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NIL$_*$-ARTINIAN RINGS

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Abstract. In this paper, we say a ring R is Nil$_*$-Artinian if any descending chain of nil ideals stabilizes. We first study Nil$_*$-Artinian properties in terms of quotients, localizations, polynomial extensions and idealizations, and then study the transfer of Nil$_*$-Artinian rings to amalgamated algebras. Besides, some examples are given to distinguish Nil$_*$-Artinian rings, Nil$_*$-Noetherian rings and Nil$_*$-coherent rings.

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1. Introduction

Throughout this paper, all rings are commutative with identity and all modules are unitary. Let R be a ring. We denote by Spec(R) the set of all prime ideals of R and by Nil(R) the nil-radical of R, that is, the set of all nilpotent elements in R. An ideal I of R is said to be a nil ideal provided that any element in I is nilpotent.

It is well-known that coherent rings with finite weak global dimensions and rings with global dimensions at most 2 are all reduced rings, i.e., rings with zero nil-radical (see [6, Corollary 4.2.4, Corollary 4.2.5]). So the nil radical is very crucial to study rings with infinite homological dimensions (also see [5] for example). Some algebraic researchers began to study rings by only consider their nil ideals. In 2014, Xiang [10] introduced the notions of Nil$_*$-coherent rings in terms of nil ideals. A ring R is said to be Nil$_*$-coherent provided that any finitely generated nil ideal is finitely presented. Later in 2017, Ismaili et al. [8] studied the Nil$_*$-coherent properties via idealization and amalgamated algebras under several assumptions. Recently, Zhang [11] defined Nil$_*$-Noetherian rings to be rings in which every nil ideal is finitely generated. He showed that the Hilbert Basis Theorem holds for Nil$_*$-Noetherian rings and also studied Nil$_*$-Noetherian properties via idealization and bi-amalgamated algebras under several assumptions.

The main motivation of this paper is to introduce and study Nil$_*$-Artinian rings. We say a ring R is Nil$_*$-Artinian if any descending chain of nil ideals stabilizes.

We study the quotient rings and localization of Nil_*-Artinian rings, and then show when a polynomial ring is Nil_*-Artinian. We also show that an idealization $R(+)M$ is a Nil_*-Artinian ring if and only if R is a Nil_*-Artinian ring and the R-module M is Artinian (see Theorem 2.9). Finally, we study the transfer of Nil_*-Artinian rings to amalgamated algebras in Theorem 3.2. In particular, we show that $A \bowtie J$ is Nil_*-Artinian if and only if A is Nil_*-Artinian (see Corollary 3.3).

2. Basic properties of Nil_*-Artinian rings

Recall that an ideal of R is said to be nil provided every element in I is nilpotent. We begin with the concept of Nil_*-Artinian rings.

Definition 2.1. A ring R is said to be a Nil_*-Artinian ring provided that any descending chain of nil ideals stabilizes, i.e., let $I_1 \supseteq I_2 \supseteq \cdots \supseteq I_n \supseteq \cdots$ be a descending chain of nil ideals, then there exists an integer k such that $I_n = I_k$ for any $n \geq k$.

Trivially, reduced rings and Artinian rings are Nil_*-Artinian. Obviously, a ring R is Nil_*-Artinian if and only if the nil-radical $\mathrm{Nil}(R)$ is an Artinian R-module.

Lemma 2.2. *Let R be a Nil_*-Artinian ring. If I is a nil ideal of R, then R/I is also Nil_*-Artinian.*

Proof. Let $\{K_i \mid i \in \mathbb{Z}^+\}$ be a family of descending chain of nil ideals of R/I. Then $K_i = J_i/I$ for some R-ideal J_i containing I. Since I is a nil ideal, each J_i is also a nil ideal of R. Hence the descending chain $\{J_i \mid i \in \mathbb{Z}^+\}$ stabilizes. □

Note that the condition "I is a nil ideal of R" in Lemma 2.2 cannot be removed.

Example 2.3. [11, Example 1.3] Let $S = k[x_1, x_2, \cdots]$ be the polynomial ring over a field k with countably infinite variables. Then S is Nil_*-Artinian. Set the quotient ring $R = S/\langle x_i^2 \mid i \geq 1\rangle$. Then $\mathrm{Nil}(R) = \langle \overline{x_1}, \overline{x_2}, \cdots\rangle$, where $\overline{x_i}$ denotes the representative of x_i in R for each i. Let $J_i = \{\langle \overline{x_i}, \overline{x_{i+1}}, \cdots\rangle\}$, then $J_1 \supseteq J_2 \supseteq \cdots$ is a descending chain which does not stop.

Proposition 2.4. *A finite direct product $R = R_1 \times \cdots \times R_n$ of rings $R_1, \cdots, R_n$ is Nil_*-Artinian if and only if each R_i is Nil_*-Artinian $(i = 1, \cdots, n)$.*

Proof. It follows by $\mathrm{Nil}(R) = \mathrm{Nil}(R_1) \times \cdots \times \mathrm{Nil}(R_n)$ and we will have $\mathrm{Nil}(R)$ is an Artinian R-module if and only if each $\mathrm{Nil}(R_i)$ is an Artinian R_i-module $(i = 1, \cdots, n)$. □

Proposition 2.5. *Let R be Nil$_*$-Artinian and S a multiplicative subset of R. Then R_S is also Nil$_*$-Artinian.*

Proof. Let $(I_1)_S \supseteq (I_2)_S \supseteq \cdots \supseteq (I_n)_S \supseteq \cdots$ be a descending chain of nil ideals of R_S. Since $\mathrm{Nil}(R_S) = \mathrm{Nil}(R)_S$, we may assume each I_i is a nil ideal of R. Since R is Nil$_*$-Artinian, there exists an integer k such that $I_n = I_k$ for any $n \geq k$. Hence $(I_n)_S = (I_k)_S$ for any $n \geq k$. Consequently, R_S is also Nil$_*$-Artinian. $\square$

Next, we will focus on the Nil$_*$-Artinian properties of polynomial rings.

Lemma 2.6. [9, Exercise 1.47] *Let R be a ring. Then $J(R[x]) = \mathrm{Nil}(R[x]) = \mathrm{Nil}(R)[x]$.*

Proposition 2.7. *Let R be a ring. If $R[x]$ is a Nil$_*$-Artinian ring, then R is a Nil$_*$-Artinian ring.*

Proof. Suppose $R[x]$ is a Nil$_*$-Artinian ring. Let $I_\bullet := \{I_i \mid i \in \mathbb{Z}^+\}$ be a descending chain of nil R-ideals. Then $I_\bullet R[x] := \{I_i R[x] \mid i \in \mathbb{Z}^+\}$ is a descending chain of nil $R[x]$-ideals, and so stabilizes. Consequently, the constant terms of the ideals in $I_\bullet R[x]$, i.e., $I_\bullet$ also stabilizes. $\square$

The following example shows the converse of Proposition 2.7 does not hold.

Example 2.8. Let $R = \mathbb{Z}_4$. Then R is an Artinian, and so is Nil$_*$-Artinian. Then $\mathrm{Nil}(\mathbb{Z}_4[x]) = \mathrm{Nil}(\mathbb{Z}_4)[x] = 2\mathbb{Z}_4[x]$ by Lemma 2.6. Note that the descending chain $\langle 2x \rangle \supsetneq \langle 2x^2 \rangle \supsetneq \cdots$ of nil ideals does not stabilize. Hence $R[x]$ is not Nil$_*$-Artinian.

Some non-reduced rings are constructed by the idealization $R(+)M$ where M is an R-module (see [7]). Let $R(+)M = R \oplus M$ as an R-module, and define

(1) $(r,m)+(s,n)=(r+s,m+n)$,
(2) $(r,m)(s,n)=(rs,sm+rn)$,

where $r, s \in R$ and $m, n \in M$. Under this construction, $R(+)M$ is a commutative ring with identity $(1,0)$.

Theorem 2.9. *Let R be a ring and M an R-module. Then $R(+)M$ is a Nil$_*$-Artinian ring if and only if R is a Nil$_*$-Artinian ring and M is an Artinian R-module.*

Proof. For necessity, since $R \cong R(+)M/0(+)M$ and $0(+)M$ is a nil ideal, R is Nil$_*$-Artinian by Lemma 2.2. Since $0(+)M$ is a nil ideal, any descending chain of sub-ideals of $0(+)M$ is stabilizing, which is equivalence to that M is an Artinian R-module.

For sufficiency, consider the exact sequence of $R(+)M$-modules: $0 \to 0(+)M \xrightarrow{i} R(+)M \xrightarrow{\pi} R \to 0$. Let $O^\bullet : O_1 \supseteq O_2 \supseteq \cdots$ be a descending chain of nil $R(+)M$-ideals. Then there is a descending chain of nil R-ideals: $\pi(O^\bullet) : \pi(O_1) \supseteq \pi(O_2) \supseteq \cdots$. Thus there exists $k \in \mathbb{Z}^+$ such that $\pi(O_n) = \pi(O_k)$ for any $n \geq k$. Similarly, $O^\bullet \cap 0(+)M : O_1 \cap 0(+)M \supseteq O_2 \cap 0(+)M \supseteq ...$ is a descending chain of nil sub-ideals of $0(+)M$ which are equivalent to some submodules of M. So there exists $k' \in \mathbb{Z}^+$ such that $O_n \cap 0(+)M = O_k \cap 0(+)M$ for any $n \geq k'$ as M is an Artinian module. Let $l = \max(k, k')$ and $n \geq l$. Consider the following natural commutative diagram with exact rows:

$$\begin{tikzcd}
0 \arrow[r] & O_n \cap 0(+)M \arrow[r] \arrow[d, equal] & O_n \arrow[r] \arrow[d, hook] & \pi(O_n) \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & O_l \cap 0(+)M \arrow[r] & O_l \arrow[r] & \pi(O_l) \arrow[r] & 0.
\end{tikzcd}$$

Then we have $O_n = O_l$ for any $n \geq l$. So $R(+)M$ is a Nil_*-Artinian ring. □

Recall from [11] that a ring R is called Nil_*-Noetherian provided that any nil ideal is finitely generated. The following example shows that Nil_*-Noetherian rings need not be Nil_*-Artinian in general.

Example 2.10. Let D be an integral domain which is not a field. Then D is not an Artinian D-module. Set $R = D(+)D$. Then R is Nil_*-Noetherian by [11, Theorem 1.8], but not Nil_*-Artinian by Theorem 2.9.

The following example shows that Nil_*-Artinian rings need not be Nil_*-Noetherian.

Example 2.11. Let $D = \mathbb{Z}$ be the ring of all integers with its quotient field $\mathbb{Q}$ and p a prime number. Let $\mathbb{Q}_p = \{x \in \mathbb{Q} \mid p^n x \in \mathbb{Z} \text{ for some } n\}$. Set $M = \mathbb{Q}_p/\mathbb{Z}$. Then M is an Artinian $\mathbb{Z}$-module but not finitely generated (see [9, Example 2.8.16]). Set $R = D(+)M$. Then R is Nil_*-Artinian by Theorem 2.9, but not Nil_*-Noetherian by [11, Theorem 1.8].

Recall from [10] that a ring R is called Nil_*-coherent provided that any finitely generated ideal in $\mathrm{Nil}(R)$ is finitely presented. Similar to the classical case, Nil_*-coherent rings are not Nil_*-Artinian in general. Indeed, let D be a coherent domain but not a field, then $R = D(+)D$ is also coherent by [1, Lemma 3.2], and hence Nil_*-coherent. Since D is not an Artinian ring, R is not Nil_*-Artinian by Theorem 2.9. The following example shows that Nil_*-Artinian rings are not Nil_*-coherent in general.

Example 2.12. [11, Example 1.11] Let $S = k[x_1, x_2, \cdots]$ be the polynomial ring over a field k with countably infinite variables. Set $R = S/\langle x_1x_i \mid i \geq 1\rangle$. Then $\mathrm{Nil}(R) = \langle \overline{x_1}\rangle$ is the only non-trivial nil ideal of R. So R is Nil$_*$-Artinian. However, since $(0 :_R \overline{x_1}) = \langle \overline{x_1}, \overline{x_2}, \cdots\rangle$ is infinitely generated, $\mathrm{Nil}(R)$ is not finitely presented. Hence R is not Nil$_*$-coherent.

By Proposition 2.5, if R is a Nil$_*$-Artinian ring, then $R_\mathfrak{p}$ is also Nil$_*$-Artinian for any prime ideal $\mathfrak{p}$ of R. However, the converse does not hold in general.

Example 2.13. Let $S = \prod\limits_{i=1}^{\infty} k$ be countably infinite direct product of a field k. Then for any prime ideal $\mathfrak{p}$ of S, we have $S_\mathfrak{p} \cong k$. Set $R = S(+)S$. Then $\mathrm{Spec}(R) = \{\mathfrak{p}(+)S \mid \mathfrak{p} \in \mathrm{Spec}(S)\}$ by [2, Theorem 3.2(2)]. Since S is not an Artinian ring, R is not Nil$_*$-Artinian by Theorem 2.9. However, by [2, Theorem 4.1(2)], we have $R_\mathfrak{p} = S_\mathfrak{p}(+)S_\mathfrak{p} = k(+)k$ which is Nil$_*$-Artinian by Theorem 2.9 again.

3. Transfer of Nil$_*$-Artinian rings to amalgamated algebras

We recall the amalgamated algebras constructed in [3]. Let $f : A \to B$ be a ring homomorphism and let I an ideal of B. The amalgamated algebra of A with B along J with respect to f is the subring of $A \times B$ given by:

$$A \bowtie^f J := \{(a, f(a) + j) \mid a \in A, j \in J\}.$$

Set $\pi_1 : A \bowtie^f J \to A$ where $\pi_1((a, f(a) + j)) = a$, $\pi_2 : A \bowtie^f J \to B$ where $\pi_2((a, f(a) + j)) = f(a) + j$. Then π_1 and π_2 are ring homomorphisms.

Lemma 3.1. [8, Lemma 5.2] *Let $f : A \to B$ be a ring homomorphism and J an ideal of B. Then*

$$\mathrm{Nil}(A \bowtie^f J) = \mathrm{Nil}(A) \bowtie^f (J \cap \mathrm{Nil}(f(A) + J)) = \mathrm{Nil}(A) \bowtie^f (J \cap \mathrm{Nil}(B)).$$

Theorem 3.2. *Let $f : A \to B$ be a ring homomorphism and J an ideal of B. If A and $f(A) + J$ are Nil$_*$-Artinian rings, then $A \bowtie^f J$ is a Nil$_*$-Artinian ring. Moreover, suppose one of the following cases holds:*

(1) $\mathrm{Ker}(f)$ *is nil and f is surjective.*
(2) $\mathrm{Ker}(f)$ *is nil and J is a nil ideal of B.*

Then the converse also holds.

Proof. Suppose A and $f(A) + J$ are Nil$_*$-Artinian rings. Let $L_\bullet := L_1 \supseteq L_2 \supseteq \cdots$ be a descending chain of nil ideals of $A \bowtie^f J$. Then $\pi_1(L_\bullet)$ and $\pi_2(L_\bullet)$ are descending chains of ideals of A and $f(A) + J$, respectively. Certainly, $\pi_1(L_\bullet)$ is composed of nil ideals of A by Lemma 3.1. Let $i \geq 1$. Then $\pi_2(L_i) = \{f(a) + j \mid$

$(a, f(a)+j) \in L_i\}$ with $a \in \mathrm{Nil}(A)$ and $j \in J \cap \mathrm{Nil}(f(A)+J)$ by Lemma 3.1. So $f(a)+j$ is nilpotent in B, and thus $\pi_2(L_i)$ is also a nil ideal. Hence there exists an integer k such $\pi_1(L_n) = \pi_1(L_k)$ and $\pi_2(L_n) = \pi_2(L_k)$ for any $n \geq k$. So $L_n = L_k$ for any $n \geq k$. Consequently, $A \bowtie^f J$ is Nil_*-Artinian.

On the other hand, suppose $A \bowtie^f J$ is Nil_*-Artinian. Let $I_\bullet := I_1 \supseteq I_2 \supseteq \cdots$ be a descending chain of nil ideals of A. Set $I_i' = \{(a, f(a)) \mid a \in I_i\}$. Then $I_\bullet' := I_1' \supseteq I_2' \supseteq \cdots$ be a descending chain of nil ideals of $A \bowtie^f J$. Thus $I_\bullet'$ stabilizes. Hence $I_\bullet$ also stabilizes, and thus A is Nil_*-Artinian. Let $K_\bullet := K_1 \supseteq K_2 \supseteq \cdots$ be a descending chain of nil ideals of $f(A)+J$. We consider the following two cases.

(1) Suppose $\mathrm{Ker}(f)$ is a nil ideal of A and f is surjective. Set $K_i' = \{(a, f(a)) \mid f(a) \in K_i\}$. We claim that a is nilpotent. Indeed, suppose $(f(a))^n = 0$. Then $a^n \in \mathrm{Ker}(f)$. Since $\mathrm{Ker}(f)$ is a nil ideal of A, a is also nilpotent. Hence K_i' is a nil ideal of $A \bowtie^f J$. Since $A \bowtie^f J$ is Nil_*-Artinian, there exists an integer k such that $K_n' = K_k'$ for any $n \geq k$. Hence $K_n = K_k$ for any $n \geq k$, and thus $f(A)+J$ is also Nil_*-Artinian.

(2) Suppose $\mathrm{Ker}(f)$ is nil and J is a nil ideal of B. Set $K_i' = \{(a, f(a)+j) \mid$ there exists $j \in J$ such that $f(a)+j \in K_i\}$. We claim that a is nilpotent. Indeed, since $f(a)+j$ and j is nilpotent, $f(a)$ is also nilpotent. Since $\mathrm{Ker}(f)$ is nil, a is nilpotent. As in (1), we can show $f(A)+J$ is Nil_*-Artinian. $\square$

Recall from [4] that, by setting $f = \mathrm{Id}_A : A \to A$ to be the identity homomorphism of A, we denote by $A \bowtie J = A \bowtie^{\mathrm{Id}_A} J$ and call it the amalgamated algebra of A along J. By Theorem 3.2, we obviously have the following result.

Corollary 3.3. *Let J be an ideal of A. Then $A \bowtie J$ is a Nil_*-Artinian ring if and only if A is a Nil_*-Artinian ring.*

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