International Electronic Journal of Algebra
Published Online: July 21, 2025
DOI: 10.24330/ieja.1747271

A NOTE ON σ-NILPOTENCY OF FINITE GROUPS

Youxin Li, Xuecheng Zhong, Wei Meng and Jiakuan Lu

Received: 17 January 2025; Accepted: 31 May 2025
Communicated by Abdullah Harmancı

Abstract. Let $\sigma = \{\sigma_i \mid i \in I\}$ be some partition of the set $\mathbb{P}$ of all primes, and $\sigma(n) = \{\sigma_i \mid i \in I, \sigma_i \cap \pi(n) \neq \emptyset\}$ for any integer n. A group G is called σ-primary if either $G = 1$ or $|\sigma(G)| = 1$. A group G is σ-nilpotent if $(H/K) \rtimes (G/C_G(H/K))$ is σ-primary for every chief factor H/K of G. In this note, we prove that G is σ-nilpotent if and only if G is a σ-full group and $\pi(|xy|) = \pi(|x||y|)$ for any two elements $x, y \in G$ such that $\sigma(|x|) \cap \sigma(|y|) = \emptyset$.

Mathematics Subject Classification (2020): 20D10, 20D20
Keywords: Finite group, σ-nilpotent, Hall σ-set

1. Introduction

Throughout this note all considered groups are finite and G will always denote a finite group. It is well-known that if G is nilpotent, then $|ab| = |a||b|$ whenever $a, b \in G$ have co-prime orders, where $|x|$ denotes the order of x in G. Conversely, Baumslag and Wiegold [3] proved that G is nilpotent if $|ab| = |a||b|$ for any $a, b \in G$ with co-prime orders. Bastos and Shumyatsky [2] got a similar sufficient and necessary condition for the nilpotency of the commutator subgroup G'. Bastos, Monetta and Shumyatsky [1] proved that the kth term of the lower central series of a finite group G is nilpotent if and only if $|ab| = |a||b|$ for any k-commutators $a, b \in G$ of coprime orders. More results can be found in [6,7,8]. In this note, we shall generalizes these results to finite σ-nilpotent groups.

Let the symbol $\pi(n)$ denote the set of all primes dividing n; as usual, $\pi(G) = \pi(|G|)$, the set of all primes dividing the order of G. Let $\sigma = \{\sigma_i \mid i \in I\}$ be some partition of the set $\mathbb{P}$ of all primes, that is,

$$\mathbb{P} = \bigcup_{i \in I} \sigma_i \text{ and } \sigma_i \cap \sigma_j = \emptyset \text{ for all } i \neq j.$$

W. Meng is supported by National Natural Science Foundation of China (12161021), Center for Applied Mathematics of Guangxi (GUET) and Guangxi Colleges and Universities Key Laboratory of Data Analysis and Computation. J.K. Lu is supported by Guangxi Natural Science Foundation Program (2024GXNSFAA010514).

We put $\sigma(n) = \{\sigma_i \mid \sigma_i \cap \pi(n) \neq \emptyset, i \in I\}$ and $\sigma(G) = \sigma(|G|)$. Without loss of a generality, we always assume that $\sigma(G) = \{\sigma_1, \ldots, \sigma_t\}$.

A group G is called σ-primary [10] if either $G = 1$ or $|\sigma(G)| = 1$. G is σ-nilpotent [4] if $(H/K) \rtimes (G/C_G(H/K))$ is σ-primary for every chief factor H/K of G.

A set S of Sylow subgroups of G is called a complete set of Sylow subgroups of G if S contains exact one Sylow p-subgroup of G for every prime $p \in \pi(G)$. By analogy with it, we say that a set $\mathcal{H} = \{H_1, \ldots, H_t\}$ of Hall subgroups of G, where H_i is σ-primary ($i = 1, \ldots, t$), is a complete Hall σ-set of G if $\gcd(|H_i|, |H_j|) = 1$ for all $i \neq j$ and $\pi(G) = \pi(H_1) \cup \cdots \cup \pi(H_t)$ (see [11,12,13]). Following [4], a group G is a σ-full group if it possesses a complete Hall σ-set.

Our main results are as follows.

Theorem 1.1. *Let G be a finite group. Then G is σ-nilpotent if and only if G is a σ-full group and $\pi(|xy|) = \pi(|x||y|)$ for any two elements $x, y \in G$ such that*
$$\sigma(|x|) \cap \sigma(|y|) = \emptyset.$$

Applying Theorem 1.1, the following theorem is immediately as $|xy| = |x||y|$ implies that $\pi(|xy|) = \pi(|x||y|)$.

Theorem 1.2. *Let G be a finite group. Then G is σ-nilpotent if and only if G is a σ-full group and $|xy| = |x||y|$ for any two elements $x, y \in G$ such that*
$$\sigma(|x|) \cap \sigma(|y|) = \emptyset.$$

Remark 1.3. Let $G = A_5$ be the alternating group of degree 5 and
$$\sigma = \{\sigma_1 = \{2, 3\}, \sigma_2 = \{5\}, \ldots\}.$$
Then G is a σ-full group. Let $H_1 \cong A_4$ be a Hall σ_1-subgroup of G and H_2 be a Sylow 5-subgroup of G. Then $\mathcal{H} = \{H_1, H_2\}$ is a complete Hall σ-set of G. However, G is not σ-nilpotent.

All unexplained notations and terminologies are standard and can be found in [5,9,14].

2. Proof of Theorem

Lemma 2.1. [10] *A group G is σ-nilpotent if and only if G has a complete Hall σ-set $\mathcal{H} = \{H_1, \ldots, H_t\}$ such that $G = H_1 \times \cdots \times H_t$.*

Proof of Theorem 1.1. Suppose that G is a σ-nilpotent group, then G has a complete Hall σ-set $\mathcal{H} = \{H_1, \ldots, H_t\}$ such that $G = H_1 \times \cdots \times H_t$ by Lemma 2.1.

Thus, G is a σ-full group and $xy = yx$ for any two elements $x, y \in G$ such that $\sigma(|x|) \cap \sigma(|y|) = \emptyset$. By well-known results, we get that $|xy| = |x||y|$. In particular, $\pi(|xy|) = \pi(|x||y|)$.

Conversely, suppose that G is a σ-full group and satisfies the theorem hypothesis. Let $\mathcal{H} = \{H_1, \dots, H_t\}$ is a complete Hall σ-set of G. Without loss of a generality, we can assume that H_i is a Hall σ_i-subgroup of G, respectively.

First, we claim that $G = H_1 \cdots H_t$. We do this by counting the number of elements on the right-hand side. Suppose that h_i, k_i are elements of H_i for $i = 1, 2, \dots, t$, and suppose that we have an equality

$$h_1 h_2 \cdots h_t = k_1 k_2 \cdots k_t.$$

Write $x = k_1 \cdots k_{t-1}$ and $y = k_t h_t^{-1} \in H_t$, then $\sigma(|x|) \subseteq \{\sigma_1, \dots, \sigma_{t-1}\}$ and $\sigma(|y|) \subseteq \{\sigma_t\}$. Consequently, we see that $\sigma(|x|) \cap \sigma(|y|) = \emptyset$. Thus, if $y \neq 1$, then $\pi(|xy|) = \pi(|x||y|)$ by hypothesis. Let $g = xy$. Then $\pi(|g|) \cap \sigma_t \neq \emptyset$. On the other hand, observe that

$$g = xy = k_1 k_2 \cdots k_t h_t^{-1} = h_1 h_2 \cdots h_{t-1}.$$

Applying theorem hypothesis again, we know that $\pi(|g|) \cap \sigma_t = \emptyset$ which is a contradiction. So we get $y = 1$ and hence $h_t = k_t$. By induction, we get $h_i = k_i$ for any $i = 1, \dots, k$. Thus, a count of the number of elements in the product $H_1 H_2 \cdots H_t$ shows that there are as many of them as there are elements in G. Therefore, we get that $G = H_1 H_2 \cdots H_t$.

In the following, we show that every subgroup H_i is normal in G. For any $h \in H$, $g \in G$, we have $|h^g| = |h|$, in particular, $\pi(|h^g|) = \pi(|h|) \subseteq \sigma_1$. By above arguments, we know that $G = H_1 H_2 \cdots H_t$. So we can let

$$h^g = h_1 h_2 \cdots h_t, \text{ where } h_i \in H_i \text{ for any } i = 1, 2, \dots, t.$$

Since $|h^g|$ is a σ_1-number, the hypothesis implies that

$$\pi(|h_1||h_2| \cdots |h_t|) = \pi(|h^g|) = \pi(|h|) \subseteq \sigma_1.$$

This leads to $\pi(|h_2|) = \ldots = \pi(|h_t|) = \emptyset$ and hence $h_2 = \ldots = h_t = 1$. So we get $h^g = h_1 \in H_1$. Therefore, H_1 is normal in G. Moreover, we can get that every H_i is normal in G. Hence $G = H_1 \times H_2 \times \cdots \times H_t$. Applying Lemma 2.1, we know that G is σ-nilpotent. The proof of theorem is completed. □

Acknowledgement. The authors would like to thank the referee for the valuable suggestions and comments.

Disclosure statement. The authors report there are no competing interests to declare.

References

[1] R. Bastos, C. Monetta and P. Shumyatsky, *A criterion for metanilpotency of a finite group*, J. Group Theory, 21(4) (2018), 713-718.

[2] R. Bastos and P. Shumyatsky, *A sufficient condition for nilpotency of the commutator subgroup*, Sib. Math. J., 57(5) (2016), 762-763.

[3] B. Baumslag and J. Wiegold, *A suffcient condition for nilpotency in a finite group*, (2014), arXiv:1411.2877 [math.GR].

[4] W. Guo and A. N. Skiba, *Finite groups with permutable complete Wielandt sets of subgroups*, J. Group Theory, 18(2) (2015), 191-200.

[5] B. Huppert, Endliche Gruppen I, Die Grundlehren der mathematischen Wissenschaften, 134, Springer-Verlag, Berlin-New York, 1967.

[6] X. Li, D. Lei and Y. Gao, *The order of the product of two elements*, Indian J. Pure Appl. Math., 53(2) (2022), 372-374.

[7] V. S. Monakhov, *The nilpotency criterion for the derived subgroup of a finite group*, Probl. Fiz. Mat. Tekh., 3(32) (2017), 58-60.

[8] V. S. Monakhov, *A metanilpotency criterion for a finite solvable group*, Proc. Steklov Inst. Math., 304(suppl. 1) (2019), 141-143.

[9] D. J. S. Robinson, A Course in the Theory of Groups, Second edition, Graduate Texts in Mathematics, 80, Springer-Verlag, New York, 1996.

[10] A. N. Skiba, *On σ-subnormal and σ-permutable subgroups of finite groups*, J. Algebra, 436 (2015), 1-16.

[11] A. N. Skiba, *A generalization of a Hall theorem*, J. Algebra Appl., 15(5) (2016), 1650085 (13 pp).

[12] A. N. Skiba, *On some results in the theory of finite partially soluble groups*, Commun. Math. Stat., 4(3) (2016), 281-309.

[13] A. N. Skiba, *Some characterizations of finite σ-soluble $P\sigma T$-groups*, J. Algebra, 495 (2018), 114-129.

[14] M. Suzuki, Group Theory II, Fundamental Principles of Mathematical Sciences, 248, Springer-Verlag, New York, 1986.

Youxin Li
School of Electronic Engineering and Automation
Guilin University of Electronic Technology
Guilin, Guangxi, 541002, P.R. China
e-mail: li15333760843@163.com
Xuecheng Zhong and **Wei Meng** (Corresponding Author)
School of Mathematics and Computing Science
Guilin University of Electronic Technology
Guilin, Guangxi, 541002, P.R. China
emails: 782144967@qq.com (X. Zhong)
mlwhappyhappy@163.com (W. Meng)
Jiakuan Lu
School of Mathematics and Statistics
Guangxi Normal University
Guilin, Guangxi, 541004, P.R. China
e-mail: jklu@gxnu.edu.cn