International Electronic Journal of Algebra
Published Online: 5 March 2023
DOI: 10.24330/ieja.1260503

(n,d)-$\mathcal{X}_R$-PHANTOM AND (n,d)-${}_R\mathcal{X}$-COPHANTOM MORPHISMS

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Received: 6 January 2023; Revised: 26 February 2023; Accepted: 2 March 2023
Communicated by Abdullah Harmancı

Abstract. Several authors have been interested in some like phantom morphisms such as d-phantoms, d-Ext-phantoms, neat-phantom morphisms, clean-cophantom morphisms, RD-phantom morphisms and RD-Ext-phantom morphisms. In this paper, we prove that these notions can be unified. We are mainly interested in proving that the majority of the existing results hold true in our general framework.

Mathematics Subject Classification (2020): 16D90, 16E30, 18G25
Keywords: (n,d)-$\mathcal{X}_R$-phantom morphism, (n,d)-${}_R\mathcal{X}$-cophantom morphism, n-$\mathcal{X}$-coherent ring

1. Introduction

Throughout this paper, unless otherwise stated, R will be an associative (not necessarily commutative) ring with identity. We denote the category of left (resp., right) R-modules by R-Mod (resp., Mod-R), ${}_R\mathcal{X}$ denotes an arbitrary class of left R-modules and $\mathcal{X}_R$ denotes a class of right R-modules, and all morphisms are morphisms of left R-modules. We denote by R-Mor the category whose objects are morphisms of left R-modules and a morphism in R-Mor between two morphisms $\alpha : M_1 \to M_2$ and $\beta : N_1 \to N_2$ is a pair of left R-modules

$$(\ M_1 \xrightarrow{d} N_1 \ , \quad M_2 \xrightarrow{s} N_2 \ )$$

such that the following diagram

$$\begin{array}{ccc} M_1 & \xrightarrow{d} & N_1 \\ {\scriptstyle\alpha}\downarrow & & \downarrow{\scriptstyle\beta} \\ M_2 & \xrightarrow[s]{} & N_2 \end{array}$$

is commutative. Recall that R-Mor is a Grothendieck category. For a module A, $pd(A)$ (resp., $fd(A)$) will denote its projective (resp., flat) dimension. For simplicity, we will write ${}_R\mathcal{M}$ (resp., $\mathcal{M}_R$) instead of R-Mod (resp., Mod-R) to refer to the class of all left (resp., right) R-modules.

Let n be a non negative integer, a left (resp., right) module A is said to be n-presented if it has a finite n-presentation, that is, there is an exact sequence of left (resp., right) R-modules

$$F_n \to F_{n-1} \to \cdots \to F_1 \to F_0 \to A \to 0$$

where each F_i is a finitely generated free R-module. The module A is said to be infinitely presented if A is n-presented for any integer $n \in \mathbb{N}$. Clearly every finitely generated projective module is n-presented for any positive integer n. A module is 0-presented (resp., 1-presented) if and only if it is a finitely generated (resp., a finitely presented) module. An n-presented module is m-presented for any $m \geq n$. Let $\mathcal{X}$ be a class of modules and $n \in \mathbb{N} \cup \{\infty\}$ be an integer, we denote by $\mathcal{X}_n$ the subclass of $\mathcal{X}$ consisting of n-presented modules, then we have a decreasing sequence

$$\mathcal{X}_\infty \subseteq \cdots \subseteq \mathcal{X}_{n+1} \subseteq \mathcal{X}_n \subseteq \cdots \subseteq \mathcal{X}_1 \subseteq \mathcal{X}_0$$

According to [2], a ring R is called left (resp., right) n-$\mathcal{X}$-coherent if the subclass $\mathcal{X}_n$ of left (resp., right) n-presented R-modules of $\mathcal{X}$ is not empty, and every R-module in $\mathcal{X}_n$ is $n+1$-presented. The ring R is called n-$\mathcal{X}$-coherent if it is both left and right n-$\mathcal{X}$-coherent, when $\mathcal{X}$ is the whole category R-Mod then left n-$\mathcal{X}$-coherent rings are exactly left n-coherent rings.

The notion of phantom morphisms has a substantial role in module theory and ideal approximation theory. The study of phantom morphisms has its roots in algebraic topology in the study of maps between CW-complexes [16]. The definition of a phantom morphism was generalized by Herzog to the category of R-modules over an associative ring R in [6]. Herzog called a morphism of left R-modules $\alpha : M \to N$ a phantom morphism if for every finitely presented R-module A and every morphism $\beta : A \to M$, $\alpha\beta$ factors through a projective module, equivalently $\mathrm{Tor}_1(A, \alpha) = 0$ for any right (finitely presented) module A. Similarly, a morphism $\alpha : M \to N$ of left R-modules is called an Ext-phantom morphism [6] if the induced morphism $\mathrm{Ext}^1(A, \alpha) = 0$ for every finitely presented left R-module A. In [12], Mao generalizes these two definitions. For a non negative integer n, α is said to be an n-phantom if $\mathrm{Tor}_n(A, \alpha) = 0$ for any finitely presented right module A and α is said to be an n-Ext-phantom if $\mathrm{Ext}^n(A, \alpha)$ is 0 for every finitely presented left R-module A. Clearly α is a 1-phantom (resp., a 1-Ext-phantom) morphism if and only if α is a phantom (resp., an Ext-phantom) morphism.

On the other hand, ideal approximation theory has been recently introduced and developed by Fu, Guil Asensio, Herzog and Torrecillas in [5]. Let $\mathcal{A}$ be any category and $\mathcal{C}$ a class of objects in $\mathcal{A}$. Recall that a morphism $\phi : X \to Y$ in $\mathcal{A}$ is a $\mathcal{C}$-precover of Y if $X \in \mathcal{C}$ and, for any morphism $f : Z \to Y$ with $Z \in \mathcal{C}$, there is a morphism $g : Z \to X$ such that $\phi g = f$. A $\mathcal{C}$-precover $\phi : X \to Y$ is said to be a $\mathcal{C}$-cover of Y if every endomorphism $g : X \to X$ such that $\phi g = \phi$ is an isomorphism. Dually, we have the definitions of a $\mathcal{C}$-preenvelope and a $\mathcal{C}$-envelope. An ideal $\mathcal{I}$ of R-Mod is a subbifunctor of $\mathrm{Hom}(-,-)$ such that for every morphism g in $\mathcal{I}$ and every morphisms f and h of R-modules we have $fgh \in \mathcal{I}$ whenever it is defined. Let $\mathcal{I}$ be an ideal of R-Mod, a morphism $\varphi : M \to N$ in $\mathcal{I}$ is an $\mathcal{I}$-precover of N if for any morphism $\psi : C \to N$ in $\mathcal{I}$ there exists $\theta : C \to M$ such that $\varphi\theta = \psi$. An $\mathcal{I}$-precover is called $\mathcal{I}$-cover if every endomorphism h of M such that $\varphi h = \varphi$ is an automorphism. An $\mathcal{I}$-preenvelope and $\mathcal{I}$-envelope are defined dually. This theory is a generalization of the classical theory of covers and envelopes (approximation theory) initiated by Enochs, Auslander and Smalø since it need to be set forth in terms of morphisms instead of objects. An important instance is about the approximation by the ideal of (n,d)-$\mathcal{X}_R$-phantom (resp., (n,d)-${}_R\mathcal{X}$-cophantom) morphisms.

In Section 2 we first introduce the concept of (n,d)-$\mathcal{X}_R$-phantom and (n,d)-${}_R\mathcal{X}$-cophantom morphisms and we give some of their properties, then we finish this section by characterizing n-$\mathcal{X}_R$-coherent rings in terms of (n,d)-$\mathcal{X}_R$-phantom left R-morphisms and (n,d)-$\mathcal{X}_R$-cophantom right R-morphisms. Section 3 is devoted to the study of (n,d)-$\mathcal{X}_R$-phantom (resp., (n,d)-${}_R\mathcal{X}$-cophantom) morphisms with respect to a subfunctor of Ext. We start by introducing (n,d)-$\mathcal{X}_R$-epimorphisms (resp., (n,d)-${}_R\mathcal{X}$-monomorphisms), afterwards we examine their connection with (n,d)-$\mathcal{X}_R$-phantom (resp., (n,d)-${}_R\mathcal{X}$-cophantom) morphisms. Section 4 will be dedicated to the existence of (n,d)-$\mathcal{X}_R$-phantom precovers (resp., preenvelopes) and (n,d)-${}_R\mathcal{X}$-cophantom precovers (resp., preenvelopes) of left R-modules. We finish this paper by taking a close look to the properties of precovers and preenvelopes by (n,d)-$\mathcal{X}_R$-phantom and (n,d)-${}_R\mathcal{X}$-cophantom morphisms under change of rings.

2. (n,d)-$\mathcal{X}_R$-phantom and (n,d)-${}_R\mathcal{X}$-cophantom morphisms

In this section we give the definitions of (n,d)-$\mathcal{X}_R$-phantom and (n,d)-${}_R\mathcal{X}$-cophantom morphisms which unifies several notions and we study their closure properties under direct sums, direct limits and direct products.

Definition 2.1. Let $\alpha : M \to N$ be a morphism of left R-modules, $n \in \mathbb{N}^* \cup \{\infty\}$ and $d \in \mathbb{N}$.
(1) Let $\mathcal{X}_R$ be a non empty class of right R-modules. The morphism α is called an (n,d)-$\mathcal{X}_R$-phantom morphism if $(\mathcal{X}_R)_n$ is not empty and the induced morphism $\mathrm{Tor}_{d+1}(A,\alpha) : \mathrm{Tor}_{d+1}(A,M) \to \mathrm{Tor}_{d+1}(A,N)$ is 0 for any R-module A of $(\mathcal{X}_R)_n$.

(2) Let ${}_R\mathcal{X}$ be a non empty class of left R-modules. The morphism α is said to be an (n,d)-${}_R\mathcal{X}$-cophantom morphism if $({}_R\mathcal{X})_n$ is not empty and $\mathrm{Ext}^{d+1}(A,\alpha)$: $\mathrm{Ext}^{d+1}(A,M) \to \mathrm{Ext}^{d+1}(A,N)$ is 0 for any R-module A of $({}_R\mathcal{X})_n$.

The right version is defined similarly: If $\alpha : M \to N$ is a morphism of right R-modules, we say that α is an (n,d)-${}_R\mathcal{X}$-phantom morphism if $({}_R\mathcal{X})_n$ is not empty and the induced morphism $\mathrm{Tor}_{d+1}(\alpha, A) : \mathrm{Tor}_{d+1}(M,A) \to \mathrm{Tor}_{d+1}(N,A)$ is 0 for any R-module A of $({}_R\mathcal{X})_n$. Similarly, we say that the morphism α is an (n,d)-$\mathcal{X}_R$-cophantom morphism if $(\mathcal{X}_R)_n$ is not empty and the induced morphism $\mathrm{Ext}^{d+1}(A,\alpha) : \mathrm{Ext}^{d+1}(A,M) \to \mathrm{Ext}^{d+1}(A,N)$ is 0 for any R-module A of $({}_R\mathcal{X})_n$.

Example 2.2. (1) $(1,0)$-$\mathcal{M}_R$-phantom (resp., $(1,0)$-${}_R\mathcal{M}$-cophantom) morphisms are the classical phantom (resp., cophantom or Ext-phantom) morphisms studied in [6] (resp., [7]).
(2) Let $d \geq 1$. $(1,d-1)$-$\mathcal{M}_R$-phantom (resp., $(1,d-1)$-${}_R\mathcal{M}$-cophantom) morphisms are d-phantom (resp., d-*Ext*-phantom) morphisms introduced and studied in [8], [12] and [13].
(3) $(\infty,0)$-$\mathcal{M}_R$-phantom (resp., $(\infty,0)$-${}_R\mathcal{M}$-cophantom) morphisms are exactly neat-phantom (resp., clean-cophantom) morphisms defined in [14].
(4) Let ${}_R\mathcal{C}$ (resp., $\mathcal{C}_R$) be the class of left (resp., right) principally cyclic modules, i.e., modules of the form R/Ra (resp., R/aR) with $a \in R$. Then $(1,0)$-$\mathcal{C}_R$-phantom (resp., $(1,0)$-${}_R\mathcal{C}$-cophantom) morphisms are RD-phantom (resp., *RD-Ext*-phantom) morphisms defined and studied in [11].

Remark 2.3. (1) If α is an (n,d)-$\mathcal{X}_R$-phantom (resp., an (n,d)-${}_R\mathcal{X}$-cophantom) morphism, then α is an (m,d)-$\mathcal{X}_R$-phantom (resp., an (m,d)-${}_R\mathcal{X}$-phantom) morphism for any $m \geqslant n$.
(2) It is clear that if $\mathcal{Y}$ is a subclass of $\mathcal{X}_R$ (resp., ${}_R\mathcal{X}$) with $\mathcal{Y}_n$ is not empty then every (n,d)-$\mathcal{X}_R$-phantom (resp., (n,d)-${}_R\mathcal{X}$-cophantom) morphism is an (n,d)-$\mathcal{Y}$-phantom (resp., an (n,d)-$\mathcal{Y}$-cophantom) morphism.
(3) We will call an (n,d)-$\mathcal{M}_R$-phantom (resp., an (n,d)-${}_R\mathcal{M}$-cophantom) morphism by (n,d)-phantom (resp., (n,d)-cophantom) morphism.

The following propositions study the closure properties of (n,d)-$\mathcal{X}_R$-phantom and (n,d)-${}_R\mathcal{X}$-cophantom morphisms under direct sums, direct limits and direct products.

Proposition 2.4. (1) *The class of (n,d)-$\mathcal{X}_R$-phantom morphisms is closed under direct sums and direct limits.*
(2) *The class of (n,d)-${}_R\mathcal{X}$-cophantom morphisms is closed under direct products.*

Proof. (1) It is a consequence of the natural isomorphisms

$$\mathrm{Tor}_{d+1}(A, \bigoplus_I M_i) \cong \bigoplus_I \mathrm{Tor}_{d+1}(A, M_i)$$

and

$$\mathrm{Tor}_{d+1}(A, \lim_I M_i) \cong \lim_I \mathrm{Tor}_{d+1}(A, M_i).$$

(2) The second assertion holds from the natural isomorphism

$$\mathrm{Ext}^{d+1}(A, \prod_I M_i) \cong \prod_I \mathrm{Ext}^{d+1}(A, M_i).$$

□

Lemma 2.5. (1) *If* $n \geq d+1$, *then* $\mathrm{Ext}^{d+1}(A, \bigoplus_{i\in I} M_i) \cong \bigoplus_{i\in I} \mathrm{Ext}^{d+1}(A, M_i)$ *for any* n*-presented left* R*-module* A *and any family* $(M_i)_{i\in I}$ *of left* R*-modules.*

(2) *If* $n > d+1$, *then* $\lim \mathrm{Ext}\,(A, M_i) \cong \mathrm{Ext}\,(A, \lim M_i)$ *for any* n*-presented left* R*-module* A *and any directed system* $(M_i)_{i\in I}$ *of left* R*-modules.*

(3) *If* $n > d+1$, *then* $\mathrm{Tor}_{d+1}(A, \prod_{i\in I} M_i) \cong \prod_{i\in I} \mathrm{Tor}_{d+1}(A, M_i)$ *for any* n*-presented right* R*-module* A *and any family* $(M_i)_{i\in I}$ *of left* R*-modules.*

The lemma above is still true if we suppose that R is n-coherent since every n-presented R-module is $(d+1)$-presented. The following proposition generalizes [12, Proposition 2.9] and [14, Proposition 3.8].

Proposition 2.6. *Suppose that* $n \geq d+1$. *Then*

(1) *The class of* (n,d)-${}_R\mathcal{X}$*-cophantom morphisms is closed under direct sums.*

(2) *If* $n > d+1$, *in particular if* R *is* n*-coherent, then the class of* (n,d)-${}_R\mathcal{X}$*-cophantom morphisms is closed under direct limits.*

(3) *If* $n > d+1$, *in particular if* R *is* n*-coherent, then the class of* (n,d)-$\mathcal{X}_R$*-phantom morphisms is closed under direct products.*

Proof. (1) Let $(\alpha_i : M_i \to N_i)_{i\in I}$ be a family of (n,d)-${}_R\mathcal{X}$-cophantom morphisms and A be an n-presented module of ${}_R\mathcal{X}$. Then by (1) of Lemma 2.5

$$\mathrm{Ext}^{d+1}(A, \bigoplus_{i\in I} M_i) \cong \bigoplus_{i\in I} \mathrm{Ext}^{d+1}(A, M_i)$$

and

$$\mathrm{Ext}^{d+1}(A, \bigoplus_{i\in I} N_i) \cong \bigoplus_{i\in I} \mathrm{Ext}^{d+1}(A, N_i).$$

So

$$\begin{array}{ccc} \mathrm{Ext}^{d+1}(A, \bigoplus_{i\in I} M_i) & \longrightarrow & \mathrm{Ext}^{d+1}(A, \bigoplus_{i\in I} N_i) \\ \downarrow \cong & & \downarrow \cong \\ \bigoplus_{i\in I} \mathrm{Ext}^{d+1}(A, M_i) & \longrightarrow & \bigoplus_{i\in I} \mathrm{Ext}^{d+1}(A, N_i) \end{array}$$

Since $\mathrm{Ext}^{d+1}(A,\alpha_i)=0$ for each $i\in I$, $\mathrm{Ext}^{d+1}(A,\bigoplus_{i\in I}\alpha_i)=0$.
(2) Similar to (1), apply (2) of Lemma 2.5.
(3) If $n>d+1$, let $(\alpha_i : M_i\to N_i)_{i\in I}$ be a family of (n,d)-$\mathcal{X}_R$-phantom morphisms and A be an n-presented module of $\mathcal{X}$. By (3) of Lemma 2.5

$$\mathrm{Tor}_{d+1}(A,\prod_{i\in I}M_i)\cong\prod_{i\in I}\mathrm{Tor}_{d+1}(A,M_i)$$

and

$$\mathrm{Tor}_{d+1}(A,\prod_{i\in I}N_i)\cong\prod_{i\in I}\mathrm{Tor}_{d+1}(A,N_i)$$

Since $\mathrm{Tor}_{d+1}(A,\alpha_i)=0$ for all $i\in I$, $\mathrm{Tor}_{d+1}(A,\prod_{i\in I}\alpha_i)$ is zero. □

For a left R-module M, $M^+ := \mathrm{Hom}_{\mathbb{Z}}(M,\mathbb{Q}/\mathbb{Z})$ denotes the character module of M which is a right R-module and a morphism of left R-modules $\alpha : M\to N$ induces a morphism $\alpha^+ : N^+\to M^+$ of right R-modules. The following result extends [9, Proposition 3.8 and Proposition 3.9], [11, Lemma 2.7], [12, Proposition 2.10] and [14, Proposition 3.3].

Proposition 2.7. *Let $\alpha : M\to N$ be a morphism of left R-modules.*

(1) *α is an (n,d)-$\mathcal{X}_R$-phantom morphism if and only if α^+ is an (n,d)-$\mathcal{X}_R$-cophantom morphism.*
(2) *If R is left n-coherent, then α is an (n,d)-${}_R\mathcal{X}$-cophantom morphism if and only if α^+ is an (n,d)-${}_R\mathcal{X}$-cophantom morphism.*

Proof. (1) For any n-presented module A of $\mathcal{X}_R$ we have natural isomorphisms

$$\mathrm{Tor}_{d+1}(A,M)^+\cong\mathrm{Ext}^{d+1}(A,M^+)$$

and

$$\mathrm{Tor}_{d+1}(A,N)^+\cong\mathrm{Ext}^{d+1}(A,N^+).$$

Then there is a commutative diagram

$$\begin{array}{ccc}
\mathrm{Tor}_{d+1}(A,N)^+ & \longrightarrow & \mathrm{Tor}_{d+1}(A,M)^+ \\
\cong\downarrow & & \downarrow\cong \\
\mathrm{Ext}^{d+1}(A,N^+) & \longrightarrow & \mathrm{Ext}^{d+1}(A,M^+)
\end{array}$$

We get the result.
(2) Since R is n-coherent, any A in $({}_R\mathcal{X})_n$ is infinitely presented, hence by [17, Theorem 9.51] and remark following it, for any module M, we have the natural isomorphism

$$\mathrm{Tor}_{d+1}(M^+,A)\cong\mathrm{Ext}^{d+1}(A,M)^+.$$

And so $\mathrm{Ext}^{d+1}(A,\alpha)=0$ if and only if $\mathrm{Tor}_{d+1}(\alpha^+,A)=0$. □

Now we give a generalization of [11, Proposition 2.10] and [12, Proposition 2.7].

Theorem 2.8. *Let $n \in \mathbb{N} \cup \{\infty\}$, $d \in \mathbb{N}$, ${}_R\mathcal{X}$ be a class of left R-modules and $\mathcal{X}_R$ be a class of right R-modules.*

(1) *Every morphism of left R-modules is an (n,d)-$\mathcal{X}_R$-phantom morphism if and only if every module of $(\mathcal{X}_R)_n$ has flat dimension at most d.*
(2) *Every morphism of left R-modules is an (n,d)-${}_R\mathcal{X}$-cophantom morphism if and only if every module of $({}_R\mathcal{X})_n$ has projective dimension at most d.*

Proof. (1) ($\Rightarrow$) Let A be an n-presented module of $\mathcal{X}_R$ and M be any left R-module, then by hypothesis Id_M is an (n,d)-$\mathcal{X}_R$-phantom morphism, so the induced morphism $\mathrm{Tor}_{d+1}(A, Id_M) : \mathrm{Tor}_{d+1}(A, M) \to \mathrm{Tor}_{d+1}(A, M)$ is zero, which means that $\mathrm{Tor}_{d+1}(A, M) = 0$ for any left R-module M, hence $fd(A) \le d$.
($\Leftarrow$) Let $\alpha : M \to N$ be a morphism of left R-modules and A be an n-presented module of $\mathcal{X}_R$, then $fd(A) \le d$, hence

$$\mathrm{Tor}_{d+1}(A, M) = \mathrm{Tor}_{d+1}(A, N) = 0.$$

So $\mathrm{Tor}_{d+1}(A, \alpha) = 0$.
(2) It is similar to (1) by using the result: $pd(A) \le d$ if and only if for any left R-module M, $\mathrm{Ext}^{d+1}(A, M) = 0$. □

Recall that a ring R is said to be a left (resp., a right) (n,d)-ring, if every n-presented left (resp., right) R-module has projective dimension at most d; R is said to be a left (resp., a right) weak (n,d)-ring, if every n-presented left (resp., right) R-module has flat dimension at most d. R is an (n,d)-ring (resp., a weak (n,d)-ring) if it is both left and right (n,d)-ring (resp., weak (n,d)-ring).

Corollary 2.9. (1) *R is a right weak (n,d)-ring if and only if every morphism of left R-modules is an (n,d)-phantom morphism.*
(2) *R is a left (n,d)-ring if and only if every morphism of left R-modules is an (n,d)-cophantom morphism.*

Let ${}_R\mathcal{X}$ (resp., $\mathcal{X}_R$) be a class of left (resp., right) R-modules. Recall that a ring R is called left (resp., right) n-${}_R\mathcal{X}$-coherent (resp., n-$\mathcal{X}_R$-coherent) if the subclass $({}_R\mathcal{X})_n$ (resp., $(\mathcal{X}_R)_n$) is not empty, and every R-module in $({}_R\mathcal{X})_n$ (resp., $(\mathcal{X}_R)_n$) is $n+1$-presented. We are going to give some results which are extensions of [11, Theorem 2.8] and [14, Proposition 3.5].

Theorem 2.10. *For a ring R and a positive integer $n \ge 1$, the following statements are equivalent:*

(1) *R is right n-$\mathcal{X}_R$-coherent;*

(2) *the class of* $(n, n-1)$-$\mathcal{X}_R$-*phantom morphisms left* R-*modules is closed under direct products;*
(3) *the class of* $(n, n-1)$-$\mathcal{X}$-*cophantom morphisms of right* R-*modules is closed under direct limits.*

Proof. (1) $\Leftrightarrow$ (2) Assume R is right n-$\mathcal{X}_R$-coherent. Let $(\alpha_i)_{i\in I}$ be a family of $(n, n-1)$-$\mathcal{X}_R$-phantom morphisms, then $(\alpha_i)_{i\in I}$ is a family of $(n+1, n-1)$-$\mathcal{X}_R$-phantom morphisms, hence by Proposition 2.6 $\prod_{i\in I}\alpha_i$ is an $(n+1, n-1)$-$\mathcal{X}_R$-phantom morphism. Since R is n-$\mathcal{X}_R$-coherent, $\prod_{i\in I}\alpha_i$ is an $(n, n-1)$-$\mathcal{X}_R$-phantom morphism. Conversely, let I be a set, since R is n-$\mathcal{X}_R$-flat as a left R-module, Id_R is an $(n, n-1)$-$\mathcal{X}_R$-phantom morphism, then by (2) the product $\prod_I Id_R = Id_{R^I}$ is an $(n, n-1)$-$\mathcal{X}_R$-phantom morphism, hence R^I is an n-$\mathcal{X}_R$-flat module, and so by [2, Theorem 2.6] R is right n-$\mathcal{X}_R$-coherent.
(1) $\Leftrightarrow$ (3) Suppose R is right n-$\mathcal{X}_R$-coherent. Let $(M_i)_{i\in I}$ and $(N_i)_{i\in I}$ be directed systems of right R-modules over a directed index set I and $(\alpha_i : M_i \to N_i)_{i\in I}$ be a family of $(n, n-1)$-${}_R\mathcal{X}$-cophantom morphisms, then $(\alpha_i)_{i\in I}$ is a family of $(n+1, n-1)$-$\mathcal{X}_R$-cophantom morphisms hence by Proposition 2.6 $\lim_I \alpha_i$ is an $(n+1, n-1)$-$\mathcal{X}_R$-cophantom morphism. Since R is right n-$\mathcal{X}_R$-coherent, $\lim_I \alpha_i$ is an $(n, n-1)$-${}_R\mathcal{X}_R$-cophantom morphism. Conversely, let $(M_{i\in I})$ be a directed system of n-$\mathcal{X}_R$-injective right R-modules, then $(Id_{M_i})_{i\in I}$ is a family of $(n, n-1)$-$\mathcal{X}_R$-cophantom morphisms, hence by (3) $\lim_I Id_i = Id_{\lim_I M_i}$ is an $(n, n-1)$-$\mathcal{X}_R$-cophantom morphism and so $\lim_I M_i$ is an n-$\mathcal{X}_R$-injective module. By [2, Theorem 2.6] R is right n-$\mathcal{X}_R$-coherent. □

Theorem 2.11. *For a ring* R *and a positive integer* $n \geq 1$*, the following statements are equivalent:*

(1) R *is right* n-$\mathcal{X}_R$-*coherent;*
(2) *a morphism* α *of right* R-*modules is an* $(n, n-1)$-$\mathcal{X}_R$-*cophantom morphism if and only if* α^+ *is an* $(n, n-1)$-$\mathcal{X}_R$-*phantom morphism;*
(3) *a morphism* α *of right* R-*modules is an* $(n, n-1)$-$\mathcal{X}_R$-*cophantom morphism if and only if* α^{++} *is an* $(n, n-1)$-$\mathcal{X}_R$-*cophantom morphism;*
(4) *a morphism* α *of left* R-*modules is an* $(n, n-1)$-$\mathcal{X}_R$-*phantom morphism if and only if* α^{++} *is an* $(n, n-1)$-$\mathcal{X}_R$-*phantom morphism.*

Proof. (1) $\Rightarrow$ (2) Since R is right n-$\mathcal{X}_R$-coherent, any $A \in (\mathcal{X}_R)_n$ is $(n+1)$-presented, so by [3, Lemma 2.7] we have $\mathrm{Tor}_n(A, M^+) \cong (\mathrm{Ext}^n(A, M))^+$, for a morphism $\alpha : M \to N$ we have the following commutative diagram

$$\begin{array}{ccc} \mathrm{Tor}_n(A, N^+) & \longrightarrow & \mathrm{Tor}_n(A, M^+) \\ \cong \downarrow & & \downarrow \cong \\ (\mathrm{Ext}^n(A, N))^+ & \longrightarrow & (\mathrm{Ext}^n(A, M))^+ \end{array}$$

$\mathrm{Ext}^n(A, \alpha) = 0$ if and only if $\mathrm{Tor}_n(A, \alpha^+) = 0$.
(2) $\Rightarrow$ (3) A morphism α is an $(n, n-1)$-$\mathcal{X}_R$-cophantom morphism if and only if "by (2)" α^+ is an $(n, n-1)$-$\mathcal{X}_R$-phantom morphism if and only if "by Proposition 2.7" α^{++} is an $(n, n-1)$-$\mathcal{X}_R$-phantom morphism.
(3) $\Rightarrow$ (4) A morphism α is an $(n, n-1)$-$\mathcal{X}_R$-phantom morphism if and only if "by Proposition 2.7" α^+ is an $(n, n-1)$-$\mathcal{X}_R$-cophantom morphism if and only if "by (4)" α^{+++} is an $(n, n-1)$-$\mathcal{X}_R$-cophantom morphism if and only if "by Proposition 2.7" α^{++} is an $(n, n-1)$-$\mathcal{X}_R$-phantom morphism.
(4) $\Rightarrow$ (1) Let M be a left R-module. M is n-$\mathcal{X}_R$-flat if and only if Id_M is an $(n, n-1)$-$\mathcal{X}_R$-phantom morphism if and only if "by (4)" $Id_M^{++} = Id_{M^{++}}$ is an $(n, n-1)$-$\mathcal{X}_R$-phantom morphism if and only if M^{++} is n-$\mathcal{X}_R$-flat. Hence by [2, Theorem 2.6] R is right n-$\mathcal{X}_R$-coherent. □

In the following theorem we assume that the class $\mathcal{X}_R$ contains all finitely generated free R-modules and closed under kernels of epimorphisms.

Theorem 2.12. *For a ring R and a positive integer $n \geq 1$, the following statements are equivalent:*

(1) *R is right n-$\mathcal{X}_R$-coherent;*
(2) *For each $m \geq n$ and each $d \geq 0$, every (m, d)-$\mathcal{X}_R$-cophantom morphism of right R-modules is an (n, d)-$\mathcal{X}_R$-cophantom morphism;*
(3) *For each $m \geq n$ and each $d \geq 0$, every (m, d)-$\mathcal{X}_R$-phantom morphism of left R-modules is an (n, d)-$\mathcal{X}_R$-phantom morphism;*
(4) *Every $(n+1, n-1)$-$\mathcal{X}_R$-cophantom morphism of right R-modules is an $(n, n-1)$-${}_R\mathcal{X}$-phantom morphism;*
(5) *Every $(n+1, n-1)$-$\mathcal{X}_R$-phantom morphism of left R-modules is an $(n, n-1)$-$\mathcal{X}_R$-phantom morphism.*

Proof. (1) $\Rightarrow$ (2) $\Rightarrow$ (4) and (1) $\Rightarrow$ (3) $\Rightarrow$ (5) are obvious.
(4) $\Rightarrow$ (5) Let $\alpha : M \to N$ be an $(n+1, n-1)$-$\mathcal{X}_R$-phantom morphism, then by Proposition 2.7 α^+ is an $(n+1, n-1)$-$\mathcal{X}_R$-cophantom morphism, hence by (4) α^+ is an $(n, n-1)$-$\mathcal{X}_R$-cophantom morphism, so α be an $(n, n-1)$-$\mathcal{X}_R$-phantom morphism.
(5) $\Rightarrow$ (1) Let $(\alpha_i)_{i\in I}$ be a family of $(n, n-1)$-$\mathcal{X}_R$-phantom morphisms, then $(\alpha_i)_{i\in I}$

is a family of $(n+1, n-1)$-$\mathcal{X}_R$-phantom morphisms, it follows from Proposition 2.6 that $\prod_{i \in I} \alpha_i$ is an $(n+1, n-1)$-$\mathcal{X}_R$-phantom morphism. By (5) $\prod_{i \in I} \alpha_i$ is an $(n, n-1)$-$\mathcal{X}_R$-phantom morphism. So the class of $(n, n-1)$-$\mathcal{X}_R$-phantom morphisms is closed under direct product, therefore by Theorem 2.10 R is right n-$\mathcal{X}_R$-coherent. □

3. (n, d)-$\mathcal{X}_R$-phantom and (n, d)-${}_R\mathcal{X}$-cophantom morphisms with respect to a subfunctor of Ext

In this section we introduce the concept of (n, d)-$\mathcal{X}_R$-epimorphisms and (n, d)-${}_R\mathcal{X}$-monomorphisms, we study some of their properties, and finally we describe the relationship between (n, d)-$\mathcal{X}_R$-phantom (resp,. (n, d)-${}_R\mathcal{X}$-cophantom) morphisms and (n, d)-$\mathcal{X}_R$-epimorphisms (resp,. (n, d)-${}_R\mathcal{X}$-monomorphisms). For example we prove that:

(1) a morphism α is an (n, d)-$\mathcal{X}_R$-phantom morphism if and only if the pullback of any epimorphism along α is an (n, d)-$\mathcal{X}_R$-epimorphism.
(2) a morphism α is an (n, d)-${}_R\mathcal{X}$-cophantom morphism if and only if the pushout of any monomorphism along α is an (n, d)-${}_R\mathcal{X}$-monomorphism.

Definition 3.1. Let $n \in \mathbb{N}^* \cup \{\infty\}$, $d \in \mathbb{N}$.

(1) A morphism $f : M \to N$ is said to be an (n, d)-$\mathcal{X}_R$-epimorphism if $(\mathcal{X}_R)_n$ is not empty and $\mathrm{Tor}_{d+1}(A, f)$ is an epimorphism for any $A \in (\mathcal{X}_R)_n$.
(2) $f : M \to N$ is said to be an (n, d)-${}_R\mathcal{X}$-monomorphism if $({}_R\mathcal{X})_n$ is not empty and $\mathrm{Ext}^{d+1}(A, f)$ is a monomorphism for any $A \in ({}_R\mathcal{X})_n$.

The right versions are defined similarly.

Example 3.2. (1) A morphism f is a $(1, d)$-$\mathcal{M}_R$-epimorphism (resp., a $(1, d)$-${}_R\mathcal{M}$-monomorphism) if and only if f is a Tor_{d+1}-epimorphism (resp., an Ext_{d+1}-monomorphism) [6].
(2) $(1, 0)$-$\mathcal{M}_R$-epimorphisms (resp., $(1, 0)$-${}_R\mathcal{M}$-monomorphisms) are exactly Tor-epimorphisms (resp., Ext-monomorphisms) [9].

Proposition 3.3. *Let $n \in \mathbb{N}^* \cup \{\infty\}$, $d \in \mathbb{N}$. A morphism $f : M \to N$ is an (n, d)-$\mathcal{X}_R$-epimorphism if and only if $f^+ : N^+ \to M^+$ is an (n, d)-${}_R\mathcal{X}$-monomorphism.*

Proof. It follows from the the commutative diagram

$$\begin{array}{ccc} \mathrm{Tor}_{d+1}(A, N)^+ & \longrightarrow & \mathrm{Tor}_{d+1}(A, M)^+ \\ \cong \downarrow & & \downarrow \cong \\ \mathrm{Ext}^{d+1}(A, N^+) & \longrightarrow & \mathrm{Ext}^{d+1}(A, M^+) \end{array}$$

where A is an n-presented module of $\mathcal{X}_R$. $\square$

Proposition 3.4. *Let $\eta : 0 \to L \overset{\alpha}{\to} M \overset{\beta}{\to} N \to 0$ be an exact sequence of left R-modules.*

(1) *β is an (n,d)-$\mathcal{X}_R$-phantom morphism if and only if α is an (n,d)-$\mathcal{X}_R$-epimorphism.*

(2) *α is an (n,d)-${}_R\mathcal{X}$-cophantom morphism if and only if β is an (n,d)-${}_R\mathcal{X}$-monomorphism.*

Proof. (1) For any $A \in (\mathcal{X}_R)_n$, by the long exact sequence of $\mathrm{Tor}(A,-)$ to the sequence η, we have the exact sequence

$$\mathrm{Tor}_{d+1}(A,L) \longrightarrow \mathrm{Tor}_{d+1}(A,M) \longrightarrow \mathrm{Tor}_{d+1}(A,N).$$

$\mathrm{Tor}_{d+1}(A,\beta) = 0$ if and only if $\mathrm{Tor}_{d+1}(A,\alpha)$ is an epimorphism.
(2) For any $A \in ({}_R\mathcal{X})_n$, by the long exact sequence of $\mathrm{Ext}(A,-)$ to the sequence η, we have the exact sequence

$$\mathrm{Ext}^{d+1}(A,L) \longrightarrow \mathrm{Ext}^{d+1}(A,M) \longrightarrow \mathrm{Ext}^{d+1}(A,N).$$

$\mathrm{Ext}^{d+1}(A,\alpha) = 0$ if and only if $\mathrm{Ext}^{d+1}(A,\beta)$ is a monomorphism. $\square$

Proposition 3.5. *Let f be an (n,d)-$\mathcal{X}_R$-epimorphism in R-Mod and $gf = h$.*

(1) *g is an (n,d)-$\mathcal{X}_R$-phantom morphism if and only if h is an (n,d)-$\mathcal{X}_R$-phantom morphism.*

(2) *g is an (n,d)-$\mathcal{X}_R$-epimorphism if and only if h is an (n,d)-$\mathcal{X}_R$-epimorphism.*

Proof. For any $A \in (\mathcal{X}_R)_n$ we have

$$\mathrm{Tor}_{d+1}(A,g)\,\mathrm{Tor}_{d+1}(A,f) = \mathrm{Tor}_{d+1}(A,h).$$

Since $\mathrm{Tor}_{d+1}(A,f)$ is an epimorphism,

(1) $\mathrm{Tor}_{d+1}(A,g) = 0$ if and only if $\mathrm{Tor}_{d+1}(A,h) = 0$,

(2) $\mathrm{Tor}_{d+1}(A,g)$ is an epimorphism if and only if $\mathrm{Tor}_{d+1}(A,h)$ is an epimorphism. $\square$

Proposition 3.6. *Let g be an (n,d)-${}_R\mathcal{X}$-monomorphism in R-Mod and $gf = h$.*

(1) *f is an (n,d)-${}_R\mathcal{X}$-cophantom morphism if and only if h is an (n,d)-${}_R\mathcal{X}$-cophantom morphism.*

(2) *f is an (n,d)-${}_R\mathcal{X}$-monomorphism if and only if h is an (n,d)-${}_R\mathcal{X}$- monomorphism.*

Proof. For any $A \in ({}_R\mathcal{X})_n$, we have

$$\mathrm{Ext}^{d+1}(A,g)\,\mathrm{Ext}^{d+1}(A,f) = \mathrm{Ext}^{d+1}(A,h).$$

Since $\mathrm{Ext}^{d+1}(A,g)$ is a monomorphism, we get the following two equivalences

(1) $\mathrm{Ext}^{d+1}(A,f) = 0$ if and only if $\mathrm{Ext}^{d+1}(A,h) = 0$,
(2) $\mathrm{Ext}^{d+1}(A,f)$ is a monomorphism if and only if $\mathrm{Ext}^{d+1}(A,h)$ is a monomorphism. □

Proposition 3.7. *Let $n \in \mathbb{N}^* \cup \{\infty\}$, $d \in \mathbb{N}$. Let $0 \to K \xrightarrow{\alpha} M \xrightarrow{f} N \to 0$ and $0 \to K \xrightarrow{\beta} L \xrightarrow{g} Q \to 0$ be two exact sequences of left R-modules with g an (n,d)-$\mathcal{X}_R$-epimorphism. Consider the following pushout:*

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & & & \\
 & & \downarrow & & \downarrow & & & & \\
0 & \longrightarrow & K & \xrightarrow{\alpha} & M & \xrightarrow{f} & N & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle\beta} & & {\scriptstyle\gamma}\downarrow & & \downarrow{\scriptstyle =} & & \\
0 & \longrightarrow & L & \xrightarrow{\phi} & H & \xrightarrow{h} & N & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle g} & & \downarrow & & & & \\
 & & Q & \xrightarrow{=} & Q & & & & \\
 & & \downarrow & & \downarrow & & & & \\
 & & 0 & & 0 & & & &
\end{array}$$

(1) *f is an (n,d)-$\mathcal{X}_R$-phantom morphism if and only if h is an (n,d)-$\mathcal{X}_R$-phantom morphism,*
(2) *f is an (n,d)-$\mathcal{X}_R$-epimorphism if and only if h is an (n,d)-$\mathcal{X}_R$-epimorphism.*

Proof. For any $A \in (\mathcal{X}_R)_n$, the exact sequence $0 \to K \xrightarrow{\beta} L \xrightarrow{g} Q \to 0$ induces an exact sequence

$$\mathrm{Tor}_{d+1}(A,L) \longrightarrow \mathrm{Tor}_{d+1}(A,Q) \longrightarrow \mathrm{Tor}_d(A,K) \longrightarrow \mathrm{Tor}_d(A,L).$$

Since $\mathrm{Tor}_{d+1}(A,g)$ is an epimorphism, $\mathrm{Tor}_d(A,\beta)$ is a monomorphism. Consider the following commutative diagram

$$\begin{array}{ccccc}
\mathrm{Tor}_{d+1}(A,M) & \longrightarrow & \mathrm{Tor}_{d+1}(A,N) & \xrightarrow{\psi} & \mathrm{Tor}_d(A,K) \\
\downarrow & & \downarrow{\scriptstyle =} & & \downarrow \\
\mathrm{Tor}_{d+1}(A,H) & \longrightarrow & \mathrm{Tor}_{d+1}(A,N) & \xrightarrow{\lambda} & \mathrm{Tor}_d(A,L)
\end{array}$$

(1) $\mathrm{Tor}_{d+1}(A,f) = 0$ if and only if ψ is a monomorphism if and only if λ is a monomorphism if and only if $\mathrm{Tor}_{d+1}(A,h) = 0$.

(2) $\mathrm{Tor}_{d+1}(A,f)$ is an epimorphism if and only if $\psi = 0$ if and only if $\lambda = 0$ if and only if $\mathrm{Tor}_{d+1}(A,h)$ is an epimorphism. □

Recall that an additive subfunctor $\mathcal{F}$ of Ext [5] is defined as follows: For every pair A and B of left R-modules, $\mathcal{F}$ associates a subgroup $\mathcal{F}(A,B)$ of $\mathrm{Ext}(A,B)$ so that for any $f : X \to A$, $g : B \to Y$, if $\eta \in \mathcal{F}(A,B)$, then $\mathrm{Ext}(f,g)(\eta) = \mathrm{Ext}(f,Y)\,\mathrm{Ext}(A,g)(\eta) \in \mathcal{F}(X,Y)$. Equivalently, $\mathcal{F}$ includes the split exact sequences and is closed under direct sums, pullbacks and pushouts by [1, Lemma 1.1]. In [13] Mao denoted the collection of exact sequences $\eta : 0 \to K \to M \to N \to 0$ such that $M \to N$ (resp., $K \to M$) is a Tor_n-epimorphism (resp., Ext^n-monomorphism) by Δ_n (resp., ∇^n). Following these notations we will denote the collection of exact sequences η such that $M \to N$ (resp., $K \to M$) is an (n,d)-$\mathcal{X}_R$-epimorphism (resp., (n,d)-${}_R\mathcal{X}$-monomorphism) by $\mathcal{X}_R$-$\Delta_{(n,d)}$ (resp., ${}_R\mathcal{X}$-$\nabla_{(n,d)}$).

Lemma 3.8. *Let R be a ring.*

(1) *The collection $\mathcal{X}_R$-$\Delta_{(n,d)}$ constitutes an additive subfunctor of* Ext.
(2) *The collection ${}_R\mathcal{X}$-$\nabla_{(n,d)}$ constitutes an additive subfunctor of* Ext.

Proof. (1) Let $0 \to K \to M \xrightarrow{f} N \to 0$ be a split exact sequence, then there exists a morphism $g : N \to M$ such that $fg = Id_M$, hence for any A in $(\mathcal{X}_R)_n$, $\mathrm{Tor}_{d+1}(A,f)\,\mathrm{Tor}_{d+1}(A,g) = Id_{\mathrm{Tor}_{d+1}(A,M)}$ and so $\mathrm{Tor}_{d+1}(A,f)$ is an epimorphism. It is clear that $\mathcal{X}_R$-$\Delta_{(n,d)}$ is closed under direct sums. Now consider an exact sequence $\eta : 0 \to K \to M \xrightarrow{\beta} N \to 0$ with β an (n,d)-$\mathcal{X}_R$-epimorphism. For any morphism $\alpha : L \to N$ we obtain the pullback η' of η along the morphism α

$$\begin{array}{ccccccccc} \eta' : 0 & \longrightarrow & K & \longrightarrow & H & \xrightarrow{\gamma} & L & \longrightarrow & 0 \\ & & \Big\| & & \Big\downarrow & & \Big\downarrow{\scriptstyle \alpha} & & \\ \eta : 0 & \longrightarrow & K & \longrightarrow & M & \xrightarrow{\beta} & N & \longrightarrow & 0 \end{array}$$

For any $A \in (\mathcal{X}_R)_n$, we get the following commutative diagram

$$\begin{array}{ccccc} \mathrm{Tor}_{d+1}(A,H) & \longrightarrow & \mathrm{Tor}_{d+1}(A,L) & \xrightarrow{\phi} & \mathrm{Tor}_d(A,K) \\ \Big\downarrow & & \Big\downarrow & & \Big\downarrow{\scriptstyle =} \\ \mathrm{Tor}_{d+1}(A,M) & \longrightarrow & \mathrm{Tor}_{d+1}(A,N) & \xrightarrow[\theta]{} & \mathrm{Tor}_d(A,K) \end{array}$$

Since $\mathrm{Tor}_{d+1}(A,\beta)$ is an epimorphism, $\theta = 0$ and so $\phi = 0$. Thus $\mathrm{Tor}_{d+1}(A,\gamma)$ is an epimorphism, hence $\mathcal{X}_R$-$\Delta_{(n,d)}$ is closed under pullbacks. For any morphism

$\lambda : K \to L$ we get the pushout η'' of η along λ

$$\begin{array}{ccccccccc} \eta : 0 & \longrightarrow & K & \longrightarrow & M & \xrightarrow{\beta} & N & \longrightarrow & 0 \\ & & {\scriptstyle\lambda}\downarrow & & \downarrow & & \downarrow{\scriptstyle =} & & \\ \eta'' : 0 & \longrightarrow & L & \longrightarrow & G & \xrightarrow{\sigma} & N & \longrightarrow & 0 \end{array}$$

which gives rise to the following commutative diagram

$$\begin{array}{ccc} \mathrm{Tor}_{d+1}(A,M) & \longrightarrow & \mathrm{Tor}_d(A,N) \\ \downarrow & & \downarrow{\scriptstyle =} \\ \mathrm{Tor}_d(A,G) & \longrightarrow & \mathrm{Tor}_d(A,N) \end{array}$$

Since $\mathrm{Tor}_{d+1}(A,\beta)$ is an epimorphism, $\mathrm{Tor}_{d+1}(A,\sigma)$ is an epimorphism and so $\mathcal{X}_R$-$\Delta_{(n,d)}$ is closed under pushouts. It follows that $\mathcal{X}_R$-$\Delta_{(n,d)}$ constitutes an additive subfunctor of Ext.
The proof of (2) is dual. $\square$

In [5], Fu, Guil Asensio, Herzog and Torrecillas introduced a relative version of phantom morphisms in the sense: For an additive subfunctor $\mathcal{F}$ of Ext, a morphism $\varphi : X \to A$ in R-Mod is an $\mathcal{F}$-phantom morphism if the pullback of any epimorphism along φ is in $\mathcal{F}$. Dually, a morphism $\psi : Y \to Z$ in R-Mod is called an $\mathcal{F}$-cophantom morphism if the pushout of any monomorphism along ψ is in $\mathcal{F}$.

A morphism $f : U \to V$ in R-Mod is called $\mathcal{F}$-projective if for any left R-module B, $\mathcal{F}(f,B) : \mathcal{F}(V,B) \to \mathcal{F}(U,B)$ is 0. A left R-module M is called $\mathcal{F}$-projective if 1_M is $\mathcal{F}$-projective. Dually, a morphism $g : X \to Y$ in R-Mod is called $\mathcal{F}$-injective if for any left R-module $B, \mathcal{F}(B,g) : \mathcal{F}(B,X) \to \mathcal{F}(B,Y)$ is 0. A left R-module N is called $\mathcal{F}$-injective if 1_N is $\mathcal{F}$-injective. The following two theorems are a generalizations of [2, Theorem 2.2], [13, Theorem 2.12] and [14, Theorem 3.6].

Theorem 3.9. *Let $n \in \mathbb{N}^* \cup \{\infty\}$ and $d \in \mathbb{N}$.*

(1) *A morphism $f : M \to N$ of left R-modules is an (n,d)-$\mathcal{X}_R$-phantom morphism if and only if f is a $\mathcal{X}_R$-$\Delta_{(n,d)}$-phantom morphism.*
(2) *A morphism $g : X \to Y$ of left R-modules is an (n,d)-${}_R\mathcal{X}$-cophantom morphism if and only if f is a ${}_R\mathcal{X}$-$\nabla_{(n,d)}$-cophantom morphism.*

Proof. (1) $\Rightarrow$) Let $\eta : 0 \to K \to Q \to N \to 0$ be any exact sequence, we get the pullback η' of η along f

$$\begin{array}{ccccccccc}
\eta' : 0 & \longrightarrow & K & \longrightarrow & H & \xrightarrow{h} & M & \longrightarrow & 0 \\
 & & \Big\downarrow{=} & & \Big\downarrow & & \Big\downarrow{f} & & \\
\eta : 0 & \longrightarrow & K & \longrightarrow & Q & \longrightarrow & N & \longrightarrow & 0
\end{array}$$

For any $A \in (\mathcal{X}_R)_n$, we have the following commutative diagram with exact rows

$$\begin{array}{ccccc}
\mathrm{Tor}_{d+1}(A,H) & \longrightarrow & \mathrm{Tor}_{d+1}(A,M) & \xrightarrow{\phi} & \mathrm{Tor}_d(A,K) \\
\Big\downarrow & & \Big\downarrow & & \Big\downarrow{=} \\
\mathrm{Tor}_{d+1}(A,Q) & \longrightarrow & \mathrm{Tor}_{d+1}(A,N) & \xrightarrow[\psi]{} & \mathrm{Tor}_d(A,K)
\end{array}$$

Since f is an (n,d)-$\mathcal{X}_R$-phantom morphism, $\phi = \psi\,\mathrm{Tor}_{d+1}(A,f) = 0$, therefore $\mathrm{Tor}_{d+1}(A,h)$ is an epimorphism and so h is an (n,d)-$\mathcal{X}_R$-epimorphism. Thus f is a $\mathcal{X}_R$-$\Delta_{(n,d)}$-phantom morphism.
$\Leftarrow$) There exists an exact sequence $\zeta : 0 \to C \to P \to N \to 0$ with P projective. We get the pullback of ζ along f :

$$\begin{array}{ccccccccc}
\zeta' : 0 & \longrightarrow & C & \longrightarrow & G & \xrightarrow{\gamma} & M & \longrightarrow & 0 \\
 & & \Big\downarrow{=} & & \Big\downarrow & & \Big\downarrow{f} & & \\
\zeta : 0 & \longrightarrow & C & \longrightarrow & P & \longrightarrow & N & \longrightarrow & 0
\end{array}$$

For any $A \in (\mathcal{X}_R)_n$, we get the following commutative diagram:

$$\begin{array}{ccc}
\mathrm{Tor}_{d+1}(A,G) & \longrightarrow & \mathrm{Tor}_{d+1}(A,M) \\
\Big\downarrow & & \Big\downarrow \\
0 = \mathrm{Tor}_{d+1}(A,P) & \longrightarrow & \mathrm{Tor}_{d+1}(A,N)
\end{array}$$

So $\mathrm{Tor}_{d+1}(A,f)\,\mathrm{Tor}_{d+1}(A,\gamma) = 0$ and $\mathrm{Tor}_{d+1}(A,f) = 0$ since $\mathrm{Tor}_{d+1}(A,\gamma)$ is an epimorphism, i.e., f is an (n,d)-$\mathcal{X}_R$-phantom morphism.
The proof of (2) is dual. □

Theorem 3.10. *Let $n \in \mathbb{N}^* \cup \{\infty\}$, $d \in \mathbb{N}$.*

(1) *If a morphism $\alpha : M \to N$ is an (n,d)-$\mathcal{X}_R$-phantom morphism, then for every morphism $\beta : F \to M$ with F $\mathcal{X}_R$-$\Delta_{(n,d)}$-projective, $\alpha\beta$ factors through a projective module.*

(2) *If a morphism $g : X \to Y$ is an (n,d)-${}_R\mathcal{X}$-cophantom morphism, then for every morphism $h : Y \to Z$ with Z ${}_R\mathcal{X}$-$\nabla_{(n,d)}$-injective, hg factors through an injective module.*

Proof. (1) There exists an exact sequence $\eta : 0 \to K \to P \to N \to 0$ with P projective. Then we get the pullback η' of η along f

$$\begin{array}{rccccccccc}
\eta' : 0 & \longrightarrow & K & \longrightarrow & H & \longrightarrow & M & \longrightarrow & 0 \\
 & & \Big\downarrow{\scriptstyle =} & & \Big\downarrow & & \Big\downarrow{\scriptstyle f} & & \\
\eta : 0 & \longrightarrow & K & \longrightarrow & P & \longrightarrow & N & \longrightarrow & 0
\end{array}$$

By Theorem 3.9 $\eta' \in \Delta^{\mathcal{X}}_{(n,d)}$. Since F is $\Delta^{\mathcal{X}}_{(n,d)}$-projective, α lifts to H. So $f\alpha$ factors through P.
The proof of (2) is dual. □

For phantom (resp., cophantom) morphisms we find that the assertions (1) and (2) are equivalent and the converse holds from the fact that every module M has a pure-projective precover (resp., a pure-injective preenevelope) and it is the same for RD-phantom and RD-Ext-phantom morphisms. In our case it is an open question: Does every left module M have an $\mathcal{X}_R$-$\Delta_{(n,d)}$-projective (resp., an ${}_R\mathcal{X}$-$\nabla_{(n,d)}$-injective) precover (resp., preenvelope)?

4. (n,d)-$\mathcal{X}_R$-phantom and (n,d)-${}_R\mathcal{X}$-cophantom precovers and preenvelopes

Lemma 4.1. *Let $0 \to X \overset{\lambda}{\to} Y \overset{\pi}{\to} Z \to 0$ be an exact sequence in R-Mod. The following conditions are equivalent:*

(1) $0 \to X \overset{\lambda}{\to} Y \overset{\pi}{\to} Z \to 0$ *is pure.*
(2) *The induced sequence* $0 \to \mathrm{Tor}_n(A, X) \to \mathrm{Tor}_n(A, Y) \to \mathrm{Tor}_n(A, Z) \to 0$ *is exact for any* $n \geq 1$ *and any right R-module A.*

If R is a left coherent ring, then the above conditions are equivalent to

(3) *The induced sequence* $0 \to \mathrm{Ext}^n(G, X) \to \mathrm{Ext}^n(G, Y) \to \mathrm{Ext}^n(G, Z) \to 0$ *is exact for any* $n \geq 1$ *and any finitely presented left R-module G.*

Lemma 4.2. *If $0 \to \psi \to \phi \to \gamma \to 0$ is pure exact in R-Mor, i.e., the following exact sequence is pure exact in R-Mor*

$$\begin{array}{rccccccccc}
\eta_1 : & 0 & \longrightarrow & K_1 & \xrightarrow{\alpha_1} & M_1 & \xrightarrow{\beta_1} & L_1 & \longrightarrow & 0 \\
 & & & \Big\downarrow{\scriptstyle \psi} & & \Big\downarrow{\scriptstyle \phi} & & \Big\downarrow{\scriptstyle \gamma} & & \\
\eta_2 : & 0 & \longrightarrow & K_2 & \xrightarrow[\alpha_2]{} & M_2 & \xrightarrow[\beta_2]{} & L_2 & \longrightarrow & 0
\end{array}$$

Then both η_1 and η_2 are pure exact in R-Mod.

Lemma 4.3. *Let $0 \to \psi \to \phi \to \gamma \to 0$ be pure exact in R-Mor. Then*

1. *If ϕ is an (n,d)-$\mathcal{X}_R$-phantom morphism then γ and ψ are also (n,d)-$\mathcal{X}_R$-phantom morphism;*
2. *If R is n-coherent and ϕ is an (n,d)-${}_R\mathcal{X}$-cophantom morphism then γ and ψ are also an (n,d)-${}_R\mathcal{X}$-cophantom morphism.*

Proof. We have a commutative diagram with pure exact rows by Lemma 4.2:

$$\begin{array}{lccccccccc}
\eta_1: & 0 & \longrightarrow & K_1 & \xrightarrow{\alpha_1} & M_1 & \xrightarrow{\beta_1} & L_1 & \longrightarrow & 0 \\
& & & \downarrow \psi & & \downarrow \phi & & \downarrow \gamma & & \\
\eta_2: & 0 & \longrightarrow & K_2 & \xrightarrow[\alpha_2]{} & M_2 & \xrightarrow[\beta_2]{} & L_2 & \longrightarrow & 0
\end{array}$$

Then we obtain the commutative diagram with split exact rows:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & L_2^+ & \xrightarrow{\beta_2^+} & M_2^+ & \xrightarrow{\alpha_2^+} & K_2^+ & \longrightarrow & 0 \\
& & \downarrow \gamma^+ & & \downarrow \phi^+ & & \downarrow \psi^+ & & \\
0 & \longrightarrow & L_1^+ & \xrightarrow[\beta_1^+]{} & M_1^+ & \xrightarrow[\alpha_1^+]{} & K_1^+ & \longrightarrow & 0
\end{array}$$

(1) Note that $\phi^+$is an (n,d)-$\mathcal{X}_R$-cophantom morphism by Proposition 2.7. For any n-presented R-module A in $\mathcal{X}_R$, we have

$$\mathrm{Ext}^{d+1}\left(A,\psi^+\right)\mathrm{Ext}^{d+1}\left(A,\alpha_2^+\right)=\mathrm{Ext}^{d+1}\left(A,\alpha_1^+\right)\mathrm{Ext}^{d+1}\left(A,\phi^+\right)=0.$$

Since $\mathrm{Ext}^{d+1}\left(A,\alpha_2^+\right)$ is an epimorphism, $\mathrm{Ext}^{d+1}\left(A,\psi^+\right)=0$. Thus, $\psi^+$is an (n,d)-$\mathcal{X}_R$-cophantom morphism and so ψ is an (n,d)-$\mathcal{X}_R$-phantom morphism by Proposition 2.7. Also, we have

$$\mathrm{Ext}^{d+1}\left(A,\beta_1^+\right)\mathrm{Ext}^{d+1}\left(A,\gamma^+\right)=\mathrm{Ext}^{d+1}\left(A,\phi^+\right)\mathrm{Ext}^{d+1}\left(A,\beta_2^+\right)=0.$$

Since $\mathrm{Ext}^{d+1}\left(A,\beta_1^+\right)$is a monomorphism, $\mathrm{Ext}^{d+1}\left(A,\gamma^+\right)=0$. Thus, $\gamma^+$is an (n,d)-$\mathcal{X}_R$-cophantom morphism. So γ is an (n,d)-$\mathcal{X}_R$-phantom morphism by Proposition 2.7.

(2) By Proposition 2.7, ϕ^+ is an (n,d)-${}_R\mathcal{X}$-phantom morphism. For any n-presented module A of ${}_R\mathcal{X}$, we obtain

$$\mathrm{Tor}_{d+1}\left(\psi^+,A\right)\mathrm{Tor}_{d+1}\left(\alpha_2^+,A\right)=\mathrm{Tor}_{d+1}\left(\alpha_1^+,A\right)\mathrm{Tor}_{d+1}\left(\phi^+,A\right)=0.$$

Since $\mathrm{Tor}_{d+1}\left(\alpha_2^+,A\right)$ is an epimorphism, $\mathrm{Tor}_{d+1}\left(\psi^+,A\right)=0$. Thus, $\psi^+$is an (n,d)-${}_R\mathcal{X}$-phantom morphism and so ψ is an (n,d)-${}_R\mathcal{X}$-cophantom morphism by Proposition 2.7. On the other hand, we get

$$\mathrm{Tor}_{d+1}\left(\beta_1^+,A\right)\mathrm{Tor}_{d+1}\left(\gamma^+,A\right)=\mathrm{Tor}_{d+1}\left(\phi^+,A\right)\mathrm{Tor}_{d+1}\left(\beta_2^+,A\right)=0.$$

Since $\operatorname{Tor}_{d+1}\left(\beta_1^+, A\right)$ is a monomorphism, $\operatorname{Tor}_{d+1}(\gamma^+, A) = 0$. Hence $\gamma^+$is an (n,d)-${}_R\mathcal{X}$-phantom morphism and so γ is an (n,d)-${}_R\mathcal{X}$-phantom comorphism by Proposition 2.7. □

Theorem 4.4. *Let $n \in \mathbb{N}^* \cup \{\infty\}$, $d \in \mathbb{N}$, and ${}_R\mathcal{X}$ (resp., $\mathcal{X}_R$) be a class of left (resp., right) R-modules.*

(1) *Every left R-module morphism has an epic (n,d)-$\mathcal{X}_R$-phantom cover in R-Mor.*
(2) *If $n > d+1$, and in particular if R is n-coherent, then every left R-module morphism has an (n,d)-$\mathcal{X}_R$-phantom preenvelope in R-Mor.*
(3) *If R is n-coherent, then every left R-module morphism has a monic (n,d)-${}_R\mathcal{X}$-cophantom preenvelope in R-Mor.*
(4) *If R is n-coherent, then every left R-module morphism has an (n,d)-${}_R\mathcal{X}$-cophantom cover.*

Proof. (1) Note that the class of (n,d)-$\mathcal{X}_R$-phantom morphisms is closed under direct limits by Proposition 2.4 and closed under pure epimorphic images by Lemma 4.3. So every left R-module morphism has an (n,d)-$\mathcal{X}_R$-phantom cover by [4, Theorem 2.6]. The (n,d)-$\mathcal{X}_R$-phantom cover is an epimorphism because every projective object of R-Mor is an (n,d)-$\mathcal{X}_R$-phantom morphism.
(2) The class of (n,d)-$\mathcal{X}_R$-phantom morphisms is closed under direct products by Proposition 2.6 and closed under pure subobjects by Lemma 4.3. Then every left R-module morphism has an (n,d)-$\mathcal{X}_R$-phantom preenvelope in R-Mor by [4, Theorem 4.1].
(3) The class of (n,d)-${}_R\mathcal{X}$-cophantom morphisms is closed under direct products by Proposition 2.4 and closed under pure subobjects by Lemma 4.3. Then every left R-module morphism has an (n,d)-$\mathcal{X}_R$-phantom preenvelope in R-Mor by [4, Theorem 4.1]. The (n,d)-${}_R\mathcal{X}$-cophantom preenvelope is a monomorphism because every injective object of R-Mor is an (n,d)-$\mathcal{X}_R$-phantom morphism.
(4) The class of (n,d)-${}_R\mathcal{X}$-cophantom morphisms is closed under direct limits by Proposition 2.6 and closed under pure epimorphic images by Lemma 4.3. Then every left R-module morphism has an (n,d)-${}_R\mathcal{X}$-cophantom cover by [4, Theorem 2.6]. □

In ideal approximation theory, the concepts of (pre)covers and (pre) envelopes for classes of objects were generalized to ideals of morphisms. An ideal $\mathcal{I}$ of R-Mod is a subbifunctor of $\operatorname{Hom}(-,-)$ such that for every morphism g in $\mathcal{I}$ and every morphisms f and h of R-modules we have $fgh \in \mathcal{I}$ whenever it is defined. Let $\mathcal{X}_R$-$\phi_{(n,d)}$ and ${}_R\mathcal{X}$-$\psi_{(n,d)}$ denote respectively the class of (n,d)-$\mathcal{X}_R$-phantom morphisms and the class of (n,d)-${}_R\mathcal{X}$-cophantom morphisms, it is clear that $\mathcal{X}_R$-$\phi_{(n,d)}$ and ${}_R\mathcal{X}$-$\psi_{(n,d)}$ are ideals of R-Mod. Let $\mathcal{I}$ be an ideal of R-Mod, a morphism $\varphi : M \to N$ in $\mathcal{I}$ is an $\mathcal{I}$-precover of N if for any morphism $\psi : C \to N$ in $\mathcal{I}$ there exists $\theta : C \to M$ such that $\varphi\theta = \psi$. An $\mathcal{I}$-precover is called $\mathcal{I}$-cover if every endomorphism h of M such that $\varphi h = \varphi$ is an automorphism. An $\mathcal{I}$-preenvelope and $\mathcal{I}$-envelope are defined dually.

The following lemma establishes the connection between an $\mathcal{I}$-cover of a left R-module and the usual $\mathcal{I}$-cover of a morphism in R-Mor.

Lemma 4.5. [10, Lemma 2.7] *Let $\mathcal{I}$ be an ideal of R-Mod and $\varphi : M \to N$ be a morphism of left R-modules. The following conditions are equivalent:*

(1) *$\varphi : M \to N$ is an $\mathcal{I}$-precover (resp., $\mathcal{I}$-cover) of N in R-Mod;*
(2) *$(\varphi, Id_N) : \varphi \to Id_N$ is an $\mathcal{I}$-precover (resp., $\mathcal{I}$-cover) of Id_N in R-Mor;*
(3) *Id_N has an $\mathcal{I}$-precover (resp., $\mathcal{I}$-cover) $(\varphi, f) : \psi \to Id_N$ in R-Mor.*

Lemma 4.6. [10, Lemma 2.6] *Let $\mathcal{I}$ be an ideal of R-Mod and $\varphi : M \to N$ be a left R-module morphism. The following conditions are equivalent:*

(1) *$\varphi : M \to N$ is an $\mathcal{I}$-prenvelope of M in R-Mod;*
(2) *$(Id_M, \varphi) : Id_M \to \varphi$ is an $\mathcal{I}$-preenvelope of Id_M in R-Mor;*
(3) *Id_M has an $\mathcal{I}$-preenvelope $(f, \varphi) : Id_M \to \psi$ in R-Mor.*

Theorem 4.7. *Let $n \in \mathbb{N}^* \cup \{\infty\}$, $d \in \mathbb{N}$.*

(1) *Every left R-module has an epic (n,d)-$\mathcal{X}_R$-phantom cover.*
(2) *If $n > d+1$, and in particular if R is n-coherent, then every left R-module has an (n,d)-$\mathcal{X}_R$-phantom preenvelope.*
(3) *If R is n-coherent, then every left R-module has a monic (n,d)-${}_R\mathcal{X}$-cophantom preenvelope.*
(4) *If R is n-coherent, then every left R-module has an (n,d)-${}_R\mathcal{X}$-cophantom cover.*

Proof. (1) Let N be a left R-module, by Theorem 4.4 Id_N has an epic (n,d)-$\mathcal{X}_R$-phantom cover in R-Mor, so by Lemma 4.5 N has an epic (n,d)-$\mathcal{X}_R$-phantom cover.
(2) Let M be left R-module, by Theorem 4.4 Id_M has an (n,d)-$\mathcal{X}_R$-phantom preenvelope. Hence by Lemma 4.6 M has an (n,d)-$\mathcal{X}_R$-phantom preenvelope in R-Mod.
(3) If R is n-coherent and M a left R-module, then by Theorem 4.4 Id_M has an (n,d)-${}_R\mathcal{X}$-cophantom preenvelope, therefore by Lemma 4.6 M has an (n,d)-${}_R\mathcal{X}$-cophantom preenvelope.
(4) Let N be a left R-module, by Theorem 4.4 Id_N has (n,d)-${}_R\mathcal{X}$-cophantom cover, then by Lemma 4.5 N has an (n,d)-${}_R\mathcal{X}$-cophantom cover. □

Theorem 4.8. *Let R be a left coherent ring and $d \geq 1$. The following assertions are equivalent:*

(1) *R is a right weak (n, d)-ring;*
(2) *Every left R-module has an epic $(n, d-1)$-phantom envelope;*
(3) *Every left R-module has a monic $(n, d-1)$-cophantom cover;*
(4) *Every pure injective left R-module has a monic $(n, d-1)$-cophantom cover.*

Proof. (1) $\Leftrightarrow$ (2) If R is a right weak (n, d)-ring and M is a left R-module, by Theorem 4.7 M has an $(n, d-1)$-phantom preenvelope $f : M \to N$. Then we get an epimorphism $\gamma : M \to \mathrm{Im}(f)$ and the inclusion $\beta : \mathrm{Im}(f) \to N$ such that $f = \beta\gamma$. For any n-presented right R-module A, the exact sequence $0 \to \mathrm{Im}(f) \xrightarrow{\beta} N \xrightarrow{\pi} L \to 0$ induces the exact sequence

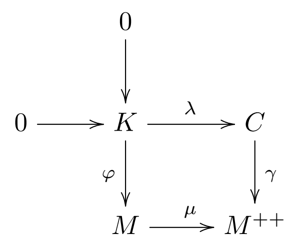
$$0 = \mathrm{Tor}_{d+1}(A, L) \longrightarrow \mathrm{Tor}_d(A, \mathrm{Im}(f)) \xrightarrow{\mathrm{Tor}_d(A,\beta)} \mathrm{Tor}_d(A, N).$$

So $\mathrm{Tor}_d(A, \beta)$ is a monomorphism. Note that

$$\mathrm{Tor}_d(A, \beta)\,\mathrm{Tor}_d(A, \gamma) = \mathrm{Tor}_d(A, f) = 0.$$

Thus $\mathrm{Tor}_d(A, \gamma) = 0$, i.e., γ is an $(n, d-1)$-phantom morphism. It is easy to verify that γ is an epic $(n, d-1)$-phantom preenvelope of M and so γ is an epic $(n, d-1)$-phantom envelope of M. Conversely, suppose (2) is satisfied, let A be an n-presented right R-module, there exists an exact sequence $0 \to K \xrightarrow{\alpha} P \to A \to 0$ with P projective. By (2) K has an epic $(n, d-1)$-phantom envelope $\varphi : K \to N$, since P is projective, α is an $(n, d-1)$-phantom morphism of right R-modules. It is now easy to see that φ is a monomorphism and so is an isomorphism. Hence K is $(n, d-1)$-flat. Since R is left coherent, any finitely presented left R-module F is n-presented and so $\mathrm{Tor}_d(K, F) = 0$. Therefore $fd(K) \leq d-1$ and so $fd(A) \leq d$, which means that R is a right weak (n, d)-ring.
(1) $\Leftrightarrow$ (3) Suppose R is a right weak (n, d)-ring, let M be a left R-module, by Theorem 4.7 M has an $(n, d-1)$-cophantom cover $f : C \to M$. Then we get an epimorphism $\lambda : C \to \mathrm{Im(f)}$ and a monomorphism $\varphi : \mathrm{Im(f)} \to M$ such that $f = \varphi\lambda$, hence $f^+ = \lambda^+\varphi^+$. Since R is coherent, by Proposition 2.7 f^+ is an $(n, d-1)$-phantom morphism. Similar to the proof (1) $\Rightarrow$ (2), we obtain that φ^+ is an $(n, d-1)$-phantom morphism and so by Proposition 2.7 φ is an $(n, d-1)$-cophantom morphism. It is easy to see that ϕ is a monic $(n, d-1)$-cophantom precover and so ϕ is a monic $(n, d-1)$-cophantom cover of M. Conversely, let A be an n-presented right R-module, there exists a short exact sequence $0 \to K \xrightarrow{g} F \to A \to 0$ with F flat, it induces an exact sequence $0 \to A^+ \to F^+ \xrightarrow{g^+} K^+ \to 0$ with F^+ injective. By (2) K^+ has monic $(n, d-1)$-cophantom cover $\rho : C \to K^+$. Since

F^+ is injective, g^+ is an $(n, d-1)$-cophantom morphism, thus ρ is an epimorphism and so is an isomorphism. Therefore K^+ is an $(n, d-1)$-injective module. Since R is left coherent, any finitely presented left R-module B is n-presented. Thus $\mathrm{Ext^d(B, K^+)} = 0$ for any finitely presented left R-module B which means that FP-$id(K^+) \leq d-1$ and so FP-$id(A^+) \leq d$. Therefore by [15, Lemma 3.1] $fd(A) \leq d$ and R is a right weak (n, d)-ring.
(3) $\Leftrightarrow$ (4) It is clear that (3) implies (4). It suffices to show that (4) implies (3). Let M be a left R-module, M^{++} is pure injective, then M^{++} has a monic $(n, d-1)$-cophantom cover $\gamma : C \to M^{++}$. Let $\mu : M \to M^{++}$ be the canonical pure monomorphism. So we get the following pullback diagram



with λ and φ monomorphisms. Since γ is an $(n, d-1)$-cophantom morphism, $\mu\varphi = \gamma\lambda$ is an $(n, d-1)$-cophantom morphism. Let A be an n-presented right R-module, A is finitely presented, since μ is a pure monomorphism and $\mathrm{Ext}^d(A, \mu)\,\mathrm{Ext}^d(A, \varphi) = \mathrm{Ext}^d(A, \gamma)\,\mathrm{Ext}^d(A, \lambda) = 0$, by Lemma 4.1, $\mathrm{Ext}^d(A, \mu)$ is a monomorphism and so $\mathrm{Ext}^d(A, \varphi) = 0$, i.e., φ is an $(n, d-1)$-cophantom morphism. Let $h : L \to M$ be an $(n, d-1)$-cophantom morphism, there exists $\psi : L \to C$ such that $\gamma\psi = \mu h$. By the universal property of a pullback diagram, there exists a morphism $\eta : L \to K$ such that $h = \varphi\eta$ and so φ is a monic $(n, d-1)$-cophantom precover of M. Therefore φ is an $(n, d-1)$-cophantom cover of M. □

5. (n,d)-$\mathcal{X}$-phantom and (n,d)-$\mathcal{X}$-cophantom morphisms under change of rings

Finally, we study (n, d)-$\mathcal{X}$-phantom and (n, d)-$\mathcal{X}$-cophantom morphisms under change of rings. Let $R \to S$ be a ring homomorphism. Then S is an R-R-bimodule in a canonical way. Moreover, any left (resp., right) S-module can be regarded as a left (resp., right) R-module, and any left (resp., right) S-module morphism can be regarded as a left (resp., right) R-module morphism.

Let ${}_RM$ be a left R-module and ${}_SN$ be a left S-module. There exists a natural S-module morphism $\mu_N : S \otimes_R N \to_S N$ defined by $\mu_N(t \otimes x) = tx$ for any $x \in N$ and $t \in S$ and a natural R-module morphism $v_M :_R M \to S \otimes_R M$ defined by

$v_M(y) = 1\otimes y$ for any $y \in M$. It is easy to check that the composition of R-module morphisms ${}_RN \overset{v_N}{\to} S \otimes_R N \overset{\mu_N}{\to} {}_SN$ is the identity and the composition of S-module morphisms $S \otimes_R M \overset{1\otimes_R v_M}{\to} S \otimes_R (S \otimes_R M) \overset{\mu_{\Theta_R M}}{\to} S \otimes_R M$ is also the identity. On the other hand, there are a natural S-module morphism η_N : ${}_SN \to \mathrm{Hom}_R(S,N)$ defined by $\eta_N(y)(t) = ty$ for any $y \in N$ and $t \in S$ and a natural R-module morphism $\varepsilon_M : \mathrm{Hom}_R(S,M) \to_R M$ defined by $\varepsilon_M(f) = f(1)$ for any $f \in \mathrm{Hom}_R(S,M)$. It is not hard to verify that the composition of R-module morphisms ${}_SN \overset{\eta_N}{\to} \mathrm{Hom}_R(S,N) \overset{\varepsilon_N}{\to} {}_RN$ is the identity and the composition of S-module morphisms is also the identity. For a class ${}_S\mathcal{X}$ (resp., $\mathcal{X}_S$) of left (resp., right) S-modules we denote ${}^R_S\mathcal{X}$ (resp., $\mathcal{X}^R_S$) the same class considered as a class of left (resp., right) R-modules.

Lemma 5.1. *Let $R \to S$ be a ring homomorphism.*

(1) *Let $\mathcal{X}_S$ be a class of right S-modules such that $X \otimes_R S \in \mathcal{X}_S$ for any element X of $\mathcal{X}_S$. If ${}_RS$ is flat and $\varphi : {}_SM \to {}_SN$ is an (n,d)-$\mathcal{X}_S$-phantom morphism in S-Mod, then $\varphi : {}_RM \to {}_RN$ is an (n,d)-$\mathcal{X}^R_S$-phantom morphism in R-Mod.*

(2) *Let $\mathcal{X}_R$ be a class of right R-modules and $\mathcal{Y}_S$ be a class of right S-modules such that $\mathcal{Y}^R_S \subseteq \mathcal{X}_R$. If S_R is finitely generated and projective and ψ : ${}_RM \to {}_RN$ is an (n,d)-$\mathcal{X}_R$-phantom morphism in R-Mod, then $1 \otimes_R \psi$: $S \otimes_R M \to S \otimes_R N$ is an (n,d)-$\mathcal{Y}_S$-phantom morphism in S-Mod.*

Proof. (1) For any $A \in (\mathcal{X}^R_S)_n$, by [17, Corollary 11.63], we get the following commutative diagram

$$\begin{array}{ccc}
\mathrm{Tor}^R_{d+1}(A,M) & \xrightarrow{\cong} & \mathrm{Tor}^S_{d+1}(A \otimes_R S, M) \\
\Big\downarrow{\scriptstyle \mathrm{Tor}^R_{d+1}(A,\varphi)} & & \Big\downarrow{\scriptstyle \mathrm{Tor}^S_{d+1}(A\otimes_R S,\varphi)} \\
\mathrm{Tor}^R_{d+1}(A,N) & \xrightarrow{\cong} & \mathrm{Tor}^S_{d+1}(A \otimes_R S, N)
\end{array}$$

Since ${}_RS$ is flat, $A\otimes_R S \in (\mathcal{X}_S)_n$. Then $\mathrm{Tor}^S_{d+1}(A\otimes_R S,\varphi) = 0$ and so $\mathrm{Tor}^R_{d+1}(A,\varphi) = 0$.

(2) For any $A \in (\mathcal{Y}_S)_n$, by [17, Corollary 11.64], we get the following commutative diagram

$$\begin{array}{ccc}
\mathrm{Tor}^R_{d+1}(A,M) & \xrightarrow{\cong} & \mathrm{Tor}^S_{d+1}(A, S \otimes_R M) \\
\Big\downarrow{\scriptstyle \mathrm{Tor}^R_{d+1}(A,\varphi)} & & \Big\downarrow{\scriptstyle \mathrm{Tor}^S_{d+1}(A,1\otimes_R\varphi)} \\
\mathrm{Tor}^R_{d+1}(A,N) & \xrightarrow{\cong} & \mathrm{Tor}^S_{d+1}(A, S \otimes_R N)
\end{array}$$

Since S_R is finitely generated and projective, $A \in (\mathcal{Y}^R_S)_n$ and so $A \in (\mathcal{X}_R)_n$. Hence $\mathrm{Tor}^R_{d+1}(A,\varphi) = 0$, thus $\mathrm{Tor}^S_{d+1}(A, 1 \otimes_R \varphi) = 0$. □

Theorem 5.2. *Let $R \to S$ be a ring homomorphism such that ${}_RS$ is flat and S_R is finitely generated and projective.*

(1) *Let $\mathcal{X}_S$ be a class of right S-modules such that $X \otimes_R S \in \mathcal{X}_S$ for any $X \in \mathcal{X}_S$. If $\varphi : {}_SM \to {}_SN$ is an (n,d)-$\mathcal{X}_S$-phantom precover in S-Mod then $\varphi : {}_RM \to {}_RN$ is an (n,d)-$\mathcal{X}_S^R$-phantom precover in R-Mod.*

(2) *If $\mathcal{X}_S$ be a class of right S-modules and $\psi : {}_RM \to {}_RN$ is an (n,d)-$\mathcal{X}_S^R$-phantom preenvelope in R-Mod then $1 \otimes_R \psi : S \otimes_R M \to S \otimes_R N$ is an (n,d)-$\mathcal{X}_S$-phantom preenvelope in S-Mod.*

Proof. (1) By Lemma 5.1 $\varphi : {}_RM \to {}_RN$ is an (n,d)-$\mathcal{X}_S^R$-phantom morphism in R-Mod. Let $f : {}_RL \to {}_RN$ be an (n,d)-$\mathcal{X}_S^R$-phantom morphism, then $1 \otimes_R f : S \otimes_R L \to {}_RS \otimes_R N$ is an (n,d)-$\mathcal{X}_S$-phantom morphism, and so $\mu_N(1 \otimes_R f) : S \otimes_R L \overset{1\otimes_R f}{\to} S \otimes_R N \overset{\mu_N}{\to}{}_S N$ is an (n,d)-$\mathcal{X}_S$-phantom morphism. By hypothesis there exists an S-morphism $\lambda : S \otimes_R L \to {}_SM$ such that $\mu_N(1 \otimes_R f) = \varphi\lambda$, i.e., such that the following diagram is commutative

$$\begin{array}{ccc} {}_RL & \xrightarrow{f} & {}_RN \\ \downarrow{\scriptstyle v_L} & & \downarrow{\scriptstyle v_N} \\ S \otimes_R L & \xrightarrow{1\otimes_R f} & S \otimes_R N \\ \downarrow{\scriptstyle \lambda} & & \downarrow{\scriptstyle \mu_N} \\ {}_SM & \xrightarrow{\varphi} & {}_SN \end{array}$$

So we have

$$\varphi(\lambda v_L) = \mu_N(1 \otimes_R f)v_L = \mu_N v_N f = f.$$

It follows that $\varphi :_R M \to_R N$ is an (n,d)-$\mathcal{X}_S^R$-phantom precover in R-Mod.

(2) By Lemma 5.1, $1 \otimes_R \psi : S \otimes_R M \to S \otimes_R M$ is an (n,d)-$\mathcal{X}_S$-phantom morphism. Let $g : S \otimes_R M \to {}_SL$ be an (n,d)-$\mathcal{X}_S$-phantom morphism in S-Mod, then $g : S \otimes_R M \to {}_RL$ is an (n,d)-$\mathcal{X}_S^R$-phantom morphism in R-Mod by Lemma 5.1. Thus $gv_M :_R M \overset{v_M}{\to} S \otimes_R M \overset{g}{\to}_R L$ is an (n,d)-$\mathcal{X}_S^R$-phantom morphism in R-Mod. Hence there exists $\alpha : {}_RN \to {}_RL$ such that the following diagram is commutative

$$\begin{array}{ccc} S \otimes_R (S \otimes_R M) & \xrightarrow{1\otimes_R g} & S \otimes_R L \\ \downarrow{\scriptstyle \mu_{S\otimes_R M}} & & \downarrow{\scriptstyle \mu_L} \\ S \otimes_R M & \xrightarrow{g} & {}_RL \\ \uparrow{\scriptstyle v_M} & & \uparrow{\scriptstyle \alpha} \\ {}_RM & \xrightarrow{\psi} & {}_RN \end{array}$$

So we have

$$(\mu_L(1 \otimes_R \alpha))(1 \otimes_R \psi) = \mu_L(1 \otimes_R \alpha\psi) = \mu_L(1 \otimes_R g v_M)$$
$$= \mu_L(1 \otimes_R g)(1 \otimes_R v_M) = g\mu_{S\otimes_R M}(1 \otimes_R v_M) = g.$$

Thus $1 \otimes_R \psi : S \otimes_R M \to S \otimes_R N$ is an (n,d)-$\mathcal{X}_S$-phantom preenvelope. □

Corollary 5.3. *Let $R \to S$ be a surjective ring homomorphism and $\mathcal{X}_S$ be a class of right S-modules. Suppose that ${}_RS$ is flat and S_R is projective. A morphism $\varphi : {}_SM \to {}_SN$ is an (n,d)-$\mathcal{X}_S$-phantom precover(resp., cover) in S-Mod if and only if $\varphi : {}_RM \to {}_RN$ is an (n,d)-$\mathcal{X}_S^R$-phantom precover (resp., cover) in R-Mod.*

Proof. Note that by [17, Corollary 10.72] $S \otimes_R B \cong_S B$ for any left S-module ${}_SB$. By a proof similar to that of Theorem 5.2 (1), $\varphi : {}_SM \to {}_SN$ is an (n,d)-$\mathcal{X}_S$-phantom precover in S-Mod if and only if $\varphi : {}_RM \to {}_RN$ is an (n,d)-$\mathcal{X}_S^R$-phantom precover in R-Mod. Suppose that $\varphi : M_S \to N_S$ is an (n,d)-$\mathcal{X}_S$-phantom cover in Mod-S. Let $\theta : {}_RM \to {}_RM$ be an R-morphism such that $\varphi\theta = \varphi$. Then $\varphi\left(\mu_M\left(\theta \otimes_R 1\right)\mu_M^{-1}\right) = \varphi\mu_M\mu_M^{-1}\theta = \varphi\theta = \varphi$. Hence $\theta = \mu_M\left(\theta \otimes_R 1\right)\mu_M^{-1}$ is an isomorphism. It follows that $\varphi : {}_RM \to {}_RN$ is an (n,d)-$\mathcal{X}_S^R$-phantom cover in R-Mod. The converse is obvious. □

Lemma 5.4. *Let $R \to S$ be a ring homomorphism.*

(1) *Let ${}_S\mathcal{X}$ be a class of left S-modules such that $S \otimes_R X \in_S \mathcal{X}$ for any $X \in {}_S\mathcal{X}$. If S_R is flat and $\varphi : {}_SM \to {}_SN$ is an (n,d)-${}_S\mathcal{X}$-cophantom morphism in S-Mod, then $\varphi : {}_RM \to {}_RN$ is an (n,d)-${}^R_S\mathcal{X}$-cophantom morphism in R-Mod.*

(2) *Let ${}_R\mathcal{X}$ be a class of left R-modules and ${}_S\mathcal{Y}$ be a class of left S-modules such that ${}^R_S\mathcal{Y} \subseteq {}_R\mathcal{X}$. If ${}_RS$ is finitely generated and projective and $\psi\colon {}_RM \to {}_RN$ is an (n,d)-${}_R\mathcal{X}$-cophantom morphism in R-Mod, then the morphism $\psi_*\colon \mathrm{Hom}_R(S,M) \to \mathrm{Hom}_R(S,N)$ is an (n,d)-${}_S\mathcal{Y}$-cophantom morphism in S-Mod.*

Proof. (1) For any ${}_RA \in ({}^R_S\mathcal{X})_n$, by [17, Corollary 11.65], we get the following commutative diagram

$$\begin{array}{ccc}
\mathrm{Ext}_S^{d+1}(S \otimes_R A, M) & \xrightarrow{\cong} & \mathrm{Ext}_R^{d+1}(A, M) \\
\Big\downarrow{\scriptstyle \mathrm{Ext}_S^{d+1}(S\otimes_R A,\varphi)} & & \Big\downarrow{\scriptstyle \mathrm{Ext}_R^{d+1}(A,\varphi)} \\
\mathrm{Ext}_S^{d+1}(S \otimes_R A, N) & \xrightarrow{\cong} & \mathrm{Ext}_R^{d+1}(A, N)
\end{array}$$

Since S_R is flat, $S \otimes_R A \in ({}_S\mathcal{X})_n$. Then $\mathrm{Ext}^S_{d+1}(S \otimes_R A, \varphi) = 0$ and so $\mathrm{Ext}^R_{d+1}(A, \varphi) = 0$.

(2) For any $A \in ({}_S\mathcal{Y})_n$, by [17, Corollary 11.66], we get the following commutative diagram

$$\begin{array}{ccc} \mathrm{Ext}_S^{d+1}(A, \mathrm{Hom}_R(S,M)) & \xrightarrow{\cong} & \mathrm{Ext}_R^{d+1}(A,M) \\ {\scriptstyle \mathrm{Ext}_S^{d+1}(A,\psi_*)}\downarrow & & \downarrow{\scriptstyle \mathrm{Ext}_R^{d+1}(A,\psi)} \\ \mathrm{Ext}_S^{d+1}(A, \mathrm{Hom}_R(S,N)) & \xrightarrow{\cong} & \mathrm{Ext}_R^{d+1}(A,N) \end{array}$$

${}_RS$ is finitely generated and projective then $A \in ({}_S^R\mathcal{Y})_n$ and so $A \in ({}_R\mathcal{X})_n$, thus $\mathrm{Ext}_R^{d+1}(A,\psi) = 0$ and so $\mathrm{Ext}_S^{d+1}(A,\psi_*) = 0$. □

Theorem 5.5. *Let $R \to S$ be a ring homomorphism such that S_R is flat and ${}_RS$ is finitely generated and projective.*

(1) *Let ${}_S\mathcal{X}$ be a class of S-modules such that $S \otimes_R X \in {}_S\mathcal{X}$ for any $X \in {}_S\mathcal{X}$. If $\varphi : {}_SM \to {}_SN$ is an (n,d)-${}_S\mathcal{X}$-cophantom preenvelope in S-Mod then $\varphi : {}_RM \to {}_RN$ is an (n,d)-${}_S^R\mathcal{X}$-cophantom preenvelope in R-Mod.*
(2) *If ${}_S\mathcal{X}$ be a class of left S-modules and $\psi : {}_RM \to {}_RN$ is an (n,d)-${}_S^R\mathcal{X}$-cophantom precover in R-Mod then $\psi_* : \mathrm{Hom}_R(S,M) \to \mathrm{Hom}_R(S,N)$ is an (n,d)-${}_S\mathcal{X}$-cophantom precover in S-Mod.*

Proof. (1) Note that $\varphi : {}_RM \to {}_RN$ is an (n,d)-${}_S^R\mathcal{X}$-cophantom morphism in R-Mod by Lemma 5.4. Let $f : {}_RM \to {}_RL$ be an (n,d)-${}_S^R\mathcal{X}$-cophantom morphism in R-Mod. Then $f_* : \mathrm{Hom}_R(S,M) \to \mathrm{Hom}_R(S,L)$ is an (n,d)-${}_S\mathcal{X}$-cophantom morphism in S-Mod by Lemma 5.4. So $f_*\eta_M : {}_SM \overset{\eta_M}{\to} \mathrm{Hom}_R(S,M) \overset{f_*}{\to} \mathrm{Hom}_R(S,L)$ is also an (n,d)-${}_S\mathcal{X}$-cophantom morphism in S-Mod. Thus there exists a left S-homomorphism $g : {}_SN \to \mathrm{Hom}_R(S,L)$ such that $g\varphi = f_*\eta_M$. So we have

$$(\varepsilon_L g)\,\varphi = \varepsilon_L f_* \eta_M = \varepsilon_L \eta_L f = f.$$

Hence $\varphi : {}_RM \to {}_RN$ is an (n,d)-${}_S^R\mathcal{X}$-cophantom preenvelope in R-Mod.
(2) By Lemma 5.4, $\psi_* : \mathrm{Hom}_R(S,M) \to \mathrm{Hom}_R(S,N)$ is an (n,d)-${}_S\mathcal{X}$-cophantom morphism in S-Mod. Let $f : {}_SL \to \mathrm{Hom}_R(S,N)$ be any (n,d)-${}_S\mathcal{X}$-cophantom morphism in S-Mod. Then $f : {}_RL \to \mathrm{Hom}_R(S,N)$ is an (n,d)-${}_S^R\mathcal{X}$-cophantom morphism in R-Mod by Lemma 5.4. Thus $\varepsilon_N f : {}_RL \overset{f}{\to} \mathrm{Hom}_R(S,N) \overset{\varepsilon_N}{\to} {}_RN$ is also an (n,d)-${}_S^R\mathcal{X}$-cophantom morphism in R-Mod. Hence there is $\alpha : {}_RL \to {}_RM$ such that $\psi\alpha = \varepsilon_N f$. Thus

$$\psi_*\left(\alpha_*\eta_L\right) = (\psi\alpha)_*\eta_L = \left(\varepsilon_N f\right)_*\eta_L = \left(\varepsilon_N\right)_*\left(f_*\eta_L\right) = \left(\varepsilon_N\right)_*\eta_{\mathrm{Hom}_R(S,N)} f = f.$$

So $\psi_* : \mathrm{Hom}_R(S,M) \to \mathrm{Hom}_R(S,N)$ is an (n,d)-${}_S\mathcal{X}$-cophantom precover in S-Mod. □

Corollary 5.6. *Let $R \to S$ be a surjective ring homomorphism with ${}_RS$ flat and S_R projective. A left S-homomorphism $\varphi : {}_SM \to {}_SN$ is an (n,d)-${}_S\mathcal{X}$-cophantom preenvelope (resp., (n,d)-${}_S\mathcal{X}$-cophantom envelope) in S-Mod if and only if $\varphi : {}_RM \to {}_RN$ is an (n,d)-${}^R_S\mathcal{X}$-cophantom preenvelope (resp., (n,d)-${}^R_S\mathcal{X}$-cophantom envelope) in R-Mod.*

Proof. Note that $\mathrm{Hom}_R(S,B) \cong {}_SB$ for any left S-module ${}_SB$. By a proof similar to that of Theorem 5.5, $\varphi\colon S_SMS$ is an (n,d)-${}_R\mathcal{X}$-cophantom preenvelope in S-Mod if and only if $\varphi : {}_RM \to {}_RN$ is an (n,d)-$\mathcal{X}_R$-cophantom preenvelope in R-Mod.

Suppose that $\varphi\colon {}_SMS_SN$ is an (n,d)-${}_R\mathcal{X}$-cophantom envelope in S-Mod. Let $\theta\colon {}_RN \to {}_RN$ be a left R-homomorphism such that $\theta\varphi = \varphi$. Then $\theta_*\varphi_* = \varphi_*$. Thus $\left(\eta_N^{-1}\theta_*\eta_N\right)\varphi = \eta_N^{-1}\theta_*\varphi_*\eta_M = \eta_N^{-1}\varphi_*\eta_M = \eta_N^{-1}\eta_N\varphi = \varphi$. Hence $\theta = \eta_N^{-1}\theta_*\eta_N$ is an isomorphism. So $\varphi : {}_RM \to {}_RN$ is an (n,d)-$\mathcal{X}_R$-cophantom envelope in R-Mod. The converse is obvious. □

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