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ON COHOMOLOGY GROUPS OF CURRENT LIE ALGEBRAS

Rosendo García-Delgado

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Abstract. In this work we state a result that relates the cohomology groups of a Lie algebra $\mathfrak{g}$ and a current Lie algebra $\mathfrak{g}\otimes\mathcal{S}$, by means of a short exact sequence similar to the universal coefficients theorem for modules, where $\mathcal{S}$ is a finite dimensional, commutative and associative algebra with unit over a field $\mathbb{F}$. Using this result we determine the cohomology group of $\mathfrak{g}\otimes\mathcal{S}$ where $\mathfrak{g}$ is a semisimple Lie algebra.

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1. Introduction

Let $\mathfrak{g}$ be a Lie algebra with bracket $[\,\cdot\,,\,\cdot\,]$ and let $\mathcal{S}$ be an associative and commutative algebra over a field $\mathbb{F}$ with product $(s,t)\mapsto s\,t$, for all s,t in $\mathcal{S}$. The skew-symmetric and bilinear map $[\,\cdot\,,\,\cdot\,]_{\mathfrak{g}\otimes\mathcal{S}}$ defined on $\mathfrak{g}\otimes\mathcal{S}$, by

$$[x\otimes s, y\otimes t]_{\mathfrak{g}\otimes\mathcal{S}}=[x,y]\otimes s\,t,\quad \text{for all } x,y\in\mathfrak{g}, \text{ and } s,t\in\mathcal{S},$$

yields a Lie algebra in $\mathfrak{g}\otimes\mathcal{S}$, which is called the current Lie algebra of $\mathfrak{g}$ by $\mathcal{S}$.

Let $\rho:\mathfrak{g}\to\mathfrak{gl}(V)$ be a representation of $\mathfrak{g}$ on a vector space V, then V is said to be a $\mathfrak{g}$-module. The representation ρ can be extended to a representation R of $\mathfrak{g}\otimes\mathcal{S}$ on the vector space $V\otimes\mathcal{S}$ by means of

$$R(x\otimes s)(v\otimes t)=\rho(x)(v)\otimes s\,t,\ \text{for all } x,y\in\mathfrak{g},\ v\in V,\ s,t\in\mathcal{S}. \tag{1}$$

Let $C(\mathfrak{g};V)=C^0(\mathfrak{g};V)\oplus\ldots\oplus C^p(\mathfrak{g};V)\oplus\ldots$ be the space of cochains from $\mathfrak{g}$ into V, where $C^0(\mathfrak{g},V)=V$ and $C^p(\mathfrak{g};V)$ is the space of the alternating p-multilinear maps of $\mathfrak{g}$ with values in V. For any $\mathfrak{g}$-mdule V and $p\geq 0$, let $\mathrm{d}:C^p(\mathfrak{g};V)\to C^{p+1}(\mathfrak{g};V)$

be the differential map given by

$$
\begin{aligned}
\mathrm{d}\,\lambda(x_1,\dots,x_{p+1}) &= \sum_{j=1}^{p+1}(-1)^{j-1}\rho(x_j)(\lambda(x_1,\dots,x_{\hat{j}},\dots,x_{p+1})) \\
&+ \sum_{j<k}(-1)^{j+k}\lambda([x_j,x_k],x_1,\dots,x_{\hat{j}},\dots,x_{\hat{k}},\dots,x_{p+1}), \qquad p>0,
\end{aligned} \tag{2}
$$

where λ is in $C^p(\mathfrak{g};V)$ and $x_1,\dots,x_{p+1}$ are in $\mathfrak{g}$. For $p=0$, we let $\mathrm{d}(v)(x)=\rho(x)(v)$ where v is in V and x is in $\mathfrak{g}$.

The aim of this work is to set a result that relates the cohomology groups $\mathcal{H}(\mathfrak{g}\otimes\mathcal{S};V\otimes\mathcal{S})$ and $\mathcal{H}(\mathfrak{g};V)$, similar to the Universal coefficient theorems for modules (see [2, Chapter VI, §3, Theorem 3.3]).

To achieve our goal, in Proposition 3.2 we introduce a map $\mathcal{T}$ between the set of cochains of $\mathfrak{g}$ and cochains of $\mathfrak{g}\otimes\mathcal{S}$, that is sort like a functor except that $\mathcal{T}(\mathrm{Id})$ is not the identity map Id (see §2 and Remark 3.1). Next we prove that there exists a surjective linear map α between $\mathcal{H}(\mathfrak{g}\otimes\mathcal{S};V\otimes\mathcal{S})$ and $\mathcal{H}(\mathfrak{g};V)\otimes\mathcal{S}$ (see Proposition 4.2). In Theorem 4.3 we determine the kernel of α and we state a result that relates the cohomology groups $\mathcal{H}(\mathfrak{g}\otimes\mathcal{S};V\otimes\mathcal{S})$ and $\mathcal{H}(\mathfrak{g};V)\otimes\mathcal{S}$ by means of a short exact sequence.

It is a well known result that if V is an irreducible $\mathfrak{g}$-module and $\mathfrak{g}$ is semisimple, then $\mathcal{H}(\mathfrak{g};V)=\{0\}$ (see [4, Theorem 24.1]). In order to illustrate the results of this work, we use this and the fact that $\mathcal{T}(\mathrm{Id})\neq\mathrm{Id}$ to determine the cohomology group $\mathcal{H}(\mathfrak{g}\otimes\mathcal{S};V\otimes\mathcal{S})$, where $\mathfrak{g}$ is a semisimple Lie algebra and V is an irreducible $\mathfrak{g}$-module (see Proposition 4.5).

The results obtained in this work are focused at knowing the cohomology group $\mathcal{H}(\mathfrak{g}\otimes\mathcal{S};V\otimes\mathcal{S})$, based on the cohomology group $\mathcal{H}(\mathfrak{g};V)$. Results in the literature include those given in [6] for the first and second cohomology groups of a current Lie algebra $\mathfrak{g}\otimes\mathcal{S}$ with coefficients in a module $V\otimes\mathcal{A}$, where V is a $\mathfrak{g}$-module and $\mathcal{A}$ is an $\mathcal{S}$-module. Other results are given in [7] (Theorem 2.1) for the second cohomology group of $\mathfrak{g}\otimes\mathcal{S}$ with coefficients in the trivial module and $\mathcal{S}$ has no unit. A description of the cohomology group $\mathcal{H}(\mathfrak{g}\otimes\mathcal{S};\mathcal{V})$, where $\mathcal{V}$ is a trivial $\mathfrak{g}\otimes\mathcal{S}$ is given in [5]. It seems that one of the first results with this focus appears in [1], where it is shown that cohomology of $\mathfrak{g}\otimes\mathcal{S}$, where $\mathcal{S}$ is a local algebra, can be reduced to cohomology of $\mathfrak{g}$. On the other hand, it is unknown if there exists a criterion for recognizing whether an arbitrary Lie algebra is a current Lie algebra. A step in this direction can be found in [3], where examples in 4-dimensional current Lie algebras are given. All vector spaces considered in this work are finite dimensional over a unique field $\mathbb{F}$ of zero characteristic.

2. The map $\mathcal{L} : C(\mathfrak{g} \otimes \mathcal{S}; V) \to C(\mathfrak{g}; V)$

The proof of the following result is standard and we omit it.

Proposition 2.1. *Let V and $\mathcal{S}$ be finite dimensional vector spaces over $\mathbb{F}$. Let $\{s_1, \dots, s_m\}$ be a basis of $\mathcal{S}$. For any X in $V \otimes \mathcal{S}$, there are unique elements $v_1, \dots, v_m$ in V such that $X = v_1 \otimes s_1 + \ldots + v_m \otimes s_m$.*

Let $\mathfrak{g} \otimes \mathcal{S}$ be the current Lie algebra of $\mathfrak{g}$ by $\mathcal{S}$, where $\mathcal{S}$ is an m-dimensional commutative and associative algebra with unit 1 over $\mathbb{F}$. We use the same symbol for the bracket on $\mathfrak{g} \otimes \mathcal{S}$ and the bracket on $\mathfrak{g}$, i.e., $[x \otimes s, y \otimes t] = [x, y] \otimes st$ for all x, y in $\mathfrak{g}$ and s, t in $\mathcal{S}$.

We fix a basis $\{s_1, \dots, s_m\}$ of $\mathcal{S}$, where $s_1 = 1$. Let $X_1, \dots, X_p$ be in $\mathfrak{g} \otimes \mathcal{S}$ and Λ in $C^p(\mathfrak{g} \otimes \mathcal{S}; V \otimes \mathcal{S})$, where $p > 0$. Since $\Lambda(X_1, \dots, X_p)$ lies in $V \otimes \mathcal{S}$, by Proposition 2.1, we write $\Lambda(X_1, \dots, X_p)$ as follows:

$$\Lambda(X_1, \dots, X_p) = \Lambda_1(X_1, \dots, X_p) \otimes s_1 + \ldots + \Lambda_m(X_1, \dots, X_p) \otimes s_m, \tag{3}$$

where $\Lambda_j(X_1, \cdots, X_p)$ belongs to V for all j. As Λ is in $C^p(\mathfrak{g} \otimes \mathcal{S}; V \otimes \mathcal{S})$, the map $(X_1, \dots, X_p) \mapsto \Lambda_j(X_1, \dots, X_p)$ belongs to $C^p(\mathfrak{g} \otimes \mathcal{S}; V)$. We denote this map by Λ_j for all $1 \leq j \leq m$.

Let $\{\omega_1, \dots, \omega_m\} \subset \mathcal{S}^*$ be the dual basis of $\{s_1, \dots, s_m\}$. For each j, the bilinear map $(v, s) \mapsto \omega_j(s)v$ yields the linear map $\widehat{\omega}_j : V \otimes \mathcal{S} \to V$, $v \otimes s \mapsto \omega_j(s)\, v$. By Proposition 2.1, we can write any X in $V \otimes \mathcal{S}$, as $X = v_1 \otimes s_1 + \ldots + v_m \otimes s_m$, then $v_j = \widehat{\omega}_j(X)$. Similarly, if Λ is in $C^p(\mathfrak{g} \otimes \mathcal{S}; V \otimes \mathcal{S})$, where $p > 0$, by (3) it follows $\Lambda_j = \widehat{\omega}_j \circ \Lambda$ for all $1 \leq j \leq m$.

For each j, define the map $\chi_j : C(\mathfrak{g} \otimes \mathcal{S}; V \otimes \mathcal{S}) \to C(\mathfrak{g} \otimes \mathcal{S}; V)$ by

$$\begin{aligned} &\chi_j(\Lambda) = \widehat{\omega}_j \circ \Lambda, \quad \text{for } \Lambda \in C^p(\mathfrak{g} \otimes \mathcal{S}; V \otimes \mathcal{S}),\ p > 0, \text{ and} \\ &\chi_j(v \otimes s) = \widehat{\omega}_j(v \otimes s), \quad \text{for } v \in V \text{ and } s \in \mathcal{S}. \end{aligned} \tag{4}$$

Then for any Λ in $C^p(\mathfrak{g} \otimes \mathcal{S}; V \otimes \mathcal{S})$, where $p \geq 0$, we have

$$\begin{aligned} &\text{If } \Lambda(X_1, \dots, X_p) = \Lambda_1(X_1, \dots, X_p) \otimes s_1 + \ldots + \Lambda_m(X_1, \dots, X_p) \otimes s_m, \\ &\text{then } \Lambda_j = \chi_j(\Lambda) \quad \text{for each } 1 \leq j \leq m. \end{aligned} \tag{5}$$

Let $\mathcal{L} : C(\mathfrak{g} \otimes \mathcal{S}; V) \to C(\mathfrak{g}; V)$ be the map defined by

$$\begin{aligned} &\mathcal{L}(v) = v, \text{ for all } v \in V, \text{ and} \\ &\mathcal{L}(\lambda)(x_1, \dots, x_p) = \lambda(x_1 \otimes 1, \dots, x_p \otimes 1), \end{aligned} \tag{6}$$

where λ is in $C^p(\mathfrak{g} \otimes \mathcal{S}; V)$, and $x_1, \dots, x_p$ are in $\mathfrak{g}$. Let D be the differential in $C(\mathfrak{g} \otimes \mathcal{S}; V \otimes \mathcal{S})$ (see (2)). In the next result, we will prove that D, d, $\mathcal{L}$ and χ_j can be inserted into a commutative diagram.

Proposition 2.2. *For each j, the following diagram is commutative*

$$\begin{array}{ccccc} C^p(\mathfrak{g}\otimes\mathcal{S};V\otimes\mathcal{S}) & \xrightarrow{\chi_j} & C^p(\mathfrak{g}\otimes\mathcal{S};V) & \xrightarrow{\mathcal{L}} & C^p(\mathfrak{g};V) \\ \Big\downarrow{\scriptstyle \mathrm{D}} & & & & \Big\downarrow{\scriptstyle \mathrm{d}} \\ C^{p+1}(\mathfrak{g}\otimes\mathcal{S};V\otimes\mathcal{S}) & \xrightarrow{\chi_j} & C^{p+1}(\mathfrak{g}\otimes\mathcal{S};V) & \xrightarrow{\mathcal{L}} & C^{p+1}(\mathfrak{g};V) \end{array} \tag{7}$$

That is $\mathrm{d}\circ\mathcal{L}\circ\chi_j=\mathcal{L}\circ\chi_j\circ\mathrm{D}$. *By* (5), *this is equivalent to*

$$\mathcal{L}\left((\mathrm{D}\,\Lambda)_j\right)=\mathrm{d}\,\mathcal{L}(\Lambda_j),\ \ \textit{for each } 1\le j\le m. \tag{8}$$

Proof. Let Λ be in $C^p(\mathfrak{g}\otimes\mathcal{S};V\otimes\mathcal{S})$ as in (5). Applying (5) to $\mathrm{D}\,\Lambda$, we obtain $(\mathrm{D}\,\Lambda)_j=\chi_j(\mathrm{D}\,\Lambda)$ for all j. Then (8) holds if and only if the diagram (7) is commutative, that is

$$\begin{aligned} &\mathcal{L}\left((\mathrm{D}\,\Lambda)_j\right)=\mathcal{L}\left(\chi_j(\mathrm{D}\,\Lambda)\right)=\mathcal{L}\circ\chi_j\circ\mathrm{D}\,\Lambda,\ \text{ and}\\ &\mathrm{d}\left(\mathcal{L}(\Lambda_j)\right)=\mathrm{d}\,\mathcal{L}(\chi_j\circ\Lambda)=\mathrm{d}\circ\mathcal{L}\circ\chi_j(\Lambda). \end{aligned} \tag{9}$$

We shall prove that $\mathcal{L}\left((\mathrm{D}\,\Lambda)_j\right)=\mathrm{d}\left(\mathcal{L}(\Lambda_j)\right)$. Let $X_i=x_i\otimes 1$, where x_i belongs to $\mathfrak{g}$ for all $1\le i\le p+1$. By (2), we have

$$\begin{aligned} &\mathrm{D}\,\Lambda(X_1,\dots,X_{p+1})=\sum_{i=1}^{p+1}(-1)^{i-1}R(X_i)(\Lambda(X_1,\dots,X_{\bar{i}},\dots,X_{p+1}))\\ &+\sum_{i<k}(-1)^{i+k}\Lambda([X_i,X_k],X_1,\dots,X_{\hat{i}},\dots,X_{\hat{k}},\dots,X_{p+1}). \end{aligned} \tag{10}$$

Applying (5) to Λ in (10) above, it follows

$$\begin{aligned} &\mathrm{D}\,\Lambda(X_1,\dots,X_{p+1})\\ &=\sum_{i=1}^{p+1}\sum_{j=1}^{m}(-1)^{i-1}R(X_i)\left(\Lambda_j(X_1,\dots,X_{\bar{i}},\dots,X_{p+1})\otimes s_j\right)\\ &+\sum_{i<k}\sum_{j=1}^{m}(-1)^{i+k}\Lambda_j([X_i,X_k],X_1,\dots,X_{\bar{i}},\dots,X_{\bar{k}},\dots,X_{p+1})\otimes s_j. \end{aligned} \tag{11}$$

Let us analyze each of the terms

$$\begin{aligned} &R(X_i)\left(\Lambda_j(X_1,\dots,X_{\bar{i}},\dots,X_{p+1})\otimes s_j\right),\ \ \text{and}\\ &\Lambda_j([X_i,X_k],X_1,\dots,X_{\bar{i}},\dots,X_{\bar{k}},\dots,X_{p+1})\otimes s_j \end{aligned}$$

given in (11). Applying the representation R (see (1)) and $\mathcal{L}$ (see (6)), we obtain

$$\begin{aligned} &R(X_i)\left(\Lambda_j(X_1,\dots,X_{\hat{i}},\dots,X_{p+1})\otimes s_j\right)\\ &=\rho(x_i)\left(\Lambda_j(X_1,\dots,X_{\hat{i}},\dots,X_{p+1})\right)\otimes s_j\\ &=\rho(x_i)\left(\mathcal{L}(\Lambda_j)(x_1,\dots,x_{\hat{i}},\dots,x_{p+1})\right)\otimes s_j \end{aligned} \tag{12}$$

as $\Lambda_j(X_1,\dots,X_{\hat{i}},\dots,X_{p+1})=\mathcal{L}(\Lambda_j)(x_1,\dots,x_{\hat{i}},\dots,x_{p+1})$. In addition,

$$\begin{aligned}&\Lambda_j([X_i,X_k],X_1,\dots,X_{\hat{i}},\dots,X_{\hat{k}},\dots,X_{p+1})\\&=\mathcal{L}(\Lambda_j)([x_i,x_k],\dots,x_{\hat{i}},\dots,x_{\hat{k}},\dots,x_{p+1}),\ \text{ for all } 1\le j\le m,\end{aligned}\tag{13}$$

as $[X_i,X_k]=[x_i,x_k]\otimes 1$. We substitute (12)-(13) in (11), to get

$$\begin{aligned}&\mathrm{D}\,\Lambda(X_1,\dots,X_{p+1})=\mathrm{D}\,\Lambda(x_1\otimes 1,\dots,x_p\otimes 1)\\&=\sum_{i=1}^{p+1}\sum_{j=1}^{m}(-1)^{i-1}\rho(x_i)\left(\mathcal{L}(\Lambda_j)(x_1,\dots,x_{\hat{i}},\dots,x_{p+1})\right)\otimes s_j\\&+\sum_{i<k}\sum_{j=1}^{m}(-1)^{i+k}\mathcal{L}(\Lambda_j)([x_i,x_k],\dots,x_{\hat{i}},\dots,x_{\hat{k}},\dots,x_{p+1})\otimes s_j.\end{aligned}\tag{14}$$

In (14) we gather the terms corresponding at each s_j and we obtain

$$\mathrm{D}\,\Lambda(x_1\otimes 1,\dots,x_p\otimes 1)=\sum_{j=1}^{m}\mathrm{d}\left(\mathcal{L}(\Lambda_j)\right)(x_1,\dots,x_{p+1})\otimes s_j.\tag{15}$$

On the other hand, applying (5) to $\mathrm{D}\,\Lambda$, we obtain

$$\mathrm{D}\,\Lambda(x_1\otimes 1,\dots,x_p\otimes 1)=\sum_{j=1}^{m}\left(\mathrm{D}(\Lambda)\right)_j(x_1\otimes 1,\dots,x_p\otimes 1)\otimes s_j.\tag{16}$$

By (6), $(\mathrm{D}\,\Lambda)_j(x_1\otimes 1,\dots,x_p\otimes 1)=\mathcal{L}\left((\mathrm{D}\,\Lambda)_j\right)(x_1,\dots,x_p)$. Then from (15) and (16), it follows $\mathcal{L}\left((\mathrm{D}\,\Lambda)_j\right)=\mathrm{d}\,\mathcal{L}(\Lambda_j)$ for each $1\le j\le m$. Therefore by (9), the diagram (7) is commutative. □

3. The map $\mathcal{T}:\mathcal{C}(\mathfrak{g})\to\mathcal{C}(\mathfrak{g}\otimes\mathcal{S})$

Let $\mathcal{C}(\mathfrak{g})$ be the set of cochains of $\mathfrak{g}$, i.e., $\mathcal{C}(\mathfrak{g})=\{C(\mathfrak{g};V)\mid V \text{ is a } \mathfrak{g}\text{-module}\}$. We define a map $\mathcal{T}:\mathcal{C}(\mathfrak{g})\to\mathcal{C}(\mathfrak{g}\otimes\mathcal{S})$ by

$$\begin{aligned}\mathcal{T}(C^p(\mathfrak{g};V))&=C^p(\mathfrak{g}\otimes\mathcal{S};V\otimes\mathcal{S}),\quad \text{for } p>0,\ \text{and}\\ \mathcal{T}(V)&=V\otimes\mathcal{S},\quad \text{where } V \text{ is a } \mathfrak{g}\text{-module}.\end{aligned}\tag{17}$$

From now on, we assume that $x,x_1,\dots,x_{p+1}$ are in $\mathfrak{g}$; $s,t,t_1,\dots,t_{p+1}$ are in $\mathcal{S}$; u is in U, v is in V, w is in W; U,V,W are finite dimensional $\mathfrak{g}$-modules. We also consider any cochain Λ as in (5). For $t_1,\dots,t_p$ in $\mathcal{S}$, we write $\widetilde{t}=t_1\cdots t_p$.

Given a linear map $f:\mathcal{V}\to\mathcal{W}$, where $\mathcal{V},\mathcal{W}$ are in $\mathcal{C}(\mathfrak{g})$, we shall define a linear map $\mathcal{T}(f):\mathcal{T}(\mathcal{U})\to\mathcal{T}(\mathcal{V})$. We shall consider four cases.

Case 1: Let $f:C^p(\mathfrak{g};V)\to C^p(\mathfrak{g};W)$ be a linear map. We define the linear map $\mathcal{T}(f):C^p(\mathfrak{g}\otimes\mathcal{S};V\otimes\mathcal{S})\to C^p(\mathfrak{g}\otimes\mathcal{S};W\otimes\mathcal{S})$ by

$$\mathcal{T}(f)(\Lambda)(x_1\otimes t_1,\dots,x_p\otimes t_p)=\sum_{j=1}^{m}f\left(\mathcal{L}(\Lambda_j)\right)(x_1,\dots,x_p)\otimes s_j\,\widetilde{t}.\tag{18}$$

Observe that by (5), we can write Λ as

$$\Lambda(x_1 \otimes t_1, \dots, x_p \otimes t_p) = \sum_{j=1}^{m} \Lambda_j(x_1 \otimes t_1, \dots, x_p \otimes t_p) \otimes s_j,$$

where Λ_j belongs to $C^p(\mathfrak{g} \otimes \mathcal{S}, V)$ for all j. Since $\mathcal{L}(\Lambda_j)$ belongs to $C^p(\mathfrak{g}, V)$, it makes sense to consider $f(\mathcal{L}(\Lambda_j))$ in (18) above.
Case 2: Now consider $p = 0$, $f : V \to W$ a linear map and $v \otimes s$ in $C^0(\mathfrak{g} \otimes \mathcal{S}; V \otimes \mathcal{S}) = V \otimes \mathcal{S}$. We define $\mathcal{T}(f) : V \otimes \mathcal{S} \to W \otimes \mathcal{S}$ by

$$\mathcal{T}(f)(v \otimes s) = f(v) \otimes s. \tag{19}$$

In this case we also denote $\mathcal{T}(f)$ by $f \otimes \mathcal{S}$.
Case 3: Let $f : V \to C^p(\mathfrak{g}; W)$ be a linear map. We define the linear map $\mathcal{T}(f) : V \otimes \mathcal{S} \to C^p(\mathfrak{g} \otimes \mathcal{S}; V \otimes \mathcal{S})$ by

$$\mathcal{T}(f)(v \otimes s)(x_1 \otimes t_1, \dots, x_p \otimes t_p) = f(v)(x_1, \dots, x_p) \otimes s\,\widetilde{t}. \tag{20}$$

Case 4: Let $f : C^p(\mathfrak{g}; V) \to W$ be a linear map. We define the linear map $\mathcal{T}(f) : C^p(\mathfrak{g} \otimes \mathcal{S}; V \otimes \mathcal{S}) \to W \otimes \mathcal{S}$ by

$$\mathcal{T}(f)(\Lambda) = f(\mathcal{L}(\Lambda_1)) \otimes s_1 + \ldots + f(\mathcal{L}(\Lambda_m)) \otimes s_m. \tag{21}$$

Remark 3.1. If Id is the identity map on $C^p(\mathfrak{g}; V)$, then $\mathcal{T}(\mathrm{Id})$ is not the identity map on $C^p(\mathfrak{g} \otimes \mathcal{S}; V \otimes \mathcal{S})$. Indeed, let Λ be in $C^p(\mathfrak{g} \otimes \mathcal{S}; V \otimes \mathcal{S})$. By definition of $\mathcal{L}$ and (18), it follows

$$\begin{aligned} \mathcal{T}(\mathrm{Id})(\Lambda)(x_1 \otimes t_1, \dots, x_p \otimes t_p) &= \sum_{j=1}^{m} \mathrm{Id}\left(\mathcal{L}(\Lambda_j)\right)(x_1, \dots, x_p) \otimes s_j\,\widetilde{t} \\ = \sum_{j=1}^{m} \mathcal{L}(\Lambda_j)(x_1, \dots, x_p) \otimes s_j \widetilde{t} &= \sum_{j=1}^{m} \Lambda_j(x_1 \otimes 1, \dots, x_p \otimes 1) \otimes s_j\,\widetilde{t}. \end{aligned} \tag{22}$$

Then $\mathcal{T}(\mathrm{Id})(\Lambda) = \Lambda$ if and only if

$$\Lambda(x_1 \otimes t_1, \dots, x_p \otimes t_p) = \sum_{j=1}^{m} \Lambda_j(x_1 \otimes 1, \dots, x_p \otimes 1) \otimes s_j\,\widetilde{t}. \tag{23}$$

As we mentioned in the introduction, we will determine the cohomology group $\mathcal{H}(\mathfrak{g} \otimes \mathcal{S}; V \otimes \mathcal{S})$, where V is an irreducible $\mathfrak{g}$-module and $\mathfrak{g}$ is a semisimple Lie algebra (see Proposition 4.5). Apart from the fact that in this case $\mathcal{H}(\mathfrak{g}; V) = \{0\}$, we find this case interesting to apply our results since the condition given in (23), is exactly the fact that helps to determine the cohomology group $\mathcal{H}(\mathfrak{g} \otimes \mathcal{S}; V \otimes \mathcal{S})$.

In the next result, we prove that $\mathcal{T}$ preserves the composition of maps.

Proposition 3.2. *The cochain complexes maps* $f : C^p(\mathfrak{g}; V) \to C^p(\mathfrak{g}; V)$ *and* $g : C^p(\mathfrak{g}; V) \to C^p(\mathfrak{g}; W)$ *yield a map* $\mathcal{T}(g\circ f)$ *from* $C^p(\mathfrak{g}\otimes\mathcal{S}; U\otimes\mathcal{S})$ *to* $C^p(\mathfrak{g}\otimes\mathcal{S}; W\otimes\mathcal{S})$ *satisfying* $\mathcal{T}(f\circ g) = \mathcal{T}(f)\circ\mathcal{T}(g)$.

We shall verify that if $f : \mathcal{U} \to \mathcal{V}$ and $g : \mathcal{V} \to \mathcal{W}$ are maps in $\mathcal{C}(\mathfrak{g})$, then $\mathcal{T}(g\circ f) = \mathcal{T}(g)\circ\mathcal{T}(f) : \mathcal{T}(\mathcal{U}) \to \mathcal{T}(\mathcal{W})$. Several cases should be considered and we only will prove one of them. The proof of the remaining cases uses the same arguments.

Claim 1. *Let* $f : C^p(\mathfrak{g}; U) \to C^p(\mathfrak{g}; V)$ *and* $g : C^p(\mathfrak{g}; V) \to C^p(\mathfrak{g}; W)$ *be maps, then* $\mathcal{T}(g\circ f) = \mathcal{T}(g)\circ\mathcal{T}(f)$ *is a map between* $C^p(\mathfrak{g}\otimes\mathcal{S}; U\otimes\mathcal{S})$ *and* $C^p(\mathfrak{g}\otimes\mathcal{S}; W\otimes\mathcal{S})$.

Proof. Let Λ be in $C^p(\mathfrak{g}\otimes\mathcal{S}; U\otimes\mathcal{S})$, and $\Theta = \mathcal{T}(f)(\Lambda)$. By (18), we have

$$\mathcal{T}(g)(\Theta)(x_1\otimes t_1, \cdots, x_p\otimes t_p) = \sum_{j=1}^{m} g(\mathcal{L}(\Theta_j))(x_1, \dots, x_p)\otimes s_j\,\widetilde{t}, \tag{24}$$

where $\Theta_j = \widehat{\omega}_j\circ\mathcal{T}(f)(\Lambda)$ (see (5)). We claim that $\mathcal{L}(\Theta_j) = f(\mathcal{L}(\Lambda_j))$. Indeed, using the definition of $\mathcal{L}$ and applying (18) to $\mathcal{T}(f)(\Lambda)$, we get

$$\begin{aligned}
&\mathcal{L}(\Theta_j)(x_1, \dots, x_p) = \Theta_j(x_1\otimes 1, \dots, x_p\otimes 1)\\
&= \widehat{\omega}_j\circ\mathcal{T}(f)(\Lambda)(x_1\otimes 1, \dots, x_p\otimes 1)\\
&= \widehat{\omega}_j\left(\sum_{k=1}^{m} f\left(\mathcal{L}(\Lambda_k)\right)(x_1, \dots, x_p)\otimes s_k\right) = f(\mathcal{L}(\Lambda_j))(x_1, \dots, x_p).
\end{aligned}$$

Then $\mathcal{L}(\Theta_j) = f(\mathcal{L}(\Lambda_j))$, for all j. Substituting this in (24), we obtain

$$\begin{aligned}
\mathcal{T}(g)\circ\mathcal{T}(f)(\Lambda)(x_1\otimes t_1, \dots, x_p\otimes t_p) &= \sum_{j=1}^{m}(g\circ f)(\mathcal{L}(\Lambda_j))(x_1, \dots, x_p)\otimes s_j\,\widetilde{t},\\
&= \mathcal{T}(g\circ f)(\Lambda)(x_1\otimes t_1, \cdots, x_p\otimes t_p).
\end{aligned}$$

In the last step above we use (18). Thus $\mathcal{T}(g\circ f) = \mathcal{T}(g)\circ\mathcal{T}(f)$. ☐

Proposition 3.3. *Let* $f : C(\mathfrak{g}; V) \to C(\mathfrak{g}; W)$ *be a map of complexes, that is* $\mathrm{d}\circ f = f\circ\mathrm{d}$ *and* $f(C^p(\mathfrak{g}; V)) \subset C^p(\mathfrak{g}; W)$ *for all* $p \geq 0$*. Then* $\mathcal{T}(f) : C(\mathfrak{g}\otimes\mathcal{S}; V\otimes\mathcal{S}) \to C(\mathfrak{g}\otimes\mathcal{S}; W\otimes\mathcal{S})$ *is a map of complexes.*

Proof. To shorten the length of expressions, we will use the notation:

$$\begin{aligned}
&\mathbf{x}\,\mathbf{t} = (x_1\otimes t_1, \dots, x_{p+1}\otimes t_{p+1}),\\
&(\mathbf{x}\,\mathbf{t})_i = (x_1\otimes t_1, \dots, x_{\hat{i}}\otimes t_{\hat{i}}, \dots, x_{p+1}\otimes t_{p+1}),\\
&(\mathbf{x}\,\mathbf{t})_{i,j} = (x_1\otimes t_1, \dots, x_{\hat{i}}\otimes t_{\hat{i}}, \dots, x_{\hat{j}}\otimes t_{\hat{j}}, \dots, x_{p+1}\otimes t_{p+1}),\\
&\mathbf{x} = (x_1, \dots, x_{p+1}), \qquad \mathbf{x}_i = (x_1, \dots, x_{\hat{i}}, \dots, x_{p+1}),\\
&\mathbf{x}_{i,j} = (x_1, \dots, x_{\hat{i}}, \dots, x_{\hat{j}}, \dots, x_{p+1}).
\end{aligned} \tag{25}$$

Let Λ be in $C^p(\mathfrak{g}\otimes\mathcal{S}; V\otimes\mathcal{S})$, $p>0$. We shall prove that $\mathrm{D}\circ\mathcal{T}(f)(\Lambda)=\mathcal{T}(f)\circ\mathrm{D}\,\Lambda$. Indeed, first we apply D to $\mathcal{T}(f)(\Lambda)$:

$$\begin{aligned}\mathrm{D}\,\mathcal{T}(f)(\Lambda)(\mathbf{x}\,\mathbf{t})&=\sum_{i=1}^{p+1}(-1)^{i-1}R(x_i\otimes t_i)\,(\mathcal{T}(f)(\Lambda))\,((\mathbf{x}\,\mathbf{t})_i)\\ &+\sum_{i<j}(-1)^{i+j}\mathcal{T}(f)(\Lambda)\,([x_i\otimes t_i,x_j\otimes t_j],(\mathbf{x}\,\mathbf{t})_{i,j}).\end{aligned}\tag{26}$$

We write A and B to denote the first and second term in (26), respectively, i.e., $\mathrm{D}\,\mathcal{T}(f)(\Lambda)(\mathbf{x}\,\mathbf{t})=A+B$. Applying (18) to $\mathcal{T}(f)(\Lambda)$ in A, we obtain

$$A=\sum_{i=1}^{p+1}\sum_{k=1}^{m}(-1)^{i-1}R(x_i\otimes t_i)\left(f(\mathcal{L}(\Lambda_k))(\mathbf{x}_i)\otimes s_k\,\hat{t}_i\right),\tag{27}$$

where $\hat{t}_i=t_1\cdots t_{i-1}t_{i+1}\cdots t_{p+1}$. In (27) above, we apply $R(x_i\otimes t_i)$ to $f(\mathcal{L}(\Lambda_k))(\mathbf{x}_i)\otimes s_k\,\hat{t}_i$ (see (1)), and we get

$$A=\sum_{i=1}^{p+1}\sum_{k=1}^{m}(-1)^{i-1}\rho(x_i)\,(f\,(\mathcal{L}(\Lambda_k))\,(\mathbf{x}_i)\otimes s_k\,\widetilde{t},\tag{28}$$

because $\widetilde{t}=t_i\hat{t}_i$. Regarding to B, we fix $i<j$; by (25), we have

$$\begin{aligned}&\mathcal{T}(f)(\Lambda)\,([x_i\otimes t_i,x_j\otimes t_j],(\mathbf{x}\,\mathbf{t})_{i,j})\\ &=\mathcal{T}(f)(\Lambda)\left([x_i,x_j]\otimes t_it_j,x_1\otimes t_1,\dots,x_{\hat{i}}\otimes t_{\hat{i}},\dots,x_{\hat{j}}\otimes t_{\hat{j}},\dots,x_{p+1}\otimes t_{p+1}\right)\\ &=\sum_{k=1}^{m}f\,(\mathcal{T}(\Lambda_k))\,([x_i,x_j],\mathbf{x}_{i,j})\otimes s_k\,\widetilde{t}\qquad\text{(we use (18))}\end{aligned}$$

because $\widetilde{t}=(t_it_j)(t_1\cdots t_{\hat{i}}\cdots t_{\hat{j}}\cdots t_{p+1})$. Hence,

$$B=\sum_{i<j}\sum_{k=1}^{m}(-1)^{i+j}f\,(\mathcal{L}(\Lambda_k))\,([x_i,x_j],\mathbf{x}_{i,j})\otimes s_k\,\widetilde{t}.\tag{29}$$

From (28) and (29), it follows:

$$\mathrm{D}\,\mathcal{T}(f)(\Lambda)(\mathbf{x}\,\mathbf{t})=A+B=\sum_{k=1}^{m}\mathrm{d}\,(f\,(\mathcal{L}(\Lambda_k)))\,(\mathbf{x})\otimes s_k\widetilde{t}.\tag{30}$$

By hypothesis, f is a map of complex, then $f\circ\mathrm{d}=\mathrm{d}\circ f$. By (8), $\mathrm{d}(\mathcal{L}(\Lambda_k))=\mathcal{L}((\mathrm{D}\,\Lambda)_k)$, then by (30), we get

$$\begin{aligned}\mathrm{D}\,\mathcal{T}(f)(\Lambda)(\mathbf{x}\,\mathbf{t})&=\sum_{k=1}^{m}f(\mathrm{d}(\mathcal{L}(\Lambda_k)))(\mathbf{x})\otimes s_k\,\widetilde{t}\\ &=\sum_{k=1}^{m}f\,((\mathcal{L}(\mathrm{D}\,\Lambda)_k))\,(\mathbf{x})\otimes s_k\,\widetilde{t}=\mathcal{T}(f)(\mathrm{D}\,\Lambda)(\mathbf{x}\,\mathbf{t}).\ \text{(We use (18).)}\end{aligned}$$

As Λ is arbitrary, it follows $\mathrm{D}\circ\mathcal{T}(f)=\mathcal{T}(f)\circ\mathrm{D}$. The proof of the case $p=0$ uses the same arguments. □

4. The map $\alpha:\mathcal{H}(\mathfrak{g}\otimes\mathcal{S};V\otimes\mathcal{S})\to\mathcal{H}(\mathfrak{g};V)\otimes\mathcal{S}$

Let V be a $\mathfrak{g}$-module. We denote the group of cocycles and coboundaries of $C(\mathfrak{g};V)$, by $\mathcal{Z}$ and $\mathcal{B}$, respectively. The cohomology group of $\mathfrak{g}$ with coefficients in V is denoted by $\mathcal{H}(\mathfrak{g};V)$. The quotient $C(\mathfrak{g};V)/\mathcal{B}$ is denoted by $\mathcal{Z}'$ and $C(\mathfrak{g};V)/\mathcal{Z}$ is denoted by $\mathcal{B}'$.

By [4, Chapter IV, §23], $\mathcal{Z}$ and $\mathcal{B}$ are $\mathfrak{g}$-modules, then $\mathcal{Z}',\mathcal{B}'$ and $\mathcal{H}(\mathfrak{g};V)$ are $\mathfrak{g}$-modules. Moreover, as in the classical and standard way, the $\mathfrak{g}$-modules $\mathcal{Z},\mathcal{B},\mathcal{Z}',\mathcal{B}'$ and $\mathcal{H}(\mathfrak{g};V)$ will be regarded as modules with zero differentiation (see [2, Chapter IV, §1]).

Lemma 4.1. *For $p=0$, $\mathcal{H}^0(\mathfrak{g}\otimes\mathcal{S};V\otimes\mathcal{S})=\mathcal{H}^0(\mathfrak{g};V)\otimes\mathcal{S}$.*

Proof. For $p=0$, we have $\mathcal{Z}^0=\mathcal{H}^0(\mathfrak{g};V)$. Let $\bar{v}$ be an element in $\mathcal{H}^0(\mathfrak{g}\otimes\mathcal{S};V\otimes\mathcal{S})\subset V\otimes\mathcal{S}$. We write $\bar{v}=v_1\otimes s_1+\ldots+v_m\otimes s_m$, where v_j belongs to V for all $1\leq j\leq m$ (see Proposition 2.1). Then

$$\begin{aligned}0=\mathrm{D}(\bar{v})(x\otimes 1)&=R(x\otimes 1)(\bar{v})\\&=\rho(x)(v_1)\otimes s_1+\ldots+\rho(x)(v_m)\otimes s_m\\&=\mathrm{d}(v_1)(x)\otimes s_1+\ldots+\mathrm{d}(v_m)(x)\otimes s_m.\end{aligned}\tag{31}$$

Whence $\mathrm{d}(v_j)=0$ and v_j belongs to $\mathcal{H}^0(\mathfrak{g};V)$ for all j, which implies that $\bar{v}$ belongs to $\mathcal{H}^0(\mathfrak{g};V)\otimes\mathcal{S}$. Hence, $\mathcal{H}^0(\mathfrak{g}\otimes\mathcal{S};V\otimes\mathcal{S})\subset\mathcal{H}^0(\mathfrak{g};V)\otimes\mathcal{S}$.

Let $\bar{v}$ be in $\mathcal{H}^0(\mathfrak{g};V)\otimes\mathcal{S}\subset V\otimes\mathcal{S}$. By Proposition 2.1, there are $v_1,\ldots,v_m$ in $\mathcal{H}^0(\mathfrak{g};V)$ such that $\bar{v}=v_1\otimes s_1+\ldots+v_m\otimes s_m$. As each v_j belongs to $\mathcal{H}^0(\mathfrak{g};v)=\mathcal{Z}^0$, then $\mathrm{d}(v_j)=0$. Thus,

$$\begin{aligned}\mathrm{D}(\bar{v})(x\otimes s)&=R(x\otimes s)(\bar{v})=\sum_{j=1}^{m}R(x\otimes s)(v_j\otimes s_j)\\&=\sum_{j=1}^{m}\rho(x)(v_j)\otimes ss_j=\sum_{j=1}^{m}\mathrm{d}(v_j)(x)\otimes ss_j=0.\end{aligned}$$

Hence $\mathrm{D}(\bar{v})=0$, and $\bar{v}$ is in $\mathcal{H}^0(\mathfrak{g}\otimes\mathcal{S};V\otimes\mathcal{S})$. Therefore $\mathcal{H}^0(\mathfrak{g}\otimes\mathcal{S};V\otimes\mathcal{S})=\mathcal{H}^0(\mathfrak{g};V)\otimes\mathcal{S}$. □

Let $\iota:\mathcal{Z}\to C(\mathfrak{g};V)$ be the inclusion map. By (20), we get a map $\mathcal{T}(\iota):\mathcal{Z}\otimes\mathcal{S}\to C(\mathfrak{g}\otimes\mathcal{S};V\otimes\mathcal{S})$. Since $\mathcal{Z}\otimes\mathcal{S}$ has zero differential, we can define a map Φ from

$\mathcal{Z}\otimes\mathcal{S}$ into $\mathcal{H}(\mathfrak{g}\otimes\mathcal{S};V\otimes\mathcal{S})$, by

$$\begin{array}{rccl} \Phi: & \mathcal{Z}\otimes\mathcal{S} & \longrightarrow & \mathcal{H}(\mathfrak{g}\otimes\mathcal{S};V\otimes\mathcal{S}) \\ & \bar{x} & \mapsto & \mathcal{T}(\iota)(\bar{x})+\mathcal{B}(\mathfrak{g}\otimes\mathcal{S};V\otimes\mathcal{S}). \end{array} \tag{32}$$

Consider $\pi':C(\mathfrak{g};V)\to\mathcal{Z}'$ defined by $\pi'(\lambda)=\lambda+\mathcal{B}$. By (21), we get a map $\mathcal{T}(\pi'):C(\mathfrak{g}\otimes\mathcal{S};V\otimes\mathcal{S})\to\mathcal{Z}'\otimes\mathcal{S}$. As $\mathcal{Z}'\otimes\mathcal{S}$ has zero differential, we can define a map Ψ from $\mathcal{H}(\mathfrak{g}\otimes\mathcal{S};V\otimes\mathcal{S})$ into $\mathcal{Z}'\otimes\mathcal{S}$ by

$$\begin{array}{rccl} \Psi: & \mathcal{H}(\mathfrak{g}\otimes\mathcal{S};V\otimes\mathcal{S}) & \longrightarrow & \mathcal{Z}'\otimes\mathcal{S} \\ & \Lambda+\mathcal{B}(\mathfrak{g}\otimes\mathcal{S};V\otimes\mathcal{S}) & \mapsto & \mathcal{T}(\pi')(\Lambda)=\sum_{j=1}^{m}(\mathcal{L}(\Lambda_j)+\mathcal{B})\otimes s_j. \end{array} \tag{33}$$

We shall prove that Ψ is well-defined. Let σ be in $C(\mathfrak{g}\otimes\mathcal{S};V\otimes\mathcal{S})$. Using (8) and (21), as well as $\pi'\circ\mathrm{d}=0$, we obtain

$$\begin{aligned} \mathcal{T}(\pi')(\mathrm{D}\,\sigma) &= \pi'\left(\mathcal{L}((\mathrm{D}\,\sigma)_1)\right)\otimes s_1+\ldots+\pi'\left(\mathcal{L}((\mathrm{D}\,\sigma)_m)\right)\otimes s_m \\ &= \pi'\left(\mathrm{d}\,\mathcal{L}(\sigma_1)\right)\otimes s_1+\ldots+\pi'\left(\mathrm{d}\,\mathcal{L}(\sigma_m)\right)\otimes s_m=0. \end{aligned}$$

Then $\mathcal{B}(\mathfrak{g}\otimes\mathcal{S};V\otimes\mathcal{S})\subset\operatorname{Ker}(\mathcal{T}(\pi'))$, hence Ψ is well-defined.

Let $\pi:\mathcal{Z}\to\mathcal{H}(\mathfrak{g};V)$ be the projection map and $\iota':\mathcal{H}(\mathfrak{g};V)\to\mathcal{Z}'$ be the inclusion map. In the next result we will prove that there exists a surjective linear map α between $\mathcal{H}(\mathfrak{g}\otimes\mathcal{S};V\otimes\mathcal{S})$ and $\mathcal{H}(\mathfrak{g};V)\otimes\mathcal{S}$.

Proposition 4.2. *Let $\mathfrak{g}\otimes\mathcal{S}$ be the current Lie algebra of $\mathfrak{g}$ by $\mathcal{S}$.*

(i) *For any $\mathfrak{g}$-module V, there exists a unique surjective linear map $\alpha:\mathcal{H}(\mathfrak{g}\otimes\mathcal{S};V\otimes\mathcal{S})\to\mathcal{H}(\mathfrak{g};V)\otimes\mathcal{S}$ that makes commutative the diagram*

$$\begin{array}{ccc} \mathcal{Z}\otimes\mathcal{S} & \xrightarrow{\pi\otimes\mathcal{S}} & \mathcal{H}(\mathfrak{g};V)\otimes\mathcal{S} \\ \Phi\downarrow & \nearrow\alpha & \downarrow\iota'\otimes\mathcal{S} \\ \mathcal{H}(\mathfrak{g}\otimes\mathcal{S};V\otimes\mathcal{S}) & \xrightarrow{\Psi} & \mathcal{Z}'\otimes\mathcal{S} \end{array} \tag{34}$$

(ii) *Let $f:C(\mathfrak{g};V)\to C(\mathfrak{g};W)$ be a map of complexes and consider $\mathcal{H}(f):\mathcal{H}(\mathfrak{g};V)\to\mathcal{H}(\mathfrak{g};W)$ the map induced by f, i.e., $\mathcal{H}(f)(\lambda+\mathcal{B})=f(\lambda)+\mathcal{B}$. Then the following diagram is commutative*

$$\begin{array}{ccc} \mathcal{H}(\mathfrak{g}\otimes\mathcal{S};V\otimes\mathcal{S}) & \xrightarrow{\mathcal{H}(\mathcal{T}(f))} & \mathcal{H}(\mathfrak{g}\otimes\mathcal{S};W\otimes\mathcal{S}) \\ \alpha_V\downarrow & & \downarrow\alpha_W \\ \mathcal{H}(\mathfrak{g};V)\otimes\mathcal{S} & \xrightarrow{\mathcal{H}(f)\otimes\mathcal{S}=\mathcal{T}(\mathcal{H}(f))} & \mathcal{H}(\mathfrak{g};W)\otimes\mathcal{S} \end{array} \tag{35}$$

where $\mathcal{H}(\mathcal{T}(f))$ is the map induced by $\mathcal{T}(f)$.

Proof. (i) Let $\eta : \mathcal{B} \to \mathcal{Z}$ be the inclusion map and $\zeta : \mathcal{Z}' \to \mathcal{B}'$ be the map defined by $\zeta(\lambda + \mathcal{B}) = \lambda + \mathcal{Z}$. We have the following short exact sequences

$$0 \longrightarrow \mathcal{H}(\mathfrak{g}; V) \xrightarrow{\iota'} \mathcal{Z}' \xrightarrow{\zeta} \mathcal{B}' \longrightarrow 0,$$

$$0 \longrightarrow \mathcal{B} \xrightarrow{\eta} \mathcal{Z} \xrightarrow{\pi} \mathcal{H}(\mathfrak{g}; V) \longrightarrow 0.$$

Since $\mathcal{S}$ is finite dimensional, the following sequences are exact

$$0 \longrightarrow \mathcal{H}(\mathfrak{g}; V) \otimes \mathcal{S} \xrightarrow{\iota' \otimes \mathcal{S}} \mathcal{Z}' \otimes \mathcal{S} \xrightarrow{\zeta \otimes \mathcal{S}} \mathcal{B}' \otimes \mathcal{S} \longrightarrow 0,$$

$$0 \longrightarrow \mathcal{B} \otimes \mathcal{S} \xrightarrow{\eta \otimes \mathcal{S}} \mathcal{Z} \otimes \mathcal{S} \xrightarrow{\pi \otimes \mathcal{S}} \mathcal{H}(\mathfrak{g}; V) \otimes \mathcal{S} \longrightarrow 0.$$

Whence, $\pi \otimes \mathcal{S}$ is surjective and $\iota' \otimes \mathcal{S}$ is injective. Since $\iota' \circ \pi = \pi' \circ \iota$, Proposition 3.2 leads to the commutativity of the following diagram:

$$\begin{array}{ccc} \mathcal{Z} \otimes \mathcal{S} & \xrightarrow{\pi \otimes \mathcal{S}} & \mathcal{H}(\mathfrak{g}; V) \otimes \mathcal{S} \\ \downarrow{\scriptstyle \mathcal{T}(\iota)} & & \downarrow{\scriptstyle \iota' \otimes \mathcal{S}} \\ C(\mathfrak{g} \otimes \mathcal{S}; V \otimes \mathcal{S}) & \xrightarrow{\mathcal{T}(\pi')} & \mathcal{Z}' \otimes \mathcal{S} \end{array} \qquad (36)$$

By (32) and (33), we have the commutative diagram

$$\begin{array}{ccccc} & & 0 & & \\ & & \downarrow & & \\ \mathcal{Z} \otimes \mathcal{S} & \xrightarrow{\pi \otimes \mathcal{S}} & \mathcal{H}(\mathfrak{g}; V) \otimes \mathcal{S} & \longrightarrow & 0 \\ \downarrow{\scriptstyle \Phi} & & \downarrow{\scriptstyle \iota' \otimes \mathcal{S}} & & \\ \mathcal{H}(\mathfrak{g} \otimes \mathcal{S}; V \otimes \mathcal{S}) & \xrightarrow{\Psi} & \mathcal{Z}' \otimes \mathcal{S} & & \end{array} \qquad (37)$$

If α and α' make commutative the diagram (37), then $(\iota' \otimes \mathcal{S}) \circ \alpha = (\iota' \otimes \mathcal{S}) \circ \alpha'$. As $\iota' \otimes \mathcal{S}$ is injective, it follows $\alpha = \alpha'$.

Claim 2. *For $p = 0$, $\alpha : \mathcal{H}(\mathfrak{g} \otimes \mathcal{S}; V \otimes \mathcal{S}) \to \mathcal{H}(\mathfrak{g}, V) \otimes \mathcal{S}$ is the identity map.*

Proof. For $p = 0$, $T(\iota) = \iota \otimes \mathcal{S}$ because $\iota : \mathcal{Z}^0 \to C^0(\mathfrak{g}; V)$ and $V = C^0(\mathfrak{g}; V)$ (see (19)). Then $\mathrm{Im}(\mathcal{T}(\iota)) = \mathrm{Im}(\iota \otimes \mathcal{S}) = \mathcal{Z}^0 \otimes \mathcal{S} = \mathcal{H}^0(\mathfrak{g}; V) \otimes \mathcal{S} = \mathcal{H}^0(\mathfrak{g} \otimes \mathcal{S}; V \otimes \mathcal{S})$. Hence, by (32), $\Phi^0 : \mathcal{Z}^0 \otimes \mathcal{S} \to \mathcal{H}^0(\mathfrak{g} \otimes \mathcal{S}; V \otimes \mathcal{S})$ is the identity map.

Similarly for $p = 0$, $\pi' : C(\mathfrak{g}; V) \to \mathcal{Z}'$ is the identity, as $C^0(\mathfrak{g}, V) = V$ and $\mathcal{Z}'^0 = V$. Since both V and $\mathcal{Z}'$ have zero differential, by (19), $\mathcal{T}(\pi') = \mathrm{Id}_V \otimes \mathcal{S} = \mathrm{Id}_{V \otimes \mathcal{S}}$ is the identity on $V \otimes \mathcal{S}$. Therefore, $\Psi^0 : \mathcal{H}^0(\mathfrak{g} \otimes \mathcal{S}; V \otimes \mathcal{S}) \to \mathcal{Z}'^0 \otimes \mathcal{S}$ is the inclusion (see (33)).

For $p = 0$, the map $\pi \otimes \mathcal{S} : \mathcal{Z}^0 \otimes \mathcal{S} \to \mathcal{H}^0(\mathfrak{g}; V) \otimes \mathcal{S}$ is the identity, as $\mathcal{Z}^0 = \mathcal{H}^0(\mathfrak{g}, V)$. Similarly, $\iota' \otimes \mathcal{S} : \mathcal{H}^0(\mathfrak{g}; V) \otimes \mathcal{S} \to {\mathcal{Z}'}^0 \otimes \mathcal{S}$ is the inclusion map, as ${\mathcal{Z}'}^0 = C^0(\mathfrak{g}, V)/\mathcal{B}^0(\mathfrak{g}, V) = V$.

In summary, for $p = 0$, we have that Φ^0 and $\pi \otimes \mathcal{S}$ are the identity maps while Ψ^0 and $\iota' \otimes \mathcal{S}$ are the inclusion maps. Since any map that makes commutative (34) is unique, we deduce that α is the identity for $p = 0$. □

Now we shall consider $p > 0$. If $\mathrm{Im}(\Psi) \subset \mathrm{Im}\,(\iota' \otimes \mathcal{S}) = \mathrm{Ker}\,(\zeta \otimes \mathcal{S})$, then there exists a map α between $\mathcal{H}(\mathfrak{g} \otimes \mathcal{S}; V \otimes \mathcal{S})$ and $\mathcal{H}(\mathfrak{g}; V) \otimes \mathcal{S}$. We shall now prove this assertion.

Claim 3. *The composition* $(\zeta \otimes \mathcal{S}) \circ \Psi$ *is zero.*

Proof. For $p > 0$, let Λ be in $C^p(\mathfrak{g} \otimes \mathcal{S}; V \otimes \mathcal{S})$ such that $\mathrm{D}\,\Lambda = 0$. By (8), $\mathcal{L}(\Lambda_j)$ belongs to $\mathcal{Z}$ for all j. By (19) and (33), we have

$$
\begin{aligned}
&(\zeta \otimes \mathcal{S}) \circ \Psi\,(\Lambda + \mathcal{B}(\mathfrak{g} \otimes \mathcal{S}; V \otimes \mathcal{S}))\\
&= (\zeta \otimes \mathcal{S})\,((\mathcal{L}(\Lambda_1) + \mathcal{B}) \otimes s_1 + \ldots + (\mathcal{L}(\Lambda_m) + \mathcal{B}) \otimes s_m)\\
&= (\mathcal{L}(\Lambda_1) + \mathcal{Z}) \otimes s_1 + \ldots + (\mathcal{L}(\Lambda_m) + \mathcal{Z}) \otimes s_m = 0.
\end{aligned}
$$

Then the composition $(\zeta \otimes \mathcal{S}) \circ \Psi$ is zero for $p > 0$. Using a similar argument, it is proved that $(\zeta \otimes \mathcal{S}) \circ \Psi = 0$ for $p = 0$. □

Claim 4. *There exists a linear map* $\alpha : \mathcal{H}(\mathfrak{g} \otimes \mathcal{S}; V \otimes \mathcal{S}) \to \mathcal{H}(\mathfrak{g}; V) \otimes \mathcal{S}$ *that makes commutative the diagram* (34).

Proof. Since $\mathrm{Im}(\Psi) \subset \mathrm{Ker}(\zeta \otimes \mathcal{S}) = \mathrm{Im}(\iota' \otimes \mathcal{S})$ (see Claim 3), for each $\overline{\Lambda}$ in $\mathcal{H}(\mathfrak{g} \otimes \mathcal{S}; V \otimes \mathcal{S})$, there exists θ in $\mathcal{H}(\mathfrak{g}; V) \otimes \mathcal{S}$ such that $\Psi(\overline{\Lambda}) = (\iota' \otimes \mathcal{S})\,(\theta)$. Since $\iota' \otimes \mathcal{S}$ is injective, θ is unique.

Define $\alpha : \mathcal{H}(\mathfrak{g} \otimes \mathcal{S}; V \otimes \mathcal{S}) \to \mathcal{H}(\mathfrak{g}; V) \otimes \mathcal{S}$, by $\alpha(\overline{\Lambda}) = \theta$, then $\Psi = (\iota' \otimes \mathcal{S}) \circ \alpha$. Since $\iota' \otimes \mathcal{S}$ is injective, $\mathrm{Ker}(\alpha) = \mathrm{Ker}(\Psi)$. As (37) is commutative, $(\iota' \otimes \mathcal{S}) \circ (\alpha \circ \Phi) = (\iota' \otimes \mathcal{S}) \circ (\pi \otimes \mathcal{S})$. Hence, $\alpha \circ \Phi = \pi \otimes \mathcal{S}$ and α makes commutative the diagram (34). □

Claim 5. *The map* $\alpha : \mathcal{H}(\mathfrak{g} \otimes \mathcal{S}; V \otimes \mathcal{S}) \to \mathcal{H}(\mathfrak{g}; V) \otimes \mathcal{S}$ *is surjective.*

Proof. Let θ be in $\mathcal{H}(\mathfrak{g}; V) \otimes \mathcal{S}$. Since $\pi \otimes \mathcal{S}$ is surjective, there exists μ in $\mathcal{Z} \otimes \mathcal{S}$ such that $(\pi \otimes \mathcal{S})\,(\mu) = \theta$. Let $\bar{\Lambda} = \Phi(\mu)$, then $\alpha(\bar{\Lambda}) = (\alpha \circ \Phi)(\mu) = (\pi \otimes \mathcal{S})(\mu) = \theta$. Whence, α is surjective. □

For $p > 0$, we shall give an explicit description of the map α. Let $\Lambda + \mathcal{B}(\mathfrak{g} \otimes \mathcal{S}; V \otimes \mathcal{S})$ be in $\mathcal{H}(\mathfrak{g} \otimes \mathcal{S}; V \otimes \mathcal{S})$, where Λ is in $C^p(\mathfrak{g} \otimes \mathcal{S}; V \otimes \mathcal{S})$. As $\alpha(\Lambda + \mathcal{B}(\mathfrak{g} \otimes \mathcal{S}; V \otimes \mathcal{S}))$

belongs to $\mathcal{H}(\mathfrak{g};V)\otimes\mathcal{S}$, by Proposition 2.1, there are μ_j in $\mathcal{Z}$ such that

$$\alpha(\Lambda+\mathcal{B}(\mathfrak{g}\otimes\mathcal{S};V\otimes\mathcal{S}))=(\mu_1+\mathcal{B})\otimes s_1+\cdots+(\mu_m+\mathcal{B})\otimes s_m. \tag{38}$$

By (34), $(\iota'\otimes\mathcal{S})\circ\alpha=\Psi$, and using (21) and (33) it follows that:

$$\begin{aligned}&(\iota'\otimes\mathcal{S})\circ\alpha(\Lambda+\mathcal{B}(\mathfrak{g}\otimes\mathcal{S};V\otimes\mathcal{S}))=\Psi\left(\Lambda+\mathcal{B}(\mathfrak{g}\otimes\mathcal{S};V\otimes\mathcal{S})\right)\\&=(\mathcal{L}(\Lambda_1)+\mathcal{B})\otimes s_1+\ldots+(\mathcal{L}(\Lambda_m)+\mathcal{B})\otimes s_m.\end{aligned} \tag{39}$$

Applying $\iota'\otimes\mathcal{S}$ to (38) we get

$$\begin{aligned}&(\iota'\otimes\mathcal{S})\circ\alpha(\Lambda+\mathcal{B}(\mathfrak{g}\otimes\mathcal{S};V\otimes\mathcal{S}))\\&=(\iota'\otimes\mathcal{S})\left(\sum_{j=1}^{m}(\mu_j+\mathcal{B})\otimes s_j\right)=\sum_{j=1}^{m}(\mu_j+\mathcal{B})\otimes s_j.\end{aligned} \tag{40}$$

From (39) and (40), it follows $\mu_j+\mathcal{B}=\mathcal{L}(\Lambda_j)+\mathcal{B}$ for all j. Hence, by (38), we obtain

$$\alpha(\Lambda+\mathcal{B}(\mathfrak{g}\otimes\mathcal{S};V\otimes\mathcal{S}))=(\mathcal{L}(\Lambda_1)+\mathcal{B})\otimes s_1+\ldots+(\mathcal{L}(\Lambda_m)+\mathcal{B})\otimes s_m. \tag{41}$$

(ii) We shall prove that if $f:C(\mathfrak{g};V)\to C(\mathfrak{g};W)$ is a map of complexes, then the following diagram is commutative:

$$\begin{array}{ccc}\mathcal{H}(\mathfrak{g}\otimes\mathcal{S};V\otimes\mathcal{S}) & \xrightarrow{\mathcal{H}(\mathcal{T}(f))} & \mathcal{H}(\mathfrak{g}\otimes\mathcal{S};W\otimes\mathcal{S})\\ \downarrow{\scriptstyle\alpha_V} & & \downarrow{\scriptstyle\alpha_W}\\ \mathcal{H}(\mathfrak{g};V)\otimes\mathcal{S} & \xrightarrow{\mathcal{T}(\mathcal{H}(f))} & \mathcal{H}(\mathfrak{g};W)\otimes\mathcal{S}\end{array} \tag{42}$$

where $\mathcal{T}(\mathcal{H}(f))=\mathcal{H}(f)\otimes\mathcal{S}$. Let $f':\mathcal{Z}'(\mathfrak{g};V)\to\mathcal{Z}'(\mathfrak{g};W)$ be the map induced by f, i.e., $f'(\lambda+\mathcal{B})=f(\lambda)+\mathcal{B}$. Then $\pi'\circ f=f'\circ\pi'$. Since diagram (34) is commutative, (33) implies that:

$$\begin{aligned}&(\iota'\otimes\mathcal{S})\circ(\alpha_W\circ\mathcal{H}(\mathcal{T}(f))\\&=((\iota'\otimes\mathcal{S})\circ\alpha_W)\circ\mathcal{H}(\mathcal{T}(f))=\Psi\circ\mathcal{H}(\mathcal{T}(f))=\mathcal{T}(f')\circ\Psi.\end{aligned} \tag{43}$$

By (19), $\mathcal{T}(f')=f'\otimes\mathcal{S}$ and by (34), $\Psi=(\iota'\otimes\mathcal{S})\circ\alpha_V$. In addition, $\mathcal{T}(\iota')=\iota'\otimes\mathcal{S}$ and since f is a map of complexes, it follows that $\iota'\circ\mathcal{H}(f)=f'\circ\iota'$. Hence:

$$\begin{aligned}&\mathcal{T}(f')\circ\Psi=\mathcal{T}(f')\circ(\mathcal{T}(\iota')\circ\alpha_V)=(\mathcal{T}(f')\circ\mathcal{T}(\iota'))\circ\alpha_V\\&=\mathcal{T}\left(f'\circ\iota'\right)\circ\alpha_V=\mathcal{T}\left(\iota'\circ\mathcal{H}(f)\right)\circ\alpha_V\\&=((\iota'\circ\mathcal{H}(f))\otimes\mathcal{S})\circ\alpha_V=(\iota'\otimes\mathcal{S})\circ((\mathcal{H}(f)\otimes\mathcal{S})\circ\alpha_V).\end{aligned} \tag{44}$$

As $\iota'\otimes\mathcal{S}$ is injective, from (43) and (44), we deduce that $\alpha_W\circ\mathcal{H}(\mathcal{T}(f))=(\mathcal{H}(f)\otimes\mathcal{S})\circ\alpha_V$. Whence, the diagram (42) is commutative. □

Let $\mathcal{R}^0 = \{0\}$ and for each $p > 0$, define $\mathcal{R}^p$ as the subspace of $C^p(\mathfrak{g}\otimes\mathcal{S}; V\otimes\mathcal{S})$ generated by all the cochains Θ satisfying $\Theta(x_1\otimes 1,\ldots,x_p\otimes 1) = 0$. Let $\mathcal{R} = \oplus_{p\geq 0}\mathcal{R}^p$ and define $\mathcal{Q}$ by

$$\mathcal{Q} = \left(\mathcal{Z}(\mathfrak{g}\otimes\mathcal{S}; V\otimes\mathcal{S})\cap\mathcal{R} + \mathcal{B}(\mathfrak{g}\otimes\mathcal{S}; V\otimes\mathcal{S})\right)/\mathcal{B}(\mathfrak{g}\otimes\mathcal{S}; V\otimes\mathcal{S}). \tag{45}$$

Now we shall state the main result of this work.

Theorem 4.3. *Let $\mathfrak{g}$ be a Lie algebra and let $\mathcal{S}$ be an m-dimensional, associative and commutative algebra with unit, over a field $\mathbb{F}$. Let $\mathfrak{g}\otimes\mathcal{S}$ be the current Lie algebra of $\mathfrak{g}$ by $\mathcal{S}$. Let $\alpha : \mathcal{H}(\mathfrak{g}\otimes\mathcal{S}; V\otimes\mathcal{S}) \to \mathcal{H}(\mathfrak{g}; V)\otimes\mathcal{S}$ be the map of Proposition 4.2. Then the following short sequence is exact*

$$0 \longrightarrow \mathcal{Q} \xrightarrow{\ \iota\ } \mathcal{H}(\mathfrak{g}\otimes\mathcal{S}; V\otimes\mathcal{S}) \xrightarrow{\ \alpha\ } \mathcal{H}(\mathfrak{g}; V)\otimes\mathcal{S} \longrightarrow 0, \tag{46}$$

where ι is the inclusion map and $\mathcal{Q}$ is the subspace defined in (45)*.*

Proof. By (3), observe that a cochain Θ belongs to $\mathcal{R}$ if and only if $\Theta_j(x_1\otimes 1,\ldots,x_p\otimes 1) = \mathcal{L}(\Theta_j)(x_1,\ldots,x_p) = 0$, for all j. Then Θ is in $\mathcal{R}$ if and only if Θ_j belongs to $\operatorname{Ker}(\mathcal{L})$ for all j.

In the proof of Claim 4, we showed that $\operatorname{Ker}(\alpha) = \operatorname{Ker}(\Psi)$. We claim that $\operatorname{Ker}(\Psi) = \mathcal{Q}$. First we assume that $p > 0$. Let $\Lambda + \mathcal{B}(\mathfrak{g}\otimes\mathcal{S}; V\otimes\mathcal{S})$ be in $\operatorname{Ker}(\Psi)$. We will find a cochain Θ in $\mathcal{R}$ such that $\Lambda + \mathcal{B}(\mathfrak{g}\otimes\mathcal{S}; V\otimes\mathcal{S}) = \Theta + \mathcal{B}(\mathfrak{g}\otimes\mathcal{S}; V\otimes\mathcal{S})$. Indeed, by (33), we have

$$\Psi(\Lambda + \mathcal{B}(\mathfrak{g}\otimes\mathcal{S}; V\otimes\mathcal{S})) = \sum_{j=1}^{m}(\mathcal{L}(\Lambda_j) + \mathcal{B})\otimes s_j = 0. \tag{47}$$

Then $\mathcal{L}(\Lambda_j)$ belongs to $\mathcal{B}$ for all j. Hence, there exists θ_j in $C^{p-1}(\mathfrak{g}; V)$ such that $\mathcal{L}(\Lambda_j) = \mathrm{d}\,\theta_j$. For each j, define Δ_j in $C^{p-1}(\mathfrak{g}\otimes\mathcal{S}; V)$ by

$$\Delta_j(x_1\otimes t_1,\ldots,x_{p-1}\otimes t_{p-1}) = \omega_1\,(t_1\cdots t_{p-1})\,\theta_j(x_1,\ldots,x_{p-1}).$$

Then $\mathcal{L}(\Delta_j) = \theta_j$ (see (6)). Let Δ in $C^{p-1}(\mathfrak{g}\otimes\mathcal{S}; V\otimes\mathcal{S})$ be defined by

$$\Delta(X_1,\ldots,X_{p-1}) = \Delta_1(X_1,\ldots,X_{p-1})\otimes s_1 + \ldots + \Delta_m(X_1,\ldots,X_{p-1})\otimes s_m,$$

where $X_1,\ldots,X_{p-1}$ are in $\mathfrak{g}\otimes\mathcal{S}$. From (8), we have

$$\mathcal{L}(\Lambda_j) = \mathrm{d}\,\theta_j = \mathrm{d}(\mathcal{L}(\Delta_j)) = \mathcal{L}((\mathrm{D}\,\Delta)_j),\ \text{ for all } 1\leq j\leq m.$$

Then there exists Θ_j in $\operatorname{Ker}(\mathcal{L})$ such that $\Lambda_j = (\mathrm{D}\,\Delta)_j + \Theta_j$. Let Θ in $C^p(\mathfrak{g}\otimes\mathcal{S}; V\otimes\mathcal{S})$ be defined by

$$\Theta(X_1,\ldots,X_p) = \Theta_1(X_1,\ldots,X_p)\otimes s_1 + \ldots + \Theta_m(X_1,\ldots,X_p)\otimes s_m,$$

for all $X_1, \dots, X_p$ in $\mathfrak{g} \otimes \mathcal{S}$. Since $\Lambda_j = (\mathrm{D}\,\Delta)_j + \Theta_j$ for each j, $\Lambda = \mathrm{D}\,\Delta + \Theta$ (see (5)). Since Θ_j belongs to $\mathrm{Ker}(\mathcal{L})$, Θ belongs to $\mathcal{R}$ and $\Lambda + \mathcal{B}(\mathfrak{g} \otimes \mathcal{S}; V \otimes \mathcal{S}) = \Theta + \mathcal{B}(\mathfrak{g} \otimes \mathcal{S}; V \otimes \mathcal{S})$ belongs to $\mathcal{Q}$.

Since $\mathrm{D}\,\Lambda = 0$, $\mathrm{D}\,\Theta = 0$. Therefore $\Lambda + \mathcal{B}(\mathfrak{g} \otimes \mathcal{S}; V \otimes \mathcal{S}) = \Theta + \mathcal{B}(\mathfrak{g} \otimes \mathcal{S}; V \otimes \mathcal{S})$ belongs to $\mathcal{Q}$, which proves that $\mathrm{Ker}(\alpha) \subset \mathcal{Q}$.

Now we affirm $\mathcal{Q} \subset \mathrm{Ker}(\alpha)$. Let $\Theta + \mathcal{B}(\mathfrak{g} \otimes \mathcal{S}; V \otimes \mathcal{S})$ be in $\mathcal{Q}$, where Θ is in $\mathcal{Z}(\mathfrak{g} \otimes \mathcal{S}; V \otimes \mathcal{S}) \cap \mathcal{R}$. As $\Theta(x_1 \otimes 1, \dots, x_p \otimes 1) = 0$, then Θ_j belongs to $\mathrm{Ker}(\mathcal{L})$ for all j. By (47), we have that $\Theta + \mathcal{B}(\mathfrak{g} \otimes \mathcal{S}; V \otimes \mathcal{S})$ belongs to $\mathrm{Ker}(\Psi) = \mathrm{Ker}(\alpha)$. Then $\mathrm{Ker}(\alpha) = \mathcal{Q}$. Since α is surjective, we deduce that the short exact sequence (46) is exact for $p > 0$.

For $p = 0$, we have $\mathcal{Q}^0 = \{0\}$, because by hypothesis, $\mathcal{R}^0 = \{0\}$. In Lemma 4.1, we showed that $\mathcal{H}^0(\mathfrak{g} \otimes \mathcal{S}; V \otimes \mathcal{S}) = \mathcal{H}^0(\mathfrak{g}; V) \otimes \mathcal{S}$ while in Claim 2, we proved that α is the identity. Therefore, $\mathrm{Ker}(\alpha) = \{0\} = \mathcal{Q}^0$ and the sequence (46) is exact for $p = 0$. □

Corollary 4.4. *Let $\mathfrak{g}$ be a Lie algebra and let $\mathcal{S}$ be a finite dimensional, associative and commutative algebra with unit over a field $\mathbb{F}$. Let V be a $\mathfrak{g}$-module. Then $\mathcal{H}(\mathfrak{g} \otimes \mathcal{S}; V \otimes \mathcal{S})$ is isomorphic to $\mathcal{H}(\mathfrak{g}; V) \otimes \mathcal{S}$ if and only if*

$$\mathcal{Z}^p(\mathfrak{g} \otimes \mathcal{S}, V \otimes \mathcal{S}) \cap \mathcal{R}^p \subset \mathcal{B}^p(\mathfrak{g} \otimes \mathcal{S}; V \otimes \mathcal{S}), \quad \textit{for all } p > 0. \tag{48}$$

Proof. By Theorem 4.3, we know $\mathrm{Ker}(\alpha) = \mathcal{Q}$. By (45), it is clear that $\mathcal{Q} = \{0\}$ if and only if (48) holds. Observe that for $p = 0$, $Q^0 = 0$ by definition. Moreover, we proved that $\mathcal{H}^0(\mathfrak{g} \otimes \mathcal{S}; V \otimes \mathcal{S}) = \mathcal{H}^0(\mathfrak{g}; V) \otimes \mathcal{S}$ (see Lemma 4.1) and that α is the identity map (see Claim 2). □

4.1. Current Lie algebras over semisimple Lie algebras. In the next result, we will determine the cohomology group $\mathcal{H}(\mathfrak{g} \otimes \mathcal{S}; V \otimes \mathcal{S})$, where $\mathfrak{g}$ is a semisimple Lie algebra and V is an irreducible $\mathfrak{g}$-module. It is a well known result that in this case $\mathcal{H}(\mathfrak{g}; V) = \{0\}$ (see [4, Theorem 24.1]). We shall use this fact in proving the following:

Proposition 4.5. *Let $\mathfrak{g}$ be a semisimple Lie algebra and V an irreducible $\mathfrak{g}$-module. Let $\mathcal{S}$ be a finite dimensional, associative and commutative algebra with unit over a field $\mathbb{F}$ of zero characteristic. Then $\mathcal{H}(\mathfrak{g} \otimes \mathcal{S}; V \otimes \mathcal{S}) = \mathcal{Q}$.*

Proof. As $\mathcal{H}(\mathfrak{g}; V) = \{0\}$, by Theorem 4.3, we get $\mathcal{H}(\mathfrak{g} \otimes \mathcal{S}; V \otimes \mathcal{S}) = \mathcal{Q}$. Now we shall verify this result without using Theorem 4.3.

Let $\Lambda + \mathcal{B}(\mathfrak{g} \otimes \mathcal{S}; V \otimes \mathcal{S})$ be in $\mathcal{H}(\mathfrak{g} \otimes \mathcal{S}; V \otimes \mathcal{S})$. Then $\mathrm{D}\,\Lambda = 0$ and $\widehat{\omega}_j \circ \mathrm{D}\,\Lambda = (\mathrm{D}\,\Lambda)_j = 0$ for all j. By Proposition 2.2, $0 = \mathcal{L}\left((\mathrm{D}\,\Lambda)_j\right) = \mathrm{d}\left(\mathcal{L}(\Lambda_j)\right)$, which implies that $\mathcal{L}(\Lambda_j)$ belongs to $\mathcal{Z} = \mathcal{B}$. Hence, there exists μ_j in $C(\mathfrak{g}; V)$ such that $\mathcal{L}(\Lambda_j) = \mathrm{d}\,\mu_j$. Then by (6), we have

$$\Lambda_j(x_1 \otimes 1, \ldots, x_p \otimes 1) = \mathrm{d}\,\mu_j(x_1, \ldots, x_p), \quad \text{for all } 1 \leq j \leq m. \tag{49}$$

Let Ω in $C(\mathfrak{g} \otimes \mathcal{S}; V \otimes \mathcal{S})$ be defined by

$$\Omega(x_1 \otimes t_1, \ldots, x_p \otimes t_p) = \sum_{j=1}^{m} \mu_j(x_1, \ldots, x_p) \otimes s_j\, \widetilde{t}, \tag{50}$$

where $\widetilde{t} = t_1 \cdots t_p$. We claim that

$$\mathrm{D}\,\Omega(x_1 \otimes t_1, \ldots, x_p \otimes t_p) = \sum_{j=1}^{m} \mathrm{d}\,\mu_j(x_1, \ldots, x_p) \otimes s_j\, \widetilde{t}. \tag{51}$$

Indeed, if we write Ω as in (5), we obtain

$$\Omega(x_1 \otimes t_1, \ldots, x_p \otimes t_p) = \sum_{j=1} \Omega_j(x_1 \otimes t_1, \ldots, x_p \otimes t_p) \otimes s_j. \tag{52}$$

Using (50), (52) and the definition of $\mathcal{L}$, it follows $\mathcal{L}(\Omega_j)(x_1, \ldots, x_p) = \Omega_j(x_1 \otimes 1, \ldots, x_p \otimes 1) = \mu_j(x_1, \ldots, x_p)$, then $\mathcal{L}(\Omega_j) = \mu_j$. From (50), this implies that

$$\Omega(x_1 \otimes t_1, \ldots, x_p \otimes t_p) = \sum_{j=1}^{m} \Omega_j(x_1 \otimes 1, \ldots, x_p \otimes 1) \otimes\ s\widetilde{t}. \tag{53}$$

On the other hand, by (8), we get

$$\mathcal{L}\left((\mathrm{D}\,\Omega)_j\right) = \mathrm{d}\,\mathcal{L}(\Omega_j) = \mathrm{d}\,\mu_j \quad \text{for all } 1 \leq j \leq m. \tag{54}$$

By Remark 3.1 and (53), we deduce that $\mathcal{T}(\mathrm{Id})(\Omega) = \Omega$. Then by Proposition 3.3, $\mathrm{D}\,\Omega = \mathrm{D}\,\mathcal{T}(\mathrm{Id})(\Omega) = \mathcal{T}(\mathrm{Id})(\mathrm{D}\,\Omega)$. Hence by (18) and (54),

$$\begin{aligned} \mathrm{D}\,\Omega(x_1 \otimes t_1, \ldots, x_p \otimes t_p) &= \mathcal{T}(\mathrm{Id})(\mathrm{D}\,\Omega)((x_1 \otimes t_1, \ldots, x_p \otimes t_p)) \\ &= \sum_{j=1}^{m} \mathcal{L}\left((\mathrm{D}\,\Omega)_j\right)(x_1, \ldots, x_p) \otimes s_j\, \widetilde{t} = \sum_{j=1}^{m} \mathrm{d}\,\mu_j(x_1, \ldots, x_p) \otimes s_j\, \widetilde{t}, \end{aligned}$$

which proves (51). Let $\Theta = \Lambda - \mathrm{D}\,\Omega$. By (49) and (51), we obtain

$$\begin{aligned} \Theta(x_1 \otimes 1, \ldots, x_p \otimes 1) &= \Lambda(x_1 \otimes 1, \ldots, x_p \otimes 1) - \mathrm{D}\,\Omega(x_1 \otimes 1, \ldots, x_p \otimes 1) \\ &= \sum_{j=1}^{m} \Lambda_j(x_1 \otimes 1, \ldots, x_p \otimes 1) \otimes s_j - \sum_{j=1}^{m} \mathrm{d}\,\mu_j(x_1, \ldots, x_p) \otimes s_j = 0. \end{aligned}$$

Then $\Theta + \mathcal{B}(\mathfrak{g} \otimes \mathcal{S}; V \otimes \mathcal{S})$ belongs to $\mathcal{Q}$ and $\mathcal{H}(\mathfrak{g} \otimes \mathcal{S}; V \otimes \mathcal{S}) = \mathcal{Q}$. □

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Rosendo García-Delgado
Center for Research in Mathematics A.C.
Mérida Campus, Yucatán, México
Carretera Sierra Papacal-Chuburna Puerto Km 5
97302 Sierra Papacal, Yucatán
e-mail: rosendo.garcia@cimat.mx