submodule* (or *Baer submodule*) [1] if $\mathrm{Ann}_R(m) \subseteq \mathrm{Ann}_R(m')$ for some $m \in N$ and $m' \in M$ implies that $m' \in N$.
(2) N is called an *fd-submodule* (or a *strongly Baer submodule*) if $\mathrm{Ann}_R(F) \subseteq \mathrm{Ann}_R(m')$ for some finite subset F of N and $m' \in M$ implies that $m' \in N$.
(3) N is said to be a *0-submodule* if $N = 0_S(M)$ for some multiplicative subset S of R.

It can be easily seen that for a submodule N of an R-module M,

$$N \text{ is a 0-submodule} \Rightarrow N \text{ is an fd-submodule} \Rightarrow N \text{ is a d-submodule.}$$

Example 2.2. Let M be an R-module.

[1] As Ebrahim Ghashghaei kindly informed the authors, in [9, Definition 1] and [3, Definition 2.3], d-submodules are also referred to as *perpetual* submodules.

(a) An ideal I of R is a d-ideal (resp., an fd-ideal or a 0-ideal) if and only if I is a d-submodule (resp., an fd-submodule or a 0-submodule) of the R-module R.

(b) If N is a d-submodule (resp., an fd-submodule) of M, then $(N :_M I)$ is also a d-submodule (resp., an fd-submodule) for every ideal I of R. In particular, $\mathrm{Ann}_M(I) = (0 :_M I)$ is an fd-submodule. Also, if S is a multiplicative subset of R, then $\mathrm{Ann}_M(S) = \{0\}$ is trivially a 0-submodule.

(c) Any intersection of d-submodules (resp., fd-submodules) is a d-submodule (resp., an fd-submodule).

(d) Every submodule N of M contains a 0-submodule. In fact, put $S := 1 + (N :_R M)$. Clearly $0_S(M)$ is a 0-submodule contained in N.

(e) Let N be a d-submodule (resp., an fd-submodule) of M and S be any multiplicative subset of R such that $(N :_R M) \cap S = \emptyset$. Then $N_S(M)$, the S-component of N, is a d-submodule (resp., an fd-submodule).

Let M be an R-module and N be a submodule of M. In [14, Lemma 3.9], it was shown that the following statements are equivalent:

(1) N is a d-submodule;

(2) $\mathrm{Ann}_M(\mathrm{Ann}_R(n)) \subseteq N$ for each $n \in N$;

(3) $N = \bigcup_{n\in N} \mathrm{Ann}_M(\mathrm{Ann}_R(n))$.

Let M be an R-module. By the trace of a submodule N in M, we mean $\mathrm{Tr}(N, M) := \{\sum \mathrm{Im}\,(f) \mid f \in \mathrm{Hom}_R(N, M)\}$.

Proposition 2.3. *Let M be an R-module and N be a submodule of M. If N is a d-submodule of M, then* $\mathrm{Tr}(N, M) = N$*.*

Proof. Suppose that N is a d-submodule of M. One can see that $N \subseteq \mathrm{Tr}(N, M)$. Now, if $f \in \mathrm{Hom}_R(N, M)$, then $\mathrm{Ann}_R(m) \subseteq \mathrm{Ann}_R(f(m))$ for each $m \in N$. It follows that $\mathrm{Im}\,(f) \subseteq N$. Thus $\mathrm{Tr}(N, M) = N$. $\square$

The converse of Proposition 2.3 is not true in general. In order to provide a counterexample, we need some terminology. Let R be an integral domain with quotient field K. Let $\mathbf{F}(R)$ be the set of nonzero fractional ideals of R. For an $I \in \mathbf{F}(R)$, define $I^{-1} = \{x \in K \mid xI \subseteq R\}$. The *$v$-operation* on R is a mapping on $\mathbf{F}(R)$ defined by $I \mapsto I_v = (I^{-1})^{-1}$. Recall that an ideal J of R is called a *Glaz-Vasconcelos ideal* (GV-ideal) if J is finitely generated and $J^{-1} = R$. We denote the set of GV-ideals by $\mathrm{GV}(R)$. The *w-operation* on R is a mapping on $\mathbf{F}(R)$ defined by $I \mapsto I_w = \{x \in K \mid Jx \subseteq I \text{ for some } J \in \mathrm{GV}(R)\}$. An $I \in \mathbf{F}(R)$ is said to be a

v-ideal (resp., *w-ideal*) if $I_v = I$ (resp., $I_w = I$). An integral domain R is called a *DW-domain* if every ideal of R is a w-ideal.

Example 2.4. Let R be an integral domain which is not DW (for a concrete example, see [10, Proposition 5.2]). Take a proper GV-ideal J of R. Since R is a w-ideal, it follows from [26, Theorem 6.1.14] that $\mathrm{Ext}^1_R(R/J, R) = \{0\}$. Thus by [19, Example 3.9 (1)], $\mathrm{Tr}(J, R) = J$. Now take $m' \in J_v \setminus J$ since $J \subsetneq J_v = R$. Then $\mathrm{Ann}_R(m) = \mathrm{Ann}_R(m') = \{0\}$ for any $0 \neq m \in J$, but $m' \notin J$. Therefore, J is not a d-ideal of R.

Let M be an R-module. We say that an ideal I of R is a d_M-ideal if for each $x, y \in R$, $\mathrm{Ann}_M(x) \subseteq \mathrm{Ann}_M(y)$ and $x \in I$ implies that $y \in I$. Also, an ideal I of R is said to be an fd_M-ideal if $\mathrm{Ann}_M(S) \subseteq \mathrm{Ann}_M(y)$ for some finite subset S of I and $y \in R$ implies that $y \in I$.

Proposition 2.5. *Let M be an R-module and N be a submodule of M. Then:*

(1) *If N is a d-submodule (resp., an fd-submodule) of M, then $(N :_R K)$ is a d_M-ideal (resp., an fd_M-ideal) for every nonempty subset K of M.*
(2) *Assume that M is a cyclic R-module. Then N is a d-submodule (resp., an fd-submodule) if and only if $(N :_R M)$ is a d_M-ideal (resp., an fd_M-ideal).*
(3) *Suppose that M has an element with zero annihilator. If N is a 0-submodule of M, then $(N :_R M)$ is a 0-ideal of R.*

Proof. (1) Assume that N is a d-submodule and let K be a nonempty subset of M. Let $x \in (N :_R K)$ and $y \in R$ such that $\mathrm{Ann}_M(x) \subseteq \mathrm{Ann}_M(y)$. Then for each $m \in K$, $\mathrm{Ann}_R(xm) \subseteq \mathrm{Ann}_R(ym)$, and so $y \in (N :_R K)$ since N is a d-submodule. Similarly, we can prove that $(N :_R K)$ is an fd_M-ideal whenever N is an fd-submodule.

(2) Assume that M is generated by m and let N be a submodule of M such that $(N :_R M)$ is a d-ideal of R. Let $xm \in N$ and $ym \in M$ such that $\mathrm{Ann}_R(xm) \subseteq \mathrm{Ann}_R(ym)$. Then $\mathrm{Ann}_M(x) \subseteq \mathrm{Ann}_M(y)$. It follows that $y \in (N :_R M)$ and so $ym \in N$.

(3) This follows from [13, Lemma 2.5]. $\square$

Remark 2.6. (1) The assertion in Proposition 2.5 (2) fails if one deletes the hypothesis that M is cyclic. For example, consider the $\mathbb{Z}$-module $M = \mathbb{Z} \times \mathbb{Z}$. Let $N = R(4, 0)$. Then it is clear that $(N :_R M) = 0$ and it is an fd_M-ideal of $\mathbb{Z}$ since M is faithful. However, N is not a d-submodule (and so it is not an fd-submodule).

(2) It is interesting that the final assertion in the statement of Proposition 2.5 would fail if $T_R(M) = M$. Indeed, let $R = \mathbb{Z}$, $M = \mathbb{Z}/4\mathbb{Z}$ and $N = \langle \bar{0} \rangle$. Clearly N is a 0-submodule, but $(N :_R M) = 4\mathbb{Z}$ is not a 0-ideal.

(3) Let N be a d-submodule (resp., an fd-submodule) of an R-module M. In general, $(N :_R M)$ need not be a d-ideal (resp., an fd-ideal). For example, let $(R, \mathfrak{m})$ be a quasi-local domain, but not a field, and let M be a nonzero R-module such that $\mathfrak{m}M = 0$. Then the zero submodule of M is an fd-submodule but $(0 :_R M) = \mathfrak{m}$ is not a d-ideal, because R is a domain.

Proposition 2.7. *Let M be a multiplication R-module. Then the following statements hold:*

1. *Let N be a submodule and P be a prime submodule of M such that $N \cap P$ is a d-submodule (resp., an fd-submodule). Then either N or P is a d-submodule (resp., an fd-submodule).*
2. *Let P and Q be prime submodules of M which do not belong to a chain. Then P and Q are both d-submodules (resp., fd-submodules) if and only if $P \cap Q$ is a d-submodule (resp., an fd-submodule).*

Proof. (1) If $N \subseteq P$, then $N = N \cap P$ is a d-submodule. Now assume that $N \nsubseteq P$ and let $m \in P$ and $m' \in M$ such that $\mathrm{Ann}_R(m) \subseteq \mathrm{Ann}_R(m')$. As M is a multiplication module, we have $N = IM$ for some ideal I of R. This implies that there exists $x \in I \backslash (P :_R M)$. Therefore $\mathrm{Ann}_R(xm) \subseteq \mathrm{Ann}_R(xm')$ and $xm \in N \cap P$. By hypothesis, we get $xm' \in P$ and hence $m' \in P$. Similarly, we can show that N or P is an fd-submodule whenever $N \cap P$ is an fd-submodule.

(2) We need only prove the converse. Assume that $P \nsubseteq Q$. Then $(P :_R M) \nsubseteq (Q :_R M)$ since M is a multiplication module. Let $m \in Q$ and $m' \in M$ such that $\mathrm{Ann}_R(m) \subseteq \mathrm{Ann}_R(m')$. It follows that $\mathrm{Ann}_R(xm) \subseteq \mathrm{Ann}_R(xm')$ where $x \in (P :_R M) \setminus (Q :_R M)$. Since $P \cap Q$ is a d-submodule and $xm \in P \cap Q$, we have $xm' \in P \cap Q \subseteq Q$ and hence either $x \in (Q :_R M)$ or $m' \in Q$. But $x \notin (Q :_R M)$, and thus $m' \in Q$. Consequently, Q is a d-submodule and so is P via a similar argument. Similarly, we can prove that P and Q are fd-submodules whenever $P \cap Q$ is an fd-submodule. $\square$

Lemma 2.8. *Let M be an R-module. For every submodule N of M, define*

$$N_{fd} := \{m \in M \mid \mathrm{Ann}_R(F) \subseteq \mathrm{Ann}_R(m) \textit{ for some finite subset } F \textit{ of } N\}.$$

Then either $N_{fd} = M$ or N_{fd} is the smallest fd-submodule containing N.

Proof. Let $m, m' \in N_{fd}$. Then there exist $F_1, F_2 \subseteq N$ such that $\mathrm{Ann}_R(F_1) \subseteq \mathrm{Ann}_R(m)$ and $\mathrm{Ann}_R(F_2) \subseteq \mathrm{Ann}_R(m')$. It is obvious that $\mathrm{Ann}_R(F_1 \cup F_2) \subseteq \mathrm{Ann}_R(m + m')$, and so $m + m' \in N_{fd}$. Clearly $rm \in N_{fd}$ for every $r \in R$ and $m \in N_{fd}$. Hence, N_{fd} is a submodule of M. Now, suppose that $F = \{m_1, \ldots, m_k\}$ is a finite subset of N_{fd} and $m \in M$ such that $\mathrm{Ann}_R(F) \subseteq \mathrm{Ann}_R(m)$. By definition, for each $i = 1, \ldots, k$, there exists a finite subset F_i of N such that $\mathrm{Ann}_R(F_i) \subseteq \mathrm{Ann}_R(m_i)$. Thus $\mathrm{Ann}_R(\bigcup_{i=1}^{k} F_i) \subseteq \mathrm{Ann}_R(F) \subseteq \mathrm{Ann}_R(m)$, and so $m \in N_{fd}$. Therefore N_{fd} is an fd-submodule of M. Finally it is clear that if N_{fd} is a proper submodule, then it is the smallest fd-submodule containing N. □

The following proposition provides a necessary and sufficient condition for an R-module which has an element with zero annihilator to satisfy the *Property T*.

Proposition 2.9. *Let M be an R-module which has an element with zero annihilator. Then M satisfies Property T if and only if N is contained in a proper fd-submodule for every submodule $N \subseteq T_R(M)$.*

Proof. Assume that M satisfies *Property T*. Let N be a submodule of M such that $N \subseteq T_R(M)$. Then $\mathrm{Ann}_R(S) \neq 0$ for every finite subset S of N, which implies that N_{fd} is a proper fd-submodule.

Conversely, let N be a finitely generated submodule of M such that $N \subseteq T_R(M)$ and let K be a proper fd-submodule containing N. Using the fact that $\mathrm{Ann}_M(\mathrm{Ann}_R(N)) \subseteq K$, we conclude that $\mathrm{Ann}_R(N) \neq 0$. □

Corollary 2.10. *Let M be an R-module which has an element with zero annihilator. Then M satisfies strong Property T if and only if $T_R(M)$ is an fd-submodule.*

Remark 2.11. It is well known from [1, Theorem 3.1] that an R-module M satisfies *strong Property T* if and only if M satisfies *Property T* and $T_R(M)$ is a submodule. Then Corollary 2.10 allows us to construct new original examples of d-submodules that are not fd-submodules. In fact, let M be an R-module that does not satisfy *Property T* with $T_R(M)$ is a proper submodule of M. Then $T_R(M)$ is a d-submodule that is not an fd-submodule.

We next give an example of a torsion module that does not satisfy *strong Property T*, which implies the condition "M has an element with zero annihilator" in Corollary 2.10 is necessary.

Example 2.12. [1, Example 3.2]. Let $R := \mathbb{Z}_2[X, Y]/(X, Y)^2 = \mathbb{Z}_2[x, y]$ and $M := F/K$, where F is the free R-module on $\{e_1, e_2\}$ and $K := \langle xe_1, ye_2, ye_1 + xe_2 \rangle$.

So $M = F/K = R\overline{e_1} + R\overline{e_2} = \{0, \overline{e_1}, (1+y)\overline{e_1}, \overline{e_2}, (1+x)\overline{e_2}, y\overline{e_1} = x\overline{e_2}, \overline{e_1}+\overline{e_2}, (1+x)(\overline{e_1}+\overline{e_2})\}$ is a torsion R-module. Then $T_R(M) = M$ is an fd-submodule but M does not satisfy strong Property T.

We say that an R-module M satisfies the *condition* $(*)$ if $M \backslash T_R(M) = \{m \in M \mid Rm = M\}$. Note that if M satisfies the condition $(*)$, then every proper submodule of M is contained in $T_R(M)$.

Theorem 2.13. *Let M be a multiplication R-module which has an element with zero annihilator. Then M satisfies Property T and the condition $(*)$ if and only if every maximal submodule of M is an fd-submodule.*

Proof. Suppose first that M satisfies *Property T* and the condition $(*)$ and let P be a maximal submodule of M. The fact that P is a proper submodule implies that $P \subseteq T_R(M)$. Thus Proposition 2.9 ensures that P_{fd} is a proper fd-submodule which contains P and hence $P = P_{fd}$.

Conversely, assume that every maximal submodule of M is an fd-submodule. Note from [7, Theorem 2.5] that every proper submodule is contained in a maximal submodule, and so every submodule in $T_R(M)$ is contained in a proper fd-submodule. Then M satisfies *Property T* by Proposition 2.9. Next, we will prove that M satisfies the condition $(*)$. If $Rm = M$ for some $m \in M$, then $\mathrm{Ann}_R(m) = 0$ since M is faithful. Now, let $m \in M$ such that $\mathrm{Ann}_R(m) = 0$. If Rm is a proper submodule of M, then there exists a maximal submodule P of M containing Rm. By hypothesis, P is an fd-submodule and so $\mathrm{Ann}_M(\mathrm{Ann}_R(m)) \subseteq P$, whence $P = M$, a contradiction. $\square$

Let R be a ring and M be an R-module. We denote by $Z_R(M) = \{x \in R \mid xm = 0 \text{ for some nonzero element } m \in M\}$, the set of zero divisors of R on M and by $T_R(M) = \{m \in M \mid xm = 0 \text{ for some nonzero } x \in R\}$, the set of torsion elements of M with respect to R. Also $Q_M(R) := S_M^{-1}R$ denotes the total quotient ring of R with respect to M, where $S_M := R \setminus Z_R(M)$ and $Q_R(M) := S_M^{-1}M$ denotes the total quotient module of M.

Recall from [17] that an R-module M is reduced if and only if for any $x \in R$ and $m \in M$, $x^2m = 0$ implies that $xm = 0$. The following proposition gives a new characterization of reduced modules in terms of d-submodules.

Proposition 2.14. *Let M be an R-module. Then the following statements are equivalent.*

(1) *M is a reduced module.*

(2) *$Q_R(M)$ is a reduced $Q_M(R)$-module.*
(3) *$\langle E_M(N)\rangle = N$ for every d-submodule N of M.*
(4) *$\langle E_M(N)\rangle = N$ for every fd-submodule N of M.*

Proof. (1) $\Leftrightarrow$ (2) This is obvious.

(1) $\Rightarrow$ (3) Assume that M is a reduced module and let N be a d-submodule of M. It is obvious that $N \subseteq \langle E_M(N)\rangle$. For the reverse inclusion let $m \in \langle E_M(N)\rangle$. Then $m = x_1m_1 + \cdots + x_km_k$ for some positive integer k and elements $x_i \in R$, $m_i \in M$ with $x_i^n m_i \in N$ for some positive integer n. As M is reduced, we have $\mathrm{Ann}_R(x_i^n m_i) = \mathrm{Ann}_R(x_i m_i)$ for each $i \in \{1, \ldots, k\}$. This implies that $x_i m_i \in N$, and so $m \in N$.

(3) $\Rightarrow$ (4) This is trivial.

(4) $\Rightarrow$ (1) Suppose that $\langle E_M(N)\rangle = N$ for every fd-submodule N of M. To show that M is a reduced module, we must prove that if $x \in R$ and $m \in M$ such that $x^2m = 0$, then $xm = 0$. But this follows from the fact that the zero submodule of M is an fd-submodule, and so $\langle E_M(0)\rangle = 0$ by the hypothesis. □

As an immediate consequence of Proposition 2.14, we give the following corollary.

Corollary 2.15. *Let M be an R-module which satisfies the radical formula. Then the following statements are equivalent.*

(1) *M is a reduced module.*
(2) *Every d-submodule of M is radical.*
(3) *Every fd-submodule of M is radical.*

The following theorem gives a class of modules in which the sum of two d-submodules is a d-submodule.

Theorem 2.16. *Let M be a faithful reduced p.p. R-module. Then the sum of two d-submodules is a d-submodule.*

We need the following lemmas in order to prove Theorem 2.16.

Lemma 2.17. *Let M be a reduced R-module. Then,*

$$\mathrm{Ann}_R(\mathrm{Ann}_M(xy)) = \mathrm{Ann}_R(\mathrm{Ann}_M(x)) \cap \mathrm{Ann}_R(\mathrm{Ann}_M(y))$$

for each $x, y \in R$.

Proof. It can be easily shown the inclusion "$\subseteq$". For the reverse, let

$$r \in \mathrm{Ann}_R(\mathrm{Ann}_M(x)) \cap \mathrm{Ann}_R(\mathrm{Ann}_M(y)) \text{ and } m \in \mathrm{Ann}_M(xy).$$

Then $ym \in \mathrm{Ann}_M(x)$ and so $rym = 0$. Similarly, we get $r^2m = 0$. As M is a reduced module, we then have $m \in \mathrm{Ann}_R(\mathrm{Ann}_M(xy))$. □

Recall that an element $e \in R$ is *weak idempotent* of an R-module M if $e^2 - e \in \mathrm{Ann}_R(M)$.

Lemma 2.18. *Let M be an R-module and let $e \in R$ be a weak idempotent of M. Then:*

(1) $\mathrm{Ann}_M(Re) = (1-e)M$.
(2) *If M is faithful, then* $\mathrm{Ann}_R(eM) = R(1-e)$.

Proof. Straightforward. □

Proof of Theorem 2.16 Let N and N' be two d-submodules of M. Let $m \in N + N'$, and pick $n \in N$ and $n' \in N'$ such that $m = n + n'$. Then $\mathrm{Ann}_R(n) \cap \mathrm{Ann}_R(n') \subseteq \mathrm{Ann}_R(m)$. Let e, f be idempotents of R such that $\mathrm{Ann}_R(n) = Re$ and $\mathrm{Ann}_R(n') = Rf$. Thus, Lemma 2.18 gives that $\mathrm{Ann}_R(n) = \mathrm{Ann}_R(\mathrm{Ann}_M(e))$ and $\mathrm{Ann}_R(n') = \mathrm{Ann}_R(\mathrm{Ann}_M(f))$. Consequently, by Lemma 2.17,

$$\mathrm{Ann}_R(\mathrm{Ann}_M(ef)) = \mathrm{Ann}_R(\mathrm{Ann}_M(e)) \cap \mathrm{Ann}_R(\mathrm{Ann}_M(f)) \subseteq \mathrm{Ann}_R(m),$$

whence $\mathrm{Ann}_M(\mathrm{Ann}_R(m)) \subseteq \mathrm{Ann}_M(ef) = (1-ef)M$. On the other hand, since N is a d-submodule, it follows that $\mathrm{Ann}_R(n) = Re$ implies $(1-e)M \subseteq \mathrm{Ann}_M(\mathrm{Ann}_R(n))$, and so $(1-e)M \subseteq N$. Similarly, we have $(1-f)M \subseteq N'$ since N' is a d-submodule. Therefore $(1-ef)m = (1-e)fm + (1-f)m \in N + N'$ for each $m \in M$, and so $\mathrm{Ann}_M(\mathrm{Ann}_R(m)) \subseteq N + N'$. Finally $N + N'$ is a d-submodule. □

We now characterize finitely generated (resp., cyclic) modules.

Proposition 2.19. *Let M be an R-module. Then M is a finitely generated* (*resp., a cyclic*) *module if and only if* $\mathrm{Ann}_R(N) = \mathrm{Ann}_R(M)$ *for some finitely generated fd-submodule* (*resp., cyclic d-submodule*) *N of M.*

Proof. The necessity is obvious. Conversely, let N be a finitely generated fd-submodule (resp., a cyclic d-submodule) of M such that $\mathrm{Ann}_R(N) = \mathrm{Ann}_R(M)$. This implies that $\mathrm{Ann}_R(N) \subseteq \mathrm{Ann}_R(m)$ for each $m \in M$. Since N is an fd-submodule (resp., a d-submodule), we have $m \in N$. Consequently, M is a finitely generated (resp., a cyclic) module. □

An R-module M is called *strongly duo* provided that $\mathrm{Tr}(N, M) = N$ for each submodule N of M (see [16]). We also recall that M is called principally quasi-injective (pq-injective for short) if each R-morphism from a principal submodule of M to M can be extended to an endomorphism of M (see [21]).

Proposition 2.20. *Let M be an R-module. Then the following statements are equivalent.*

(1) *Every submodule of M is a d-submodule.*
(2) *Every cyclic submodule of M is a d-submodule.*
(3) *M satisfies the condition $(*)$ and every submodule contained in $T_R(M)$ is a d-submodule.*
(4) *M is a strongly duo module.*
(5) *M is a pq-injective module.*

Proof. (1) $\Rightarrow$ (2) This is clear.

(2) $\Rightarrow$ (1) Let N be a submodule of M. By hypothesis, every cyclic submodule contained in N is a d-submodule. In other words, if $\mathrm{Ann}_R(m) \subseteq \mathrm{Ann}_R(m')$ for some $m \in N$ and $m' \in M$, then $m' \in Rm \subseteq N$.

(2) $\Leftrightarrow$ (3) This is obvious.

(2) $\Leftrightarrow$ (4) This follows from [16, Theorem 2.1].

(4) $\Leftrightarrow$ (5) See [16, Theorem 3.5]. □

Recall that an R-module M is said to be *prime* if the zero submodule of M is a prime submodule of M.

Corollary 2.21. *Let M be an R-module. Then M is a prime strongly duo module if and only if M is a simple module.*

Proof. The sufficiency is trivial. Note that an R-module M is prime if and only if $\mathrm{Ann}_R(N) = \mathrm{Ann}_R(M)$ for every nonzero submodule N of M. If M is a prime strongly duo R-module, then by Propositions 2.19 and 2.20, $M = Rm$ for each nonzero $m \in M$. Consequently, M is a simple module, as desired. □

Let M be an R-module. We say that M is *perfect* if M satisfies the descending chain condition (DCC) on cyclic submodules (see [5]). Now we consider the set $\mathcal{A}_M := \{\mathrm{Ann}_R(m) \mid m \in M\}$.

Proposition 2.22. *Let M be a strongly duo module satisfying ACC on $\mathcal{A}_M$. Then M is a perfect module.*

Proof. Let $Rm_1 \supseteq Rm_2 \supseteq \cdots$ be a descending chain of cyclic submodules of M. By hypothesis, $\mathrm{Ann}_R(m_k) = \mathrm{Ann}_R(m_{k+1})$ for some positive integer k. As M is a strongly duo module, we have $Rm_k = Rm_{k+1}$. This implies that M is a prefect module, as desired. □

Corollary 2.23. *Let R be a Noetherian ring. Then every finitely generated strongly duo module over R is an Artinian module.*

Proof. If M is a finitely generated strongly duo module over a Noetherian ring R, then M is a Noetherian perfect module and hence M is Artinian by [6, Proposition 4.12]. □

The following example shows that the converse of the previous corollary is not true in general.

Example 2.24. Consider the $\mathbb{Z}$-module $M = \mathbb{Z}_{p^\infty}$, where p is a prime number. Then, M is an Artinian module which is not Noetherian.

We now characterize finitely generated von Neumann regular modules.

Theorem 2.25. *Let M be a finitely generated R-module. Consider the following conditions:*

(1) *M is a reduced multiplication module and every submodule is a 0-submodule.*
(2) *M is a reduced multiplication module and every submodule is an fd-submodule.*
(3) *M is a reduced multiplication module and satisfies any one of the conditions of Proposition 2.20.*
(4) *M is a von Neumann regular module.*

Then:

(i) (1) $\Rightarrow$ (2) $\Leftrightarrow$ (3) $\Leftrightarrow$ (4).
(ii) *If M is a cyclic module, then the above conditions are equivalent.*

Proof. (i) (1) $\Rightarrow$ (2) and (2) $\Rightarrow$ (3) are clear.

(3) $\Rightarrow$ (4) This follows from [14, Theorem 3.10].

(4) $\Rightarrow$ (2) Assume that M is a von Neumann regular module. Let N be a submodule of M and let F be a finite subset of N. By hypothesis, there exists a weak idempotent $e \in R$ of M such that $\langle F\rangle = eM$. Then $R(1-e) \subseteq \mathrm{Ann}_R(F)$ by Lemma 2.18. Thus $\mathrm{Ann}_M(\mathrm{Ann}_R(F)) \subseteq \mathrm{Ann}_M(1-e) = eM$ which implies that $\mathrm{Ann}_M(\mathrm{Ann}_R(F)) \subseteq N$. So N is an fd-submodule.

(ii) (4) $\Rightarrow$ (1) Suppose that M is a cyclic von Neumann regular module. Let N be a submodule of M. It will first be shown that $S := \{r \in R \mid \mathrm{Ann}_R(m) \subseteq$

$\mathrm{Ann}_R(\mathrm{Ann}_M(r))$ for some $m \in N\}$ is a multiplicative subset of R. If $r, s \in S$, then there exist $m, m' \in N$ such that $\mathrm{Ann}_R(m) \subseteq \mathrm{Ann}_R(\mathrm{Ann}_M(r))$ and $\mathrm{Ann}_R(m') \subseteq \mathrm{Ann}_R(\mathrm{Ann}_M(s))$. It follows by Lemma 2.17 that

$$\mathrm{Ann}_R(m) \cap \mathrm{Ann}_R(m') \subseteq \mathrm{Ann}_R(\mathrm{Ann}_M(rs)).$$

By hypothesis, there exists a weak idempotent $e \in R$ such that $Rm + Rm' = eM$ and so $Rm+Rm'$ is a cyclic submodule of M. Thus $\mathrm{Ann}_R(m'') \subseteq \mathrm{Ann}_R(\mathrm{Ann}_M(rs))$ for some $m'' \in N$. We complete the proof by showing that $N = 0_S(M)$. Since N is a d-submodule, we have $0_S(M) \subseteq N$. In fact, if $m \in 0_S(M)$, then $sm = 0$ for some $s \in S$. Consequently, $\mathrm{Ann}_R(n) \subseteq \mathrm{Ann}_R(m)$ for some $n \in N$, and so $m \in N$. Now, let $m \in N$. By assumption, we get $Rm = eM$ for some weak idempotent $e \in R$. Then $(1-e) \in \mathrm{Ann}_R(m)$. Also, it is clear to see that $(1-e) \in S$, and so $m \in 0_S(M)$. □

We remark that finitely generated von Neumann regular modules are characterized in [14, Theorem 3.10]. In particular, it was shown that for a finitely generated module M, M is von Neumann regular if and only if M is a reduced multiplication module in which every submodule is a d-submodule.

Corollary 2.26. *Let M be a finitely generated von Neumann regular R-module. Then $End_R(M)$ is a von Neumann regular ring.*

Proof. By Theorem 2.25 and [16, Theorem 5.5]. □

Corollary 2.27. *If R is a PID which is not a field, then finitely generated von Neumann regular R-modules are precisely non-faithful cyclic reduced R-modules.*

Proof. Combining Theorem 2.25 with [16, Corollary 3.8]. □

Example 2.28. Finitely generated von Neumann regular $\mathbb{Z}$-modules are precisely $\mathbb{Z}/n\mathbb{Z}$, where $n > 1$ is square-free.

Let M_i be an R_i-module for each $i = 1, 2$. Set $M := M_1 \times M_2$ and $R := R_1 \times R_2$. Then M is clearly an R-module with componentwise addition and scalar multiplication. Also every submodule N of M is of the form $N = N_1 \times N_2$, where N_i is a submodule of M_i.

Proposition 2.29. *Let M_i be an R_i-module for each $i = 1, 2$. Set $M := M_1 \times M_2$, $R := R_1 \times R_2$, and $N := N_1 \times N_2$ be a submodule of M. Then:*

(1) *N is a d-submodule (resp., an fd-submodule) of M if and only if N_i is a d-submodule (resp., an fd-submodule) for each i.*

(2) *N is a 0-submodule of M if and only if N_i is a 0-submodule of M_i for each i.*

Proof. (1) It suffices to see that $\mathrm{Ann}_R(m_1, m_2) = \mathrm{Ann}_{R_1}(m_1) \times \mathrm{Ann}_{R_2}(m_2)$.

(2) Assume that N is a 0-submodule of M. Then $N = 0_S(M)$ for some multiplicative subset S of R. Put $S_1 := \{s_1 \in R_1 \mid (s_1, s_2) \in S$ for some $s_2 \in R_2\}$ and $S_2 := \{s_2 \in R_2 \mid (s_1, s_2) \in S$ for some $s_1 \in R_1\}$. It is clear that S_1 and S_2 are multiplicative subsets of R_1 and R_2, respectively. Also one can easily check that $0_S(M) = 0_{S_1}(M_1) \times 0_{S_2}(M_2)$. This implies that N is a 0-submodule of M if and only if N_1 and N_2 are 0-submodules, as desired. □

Proposition 2.30. *Let X be an indeterminate over R and let M be an R-module. Then N is an fd-submodule if and only if $N[X]$ is an fd-submodule of $M[X]$ as an $R[X]$-module.*

Proof. Let N be an fd-submodule of M and let $F = \{f_1, \dots, f_n\} \subseteq N[X]$ and $f \in M[X]$ such that $\mathrm{Ann}_{R[X]}(F) \subseteq \mathrm{Ann}_{R[X]}(f)$. Now put $f := \sum_{j=0}^{p} m'_j X^j$. Then it follows easily that $\mathrm{Ann}_R(C(F)) \subseteq \mathrm{Ann}_R(m'_1, \dots, m'_p)$ where $C(F)$ is the set of all coefficients of elements of F. Hence $\mathrm{Ann}_R(C(F)) \subseteq \mathrm{Ann}_R(m'_j)$ for each $j = 0, \dots, p$, and therefore $C(F) \subseteq N$ implies that $m'_i \in N$ for each $j = 0, \dots, p$. This shows that $f \in N[X]$.

Conversely, suppose that $N[X]$ is an fd-submodule. Let $\{m_1, \dots, m_n\} \subseteq N$ and $m' \in M$ such that $\bigcap_{i=1}^{n} \mathrm{Ann}_R(m_i) \subseteq \mathrm{Ann}_R(m')$. Then

$$\bigcap_{i=1}^{n} \mathrm{Ann}_{R[X]}(m_i X) \subseteq \mathrm{Ann}_{R[X]}(m' X)$$

in $M[X]$ and $\{m_1 X, \dots, m_n X\} \subseteq N[X]$ implies that $m' X \in N[X]$, and so $m' \in N$. □

Let $f : M \to M'$ be an R-module homomorphism. By the extension of a submodule N of M, we mean $N^e = f(N)$ and by the contraction of a submodule K of M', we mean $K^c = f^{-1}(K)$.

Proposition 2.31. *Let $f : M \to M'$ be an R-morphism. Then every d-submodule (resp., fd-submodule) of M' contracts to a d-submodule (resp., an fd-submodule) of M if and only if $\ker(f)$ is a d-submodule (resp., an fd-submodule) of M.*

Proof. The necessity is obvious. Conversely, suppose that $\ker(f)$ is a d-submodule and let K be a d-submodule of M'. If $\mathrm{Ann}_R(m) \subseteq \mathrm{Ann}_R(m')$ for some $m \in K^c$ and $m' \in M$, then $\mathrm{Ann}_R(rm) \subseteq \mathrm{Ann}_R(rm')$ for each $r \in \mathrm{Ann}_R(f(m))$. By hypothesis, we then have $rm' \in \ker(f)$ for each $r \in \mathrm{Ann}_R(f(m))$ and so $\mathrm{Ann}_R(f(m)) \subseteq$

$\mathrm{Ann}_R(f(m'))$. As K is a d-submodule, it follows that $f(m') \in K$. Similarly, we can prove that every fd-submodule of M' contracts to an fd-submodule of M whenever $\ker(f)$ is an fd-submodule of M. □

Corollary 2.32. *Let R be a commutative ring, M be an R-module, and N be a d-submodule (resp., an fd-submodule) of M. Then every d-submodule (resp., fd-submodule) of the R-module M/N contracts to a d-submodule (resp., an fd-submodule) of M.*

Corollary 2.33. *Let R be a commutative ring, S be a multiplicative subset of R, M be an R-module, and $f : M \to S^{-1}M$ be the natural morphism of R-modules. Then every d-submodule (resp., fd-submodule) of $S^{-1}M$ contracts to a d-submodule (resp., an fd-submodule) of M.*

Proposition 2.34. *Let R be a commutative ring, S be a multiplicative subset of R, M be an R-module, and $f : M \to S^{-1}M$ be the natural morphism of R-modules. If N is a d-submodule (resp., an fd-submodule) of M with $S \cap (N :_R M) = \emptyset$, then N^{ec} is also a d-submodule (resp., an fd-submodule) containing N.*

Proof. We just observe that $N^{ec} = N_M(S)$, see Example 2.2. □

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