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ON REGULARLY COHERENT MODULES AND REGULARLY NOETHERIAN MODULES

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Abstract. The concepts of regular Noetherianity and regular coherence, which extend the classical notions of Noetherian and coherent rings, have been fundamental in the study of algebraic structures. In this paper, we aim to expand these notions to the realm of module theory. Specifically, we introduce and explore weak versions of injective, flat, and projective modules, which we term as reg-injective, reg-flat, and reg-projective modules. We provide analogues of classical results and establish their properties, offering examples to illustrate modules that meet these new criteria but not their classical counterparts. Additionally, we define and study regularly Noetherian and regularly coherent modules, characterizing their properties and examining their stability under various ring constructions. Our results contribute new examples and broaden the current understanding of these algebraic concepts.

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1. Introduction

All rings considered in this paper are assumed to be commutative with non-zero identity. We use $\mathrm{Nil}(R)$ to denote the set of nilpotent elements of R and $Z(R)$ to denote the set of zero-divisors of R. The multiplicative subsets S considered in this paper do not intersect the set of zero-divisors. An R-module M is called a torsion module if for every $x \in M$, there exists $s \in R \setminus Z(R)$ such that $sx = 0$. An R-module M is said to be uniformly torsion (or u-torsion for short) if there exists $s \in R \setminus Z(R)$ such that $sM = 0$. An R-submodule N of an R-module M is called a reg-submodule if M/N is a torsion R-module. Recall from [8] that a ring R is said to be Noetherian (resp., coherent) if every ideal of R is finitely generated (resp.,

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every finitely generated ideal of R is finitely presented). An R-module M is said to be Noetherian if M is a finitely generated R-module and every submodule of M is finitely generated. An R-module M is said to be coherent if M is a finitely generated R-module and every finitely generated submodule of M is finitely presented.

The notion of coherence of rings, which we defined in the first paragraph of this introduction, generalizes Noetherian rings and other important classes of rings defined by finiteness conditions. Many algebraists have studied coherent rings in terms of various modules. In 1960, S. U. Chase [2, Theorem 2.1] showed that a ring is coherent if and only if the class of flat modules is closed under direct products. In 1970, B. Stenström [12, Theorem 3.2] demonstrated that a ring is coherent if and only if every direct limit of absolutely pure modules is absolutely pure. In 1982, E. Matlis [10, Theorem 1] proved that a ring R is coherent if and only if $\mathrm{Hom}_R(M, E)$ is flat for any injective modules M and E.

Recall from [6, Definition 2.2.25] that a ring is called regularly Noetherian (or reg-Noetherian for short) if every regular ideal is finitely generated. Additionally, according to [16, Definition 3.1], a ring is termed regularly coherent (or reg-coherent for short) if every finitely generated regular ideal is finitely presented. Later, M. Chhiti and S. E. Mahdou [4, 5] further studied these two notions. Specifically, they investigated the stability of these properties under localization and homomorphic images, as well as their transfer through various ring constructions such as trivial ring extensions, pullbacks, and amalgamated duplication along an ideal. Their results produced examples that contribute new and original families of rings satisfying these properties, thereby enriching the existing literature.

To advance this study, Section 2 introduces and examines the weak versions of injective, flat, and projective modules, defined as follows: An R-module E is called reg-injective if E satisfies the definition of injective modules for inclusions $0 \to A \to B$, where A is a reg-submodule of B. We then establish an analogue of the Bear criterion using regular ideals to characterize reg-injective modules. An example of a reg-injective module that is not injective is given in Example 2.12.

An R-module M is called reg-flat if the functor $-\otimes_R M$ preserves exact sequences $0 \to A \to B$, where A is a reg-submodule of B. Theorem 2.16 characterizes reg-flat modules similarly to the classical case. Example 2.21 provides an example of a reg-flat module that is not flat.

Reg-projective modules are defined as those M satisfying $\mathrm{Ext}^1_R(M, N) = 0$ for every u-torsion R-module N. Theorems 2.25 and 2.26 establish that reg-projective modules are reg-flat, and that finitely presented reg-flat modules are reg-projective. Remark 2.27 gives an example of a reg-projective module that is not projective.

In Sections 3 and 4, we introduce and study two new classes of modules called reg-Noetherian and reg-coherent. An R-module M is said to be reg-Noetherian if M is finitely generated and every reg-submodule of M is finitely generated. An R-module M is said to be reg-coherent if M is finitely generated and each finitely generated reg-submodule of M is finitely presented. In these sections, we provide properties that characterize these modules and explore their relationships and implications.

2. On reg-specific modules

We begin this section with the following definition.

Definition 2.1. Let R be a ring and M be an R-module. An R-submodule N of M is said to be a reg-submodule of M if M/N is a torsion R-module. This means that for every $x \in M$, there exists $s \in R \setminus Z(R)$ such that $sx \in N$.

Recall from [17] that an R-module is said to be a uniformly S-torsion R-module, where S is a multiplicative subset of R, if $sM = 0$ for some $s \in S$. In particular, when $S = R \setminus Z(R)$, a u-S-torsion R-module will be referred to as a u-torsion R-module in this paper.

Proposition 2.2. *Let R and T be rings, M be an $(R$-$T)$-bimodule, and N be a T-module. If M is a u-torsion R-module, then so is* $\mathrm{Hom}_T(M, N)$.

Proof. We can immediately establish that the following map

$$\varphi : \begin{array}{rcl} R \times \mathrm{Hom}_T(M, N) & \longrightarrow & \mathrm{Hom}_T(M, N) \\ (s, f) & \longmapsto & sf \end{array}$$

such that $sf(x) = f(sx)$ for every $x \in M$, is a modulation of $\mathrm{Hom}_T(M, N)$ over R. Since M is a u-torsion R-module, there exists some $s \in R \setminus Z(R)$ such that $sM = 0$, and therefore $s\mathrm{Hom}_T(M, N) = 0$, as desired. □

Denote $M^+ := \mathrm{Hom}_{\mathbb{Z}}(M, \mathbb{Q}/\mathbb{Z})$ as the character module of a module M.

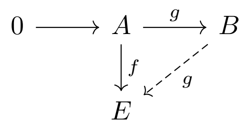
Corollary 2.3. *Let R be a ring and M be an R-module. If M is a u-torsion R-module, then so is M^+.*

Proof. This follows immediately from Proposition 2.2. □

Example 2.4. For every regular ideal I of a ring R, the R-module $\frac{R}{I}$ is a u-torsion R-module.

A weak version of injectivity is defined as follows:

Definition 2.5. An R-module E is said to be *reg-injective* if given R-modules $A \subset B$, where A is a reg-submodule of B, and a homomorphism $f : A \to E$, there exists a homomorphism $g : B \to E$ such that $g|_A = f$, that is, such that



is a commutative diagram.

Remark 2.6.
(1) It is easy to see that every injective module is reg-injective. We will see in Example 2.12 below that the converse is not true in general.
(2) The concept of reg-injective modules was first introduced in [15] as follows: An R-module E is called reg-injective if $\mathrm{Ext}^1_R(N, M) = 0$ for every torsion R-module N.

The following Theorem 2.7 is an analogue of the well-known Baer's Criterion. This is actually the definition of reg-injective modules as given in [11, Definition 2.1].

Theorem 2.7. (Reg-Baer's Criterion) *Let R be a ring. An R-module E is reg-injective if and only if for all regular ideals I of R, every homomorphism $f : I \to E$ can be extended to R.*

Proof. The necessity follows immediately from Example 2.4 and Definition 2.5. We now prove the sufficiency. Let $A \subset B$ be R-modules such that B/A is a torsion R-module, and let $f : A \to E$ be a homomorphism.

Consider the collection $\mathscr{C}$ of all pairs (C, g), where $A \subset C \subset B$, C is a reg-submodule of B, and $g : C \to E$ extends f, i.e., $g|_A = f$. This collection is nonempty since $(A, f) \in \mathscr{C}$. We partially order $\mathscr{C}$ by $(C, g) \leq (C', g')$ if $C \subset C'$ and $g'|_C = g$. By Zorn's Lemma, $\mathscr{C}$ has a maximal element (C_0, g_0).

Suppose $C_0 \neq B$. Choose $x \in B \setminus C_0$ and define $I = \{r \in R : rx \in C_0\}$. Since B/C_0 is a torsion R-module, there exists $s \in R \setminus Z(R)$ such that $sx \in C_0$. Thus, $s \in I$, and I is a regular ideal of R.

Define a map $h : I \to E$ by $h(r) = g_0(rx)$. Since h is a homomorphism, it can be extended to a homomorphism $h' : R \to E$ by assumption and Example 2.4. Now, define $\bar{g} : C_0 + Rx \to E$ by

$$\bar{g}(c_0 + rx) = g_0(c_0) + h'(r).$$

To show that $\bar{g}$ is well-defined, suppose $c_0+rx = c_0'+r'x$. Then $c_0-c_0' = (r'-r)x$, and since $r'-r \in I$, we obtain

$$g_0(c_0-c_0') = g_0((r'-r)x) = h(r'-r) = h'(r'-r).$$

Thus, $g_0(c_0)+h'(r) = g_0(c_0')+h'(r')$, proving that $\bar{g}$ is well-defined.

Furthermore, $\bar{g}(a) = g_0(a) = f(a)$ for all $a \in A$, so $(C_0+Rx, \bar{g}) \in \mathscr{C}$. This contradicts the maximality of (C_0, g_0) since $C_0 \subsetneqq C_0+Rx$. Hence, we conclude that $C_0 = B$. □

From Definition 2.5 and Theorem 2.7, we can easily state the following corollaries.

Corollary 2.8. *An R-module E is reg-injective if and only if $Ext_R^1\left(\frac{R}{I}, E\right) = 0$ for every regular ideal I of R.*

Proof. This follows directly from the short exact sequence $0 \to I \to R \to \frac{R}{I} \to 0$. □

Corollary 2.9. *Let R be a ring and $\{E_i\}_{i\in\Gamma}$ be a family of R-modules. Then $\prod_{i\in\Gamma} E_i$ is a reg-injective R-module if and only if for every $i \in \Gamma$, E_i is a reg-injective R-module.*

Proof. This follows straightforwardly from Corollary 2.8 and the natural isomorphism

$$\mathrm{Ext}_R^1(R/I, \prod_{i\in\Gamma} E_i) \cong \prod_{i\in\Gamma} \mathrm{Ext}_R^1(R/I, E_i),$$

for every regular ideal I of R. □

Recall that in classical homology, every injective module is divisible. The following Theorem 2.10 provides an analogue of the well-known relationship between reg-injectivity and the divisibility of modules.

Theorem 2.10. *Let R be a ring and E be an R-module. If E is reg-injective, then E is a divisible R-module.*

Proof. Let E be a reg-injective module and let r be a non-zero-divisor in R. If $r \in U(R)$, then clearly $E = rE$. Suppose instead that $r \notin U(R)$. In this case, Rr is a proper regular ideal of R. Since E is reg-injective, we have

$$\mathrm{Ext}_R^1\left(\frac{R}{Rr}, E\right) = 0.$$

By [14, Exercise 3.2 (1)], it follows that $E = rE$, which shows that E is a divisible module, as desired. □

Recall from [6, Definition 2.2.19] that a ring R is said to be a reg-principal ideal ring if every regular ideal of R is principal.

In classical homology, if R is a principal ideal domain, then every divisible module is injective. The following Theorem 2.11 establishes the analog of this result.

Theorem 2.11. *Let R be a reg-principal ideal ring. Then the following statements are equivalent for an R-module E:*

(1) *E is a reg-injective module.*
(2) *E is a divisible module.*

Proof. (1) $\Rightarrow$ (2) This follows directly from Theorem 2.10.

(2) $\Rightarrow$ (1) Let I be a regular ideal of R. Then we can write $I = Rr$ for some regular element $r \in R$. Since E is divisible, we have $E/rE = 0$, which implies

$$\mathrm{Ext}^1_R(R/Rr, E) = 0$$

by [14, Exercise 3.2 (1)]. Hence, E is reg-injective. □

Let R be a ring and M an R-module. We denote by $R \propto M$ the trivial extension (or idealization) of M over R.

Now, we present an example of a reg-injective module that is not injective.

Example 2.12. If $R = \mathbb{Z} \propto \mathbb{Q}$, then the nilradical of R, given by $\mathrm{Nil}(R) := 0 \propto \mathbb{Q}$, provides an example of a reg-injective module that is not injective.

Proof. It is straightforward to see that R is a reg-principal ideal ring with $\mathrm{Nil}(R) = Z(R)$, which forms a divisible ideal of R. Consequently, $\mathrm{Nil}(R)$ is a divisible R-module and, by Theorem 2.11, it follows that $\mathrm{Nil}(R)$ is reg-injective.

Suppose, for contradiction, that $\mathrm{Nil}(R)$ is an injective module. Then the exact sequence

$$0 \to \mathrm{Nil}(R) \to R \to \mathbb{Z} \to 0$$

would split, implying that $\mathrm{Nil}(R)$ is a projective ideal. However, this contradicts [14, Proposition 6.7.12]. Thus, $\mathrm{Nil}(R)$ is not injective. □

It is worth mentioning that X. Zhang provided a complete characterization of rings over which every reg-injective module is injective (see [18, Theorem 4.3]).

Now, we introduce a new generalization of the classical flatness.

Definition 2.13. Let R be a ring. An R-module F is said to be reg-flat if given any exact sequence $0 \to A \to B$ of R-modules such that B/A is a torsion R-module, the tensored sequence $0 \to A \otimes_R F \to B \otimes_R F$ is exact.

Remark 2.14. (1) It is easy to see that every flat module is reg-flat.

(2) The concept of reg-flat modules was first introduced in [16] as follows: An R-module F is called reg-flat if for any regular ideal I of R, the sequence $0 \to F \otimes_R I \to F \otimes_R I$ is exact. Equivalently, $\mathrm{Tor}_1^R(F, R/I) = 0$.

The following Theorem 2.15 generalizes [14, Theorem 2.5.5] and characterizes reg-flatness by reg-injectivity.

Theorem 2.15. *Let R and T be rings, and let M be an (R,T)-bimodule. Then M is a reg-flat R-module if and only if $\mathrm{Hom}_T(M,E)$ is a reg-injective R-module for every injective T-module E.*

Proof. The sufficiency is established in [16, Theorem 2.11], but we provide a proof for completeness.

Assume that M is a reg-flat R-module, and let $A \subset B$ be a submodule of an R-module B such that B/A is torsion. Since M is reg-flat, the sequence

$$0 \to A \otimes_R M \to B \otimes_R M$$

is exact. Consider the following commutative diagram:

$$\begin{array}{ccccc} \mathrm{Hom}_T(B \otimes_R M, E) & \longrightarrow & \mathrm{Hom}_T(A \otimes_R M, E) & \longrightarrow & 0 \\ \downarrow\cong & & \downarrow\cong & & \\ \mathrm{Hom}_R(B, \mathrm{Hom}_T(M,E)) & \longrightarrow & \mathrm{Hom}_R(A, \mathrm{Hom}_T(M,E)) & \longrightarrow & 0. \end{array}$$

Since E is an injective T-module, the top row is exact, implying that $\mathrm{Hom}_T(M,E)$ is a reg-injective R-module.

Conversely, assume that $\mathrm{Hom}_T(M,E)$ is reg-injective for every injective T-module E. Let $A \subset B$ be a submodule such that B/A is torsion. Define $K = \mathrm{Ker}(A \otimes_R M \to B \otimes_R M)$. Then we have the exact sequence

$$0 \to K \to A \otimes_R M \to B \otimes_R M.$$

Consider the following commutative diagram:

$$\begin{array}{ccccccc} \mathrm{Hom}_R(B \otimes_R M, E) & \longrightarrow & \mathrm{Hom}_R(A \otimes_R M, E) & \longrightarrow & \mathrm{Hom}_R(K,E) & \longrightarrow & 0 \\ \downarrow\cong & & \downarrow\cong & & \downarrow & & \\ \mathrm{Hom}_R(B, \mathrm{Hom}_T(M,E)) & \longrightarrow & \mathrm{Hom}_R(A, \mathrm{Hom}_T(M,E)) & \longrightarrow & 0 & \longrightarrow & 0. \end{array}$$

Since E is an injective T-module, the top row is exact. By assumption, $\mathrm{Hom}_R(M,E)$ is reg-injective, ensuring that the bottom row is also exact. Consequently, $\mathrm{Hom}_R(K,E) = 0$, which implies $K = 0$. Thus, the sequence

$$0 \to A \otimes_R M \to B \otimes_R M$$

is exact, proving that M is reg-flat. □

The following Theorem 2.16 completely characterizes reg-flatness.

Theorem 2.16. *Let R be a ring. The following conditions are equivalent for an R-module M:*

(1) *M is reg-flat.*
(2) $\mathrm{Tor}_1^R(P, M) = 0$ *for every torsion R-module P.*
(3) $\mathrm{Tor}_1^R(R/I, M) = 0$ *for every regular ideal I of R.*
(4) $0 \to I \otimes_R M \to R \otimes_R M$ *is an exact sequence for every regular ideal I of R.*
(5) $I \otimes_R M \cong IM$ *for every regular ideal I of R.*
(6) $- \otimes_R M$ *is exact for every exact R-sequence $0 \to N \to F \to C \to 0$ where N, F, C are finitely generated, C is a torsion R-module, and F is free.*
(7) $- \otimes_R M$ *is exact for every exact R-sequence $0 \to N \to F \to C \to 0$ where C is a torsion R-module, and F is free.*
(8) $\mathrm{Tor}_1^R(R/I, M) = 0$ *for every finitely generated regular ideal I of R.*
(9) $0 \to I \otimes_R M \to R \otimes_R M$ *is an exact sequence for every finitely generated regular ideal I of R.*
(10) $I \otimes_R M \cong IM$ *for every finitely generated regular ideal I of R.*
(11) $\mathrm{Ext}_R^1(R/I, M^+) = 0$ *for every regular ideal I of R.*
(12) *Let $0 \to K \to F \xrightarrow{g} M \to 0$ be an exact sequence of R-modules, where F is free. Then $K \cap FI = IK$ for every regular ideal I of R.*
(13) *Let $0 \to K \to F \xrightarrow{g} M \to 0$ be an exact sequence of R-modules, where F is free. Then $K \cap FI = IK$ for every finitely generated regular ideal I of R.*
(14) *Let $\xi : 0 \to A \to B \to M \to 0$ be an exact sequence. Then, for any regular ideal I of R, the sequence $R/I \otimes \xi$ remains exact.*
(15) *There exists an exact sequence $\xi : 0 \to A \to P \to M \to 0$, where P is a projective R-module, such that for any regular ideal I of R, the sequence $R/I \otimes \xi$ remains exact.*

Proof. (1) ⇒ (2) Assume that M is a reg-flat module, and let P be a torsion R-module. Then there exists a short exact sequence of R-modules

$$0 \to A \to F \to P \to 0,$$

where F is a free R-module. Since M is reg-flat, the sequence

$$0 \to A \otimes_R M \to F \otimes_R M$$

is exact. Applying the Tor functor to $0 \to A \to F \to P \to 0$ gives

$$0 \to \mathrm{Tor}_1^R(P, M) \to A \otimes_R M \to F \otimes_R M.$$

By [14, Theorem 1.9.9 (Five Lemma)], it follows that $\mathrm{Tor}_1^R(P, M) = 0$.

(2) $\Rightarrow$ (3), (3) $\Leftrightarrow$ (4), and (4) $\Leftrightarrow$ (5) follow similarly to the classical case.

(4) $\Rightarrow$ (1) For any injective R-module E, consider the following commutative diagram:

$$\begin{array}{ccccc}
\mathrm{Hom}_R(R \otimes_R M, E) & \longrightarrow & \mathrm{Hom}_R(I \otimes_R M, E) & \longrightarrow & 0 \\
\downarrow\cong & & \downarrow\cong & & \\
\mathrm{Hom}_R(B, \mathrm{Hom}_T(M, E)) & \longrightarrow & \mathrm{Hom}_R(A, \mathrm{Hom}_T(M, E)) & \longrightarrow & 0.
\end{array}$$

By assumption, the top row is exact, implying that the bottom row is exact. By Theorem 2.7, $\mathrm{Hom}_R(M, E)$ is an injective module. By Theorem 2.15, M is reg-flat.

(2) $\Leftrightarrow$ (7) and (7) $\Rightarrow$ (6) follow similarly to the classical case.

(6) $\Rightarrow$ (8) Let I be a finitely generated regular ideal of R, and consider the short exact sequence

$$0 \to I \to R \to \frac{R}{I} \to 0.$$

Applying the Tor functor to the above sequence and using the hypothesis, we obtain the exact sequences

$$0 \to \mathrm{Tor}_1^R(\frac{R}{I}, M) \to I \otimes_R M \to R \otimes_R M$$

and

$$0 \to I \otimes_R M \to R \otimes_R M.$$

By [14, Theorem 1.9.9 (Five Lemma)], it follows that (8) holds.

(8) $\Leftrightarrow$ (9) and (9) $\Leftrightarrow$ (10) follow similarly to the classical case.

(10) $\Rightarrow$ (5) Let I be a regular ideal, and consider the natural R-epimorphism

$$\sigma_I : I \otimes_R M \to IM$$

given by $\sigma_I(a \otimes x) = ax$ for all $(a, x) \in I \times M$. To show that σ_I is an R-monomorphism, let $\sum_{i=1}^n a_i \otimes x_i \in \ker(\sigma_I)$, and consider a finitely generated regular ideal J containing the a_i as generators for all $1 \leq i \leq n$. Then the restriction of σ_I over J is an R-monomorphism by assumption, implying that

$$\sum_{i=1}^n a_i \otimes x_i = 0,$$

which shows that σ_I is an R-monomorphism. Therefore, (5) holds.

(11) ⇔ (3) This follows from the natural isomorphism

$$\mathrm{Tor}_1^R(\frac{R}{I}, M)^+ \cong \mathrm{Ext}_R^1(\frac{R}{I}, M^+).$$

(3) ⇔ (12) and (12) ⇔ (13) follow similarly to the classical case.

(4) ⇔ (14) ⇔ (15) See [16, Proposition 2.12]. □

We remark that the equivalences of (1), (2), (3), (4), (5), (8), (10), and (11) in Theorem 2.16 are proved in [16, Proposition 2.12].

Corollary 2.17. *Let R be a ring with a multiplicative subset S. If M is a reg-flat R-module, then $S^{-1}M$ is a reg-flat $S^{-1}R$-module.*

Proof. First, it is easy to verify that an ideal I of R is regular if and only if $S^{-1}I$ is a regular ideal of $S^{-1}R$. The proof of this corollary then follows immediately from Theorem 2.16 and [14, Theorem 3.4.12]. □

Remark 2.18. It is easy to see that a ring R satisfies $Z(R) = \mathrm{Nil}(R)$ if and only if $(R \setminus \mathfrak{p}) \cap Z(R) = \emptyset$ for all $\mathfrak{p} \in \mathrm{Spec}(R)$.

Corollary 2.19. *Let R be a ring with $\mathrm{Nil}(R) = Z(R)$ and M be an R-module. The following statements are equivalent:*

(1) *M is reg-flat.*
(2) *$M_{\mathfrak{p}}$ is reg-flat over $R_{\mathfrak{p}}$ for every prime ideal $\mathfrak{p}$ of R.*
(3) *$M_{\mathfrak{m}}$ is reg-flat over $R_{\mathfrak{m}}$ for every maximal ideal $\mathfrak{m}$ of R.*

Proof. By Corollary 2.17, it suffices to prove that (3) ⇒ (2). Let I be a regular ideal of R, and let $\mathfrak{m}$ be a maximal ideal of R. Since R is assumed to be a ring with $\mathrm{Nil}(R) = Z(R)$, it follows that $I_{\mathfrak{m}}$ remains a regular ideal in $R_{\mathfrak{m}}$. Consequently, we obtain the following short exact sequence:

$$0 \to I_{\mathfrak{m}} \otimes_{R_{\mathfrak{m}}} M_{\mathfrak{m}} \to R_{\mathfrak{m}} \otimes_{R_{\mathfrak{m}}} M_{\mathfrak{m}} \to \left(\frac{R}{I}\right)_{\mathfrak{m}} \otimes_{R_{\mathfrak{m}}} M_{\mathfrak{m}} \to 0.$$

By [14, Theorem 1.5.21], this short exact sequence induces the exact sequence

$$0 \to I \otimes_R M \to R \otimes_R M \to \frac{R}{I} \otimes_R M \to 0.$$

Thus, M is a reg-flat R-module, as required. □

Theorem 2.20. *Let $f : R \to T$ be a surjective ring homomorphism such that $f(Z(R)) \subset Z(T)$, and let M be a T-module. If M is a reg-flat R-module, then $M \otimes_R T$ is a reg-flat T-module.*

Proof. Let X be a torsion T-module and let $x \in X$. Then there exists $s \in T \setminus Z(T)$ such that $sx = 0$. Since the surjectivity of f implies that $f(n) = s$ for some $n \in R$, and $f(Z(R)) \subset Z(T)$, we have $n \in R \setminus Z(R)$ and so $nx = 0$, which means X is a torsion R-module.

Let $0 \to A \to B \to X \to 0$ be a short exact sequence of T-modules. This induces the short exact sequence of R-modules:

$$0 \to A \otimes_T T \to B \otimes_T T \to X \otimes_T T \to 0.$$

Since $X \otimes_T T$ is a torsion R-module and M is a reg-flat R-module, from Theorem 2.16 we get:

$$0 \to A \otimes_T T \otimes_R M \to B \otimes_T T \otimes_R M \to X \otimes_T T \otimes_R M \to 0,$$

which is exact as a sequence of T-modules. This implies that $M \otimes_R T$ is a reg-flat T-module. □

Now, we provide an example of a reg-flat module that is not flat. In [16, Remark 2.14], an example of a reg-flat module that is not flat is provided implicitly.

Example 2.21. Let K be a field. Then $\mathrm{Nil}(K \propto K) = 0 \propto K$ is an example of a reg-flat $(K \propto K)$-module that is not flat. In fact, the only regular ideal of $K \propto K$ is $K \propto K$ itself. Thus, $\mathrm{Nil}(K \propto K)$ is trivially reg-flat. However, $K \propto K$ is a Noetherian ring. Therefore, $\mathrm{Nil}(K \propto K)$ is a finitely presented module. If $\mathrm{Nil}(K \propto K)$ were flat, then it would be projective, which contradicts [14, Proposition 6.7.12].

Definition 2.22. Let R be a ring. An R-module P is said to be reg-projective if $\mathrm{Ext}^1_R(P, N) = 0$ for every u-torsion R-module N. In particular, every projective module is reg-projective.

Proposition 2.23. *Let R be a ring and $\{P_i\}_{i \in I}$ be a family of R-modules. Then $\bigoplus_{i \in I} P_i$ is reg-projective if and only if every P_i is reg-projective.*

Proof. Let N be a u-torsion module. We have the following R-isomorphism:

$$\mathrm{Ext}^1_R(\bigoplus_{i \in I} P_i, N) \cong \prod_{i \in I} \mathrm{Ext}^1_R(P_i, N).$$

Therefore, $\bigoplus_{i \in I} P_i$ is reg-projective if and only if every P_i is reg-projective. □

Next, Theorem 2.24 provides an analogue of the well-known characterization of projective modules by short exact sequences [14, Theorem 2.3.3].

Theorem 2.24. *Let R be a ring. Then the following statements hold for an R-module P:*

(1) *If P is reg-projective, then every exact sequence $0 \to A \to B \xrightarrow{g} P \to 0$ is split for every u-torsion R-module A.*
(2) *P is reg-projective if and only if every exact sequence $0 \to A \to B \to C \to 0$, where A is a u-torsion R-module, induces the exact sequence*

$$0 \to \mathrm{Hom}_R(P, A) \to \mathrm{Hom}_R(P, B) \to \mathrm{Hom}_R(P, C) \to 0.$$

Proof. (1) Let $0 \to A \to B \to P \to 0$ be an exact sequence where A is u-torsion. Since P is reg-projective, $\mathrm{Ext}^1_R(P, A) = 0$. Hence, we have the exact sequence

$$0 \to \mathrm{Hom}_R(P, A) \to \mathrm{Hom}_R(P, B) \longrightarrow \mathrm{Hom}_R(P, P) \to 0,$$

showing that the sequence is split.

(2) Assume P is reg-projective. Then $\mathrm{Ext}^1_R(P, A) = 0$ for any u-torsion module A, giving us the exact sequence

$$0 \to \mathrm{Hom}_R(P, A) \to \mathrm{Hom}_R(P, B) \to \mathrm{Hom}_R(P, C) \to 0.$$

Conversely, assume that every short exact sequence $0 \to A \to B \to C \to 0$, where A is u-torsion, induces the exact sequence

$$0 \to \mathrm{Hom}_R(P, A) \to \mathrm{Hom}_R(P, B) \to \mathrm{Hom}_R(P, C) \to 0.$$

We claim P is reg-projective. Let A be a u-torsion module and E an injective module containing A. By hypothesis, the short exact sequence $0 \to A \to E \to E/A \to 0$ induces

$$0 \to \mathrm{Hom}_R(P, A) \to \mathrm{Hom}_R(P, E) \to \mathrm{Hom}_R(P, E/A) \to 0.$$

The commutative diagram with exact rows

$$\begin{array}{ccccccc}
\mathrm{Hom}_R(P, E) & \longrightarrow & \mathrm{Hom}_R(P, E/A) & \longrightarrow & \mathrm{Ext}^1_R(P, A) & \longrightarrow & 0 \\
\| & & \| & & \downarrow \cong & & \\
\mathrm{Hom}_R(P, E) & \longrightarrow & \mathrm{Hom}_R(P, E/A) & \longrightarrow & 0 & \longrightarrow & 0
\end{array}$$

implies $\mathrm{Ext}^1_R(P, A) = 0$, hence P is reg-projective. □

In classical homology, every projective module is flat. The following Theorem 2.25 shows the same result in the context of reg-projective modules.

Theorem 2.25. *Let R be a ring and P be an R-module. If P is a reg-projective module, then P is reg-flat.*

Proof. Let P be a reg-projective module and I be a regular ideal of R. By Corollary 2.3 and Example 2.4, we get $(R/I)^+$ is a u-torsion R-module and so $\mathrm{Ext}^1_R(P,(R/I)^+) = 0$. Using the duality formula $\mathrm{Ext}^1_R(P,(R/I)^+) \cong \mathrm{Tor}^R_1(P,R/I)^+$, we immediately get $\mathrm{Tor}^R_1(P,R/I) = 0$, as desired. Thus, P is reg-flat by Theorem 2.16. □

In classical homology, every finitely presented flat module is projective. The following Theorem 2.26 shows the dual of this result.

Theorem 2.26. *Let R be a ring. Then every finitely presented reg-flat module is reg-projective.*

Proof. Let F be a finitely presented reg-flat R-module, and consider an exact sequence of R-modules

$$B \to C \to 0,$$

where $K = \ker(B \to C)$ is u-torsion. We aim to show that the sequence

$$\mathrm{Hom}_R(F,B) \to \mathrm{Hom}_R(F,C) \to 0$$

is exact, or equivalently, by [7, Lemma 3.2.8], that the sequence

$$0 \to \mathrm{Hom}_R(F,C)^+ \to \mathrm{Hom}_R(F,B)^+$$

is exact. Consider the following commutative diagram:

$$\begin{array}{ccccc} 0 & \longrightarrow & F\otimes_R C^+ & \longrightarrow & F\otimes_R B^+ \\ & & \downarrow \cong & & \downarrow \cong \\ 0 & \longrightarrow & \mathrm{Hom}_R(F,C)^+ & \longrightarrow & \mathrm{Hom}_R(F,B)^+. \end{array}$$

The top row is exact since F is reg-flat and K^+ is u-torsion by Corollary 2.3. The vertical maps are isomorphisms by [7, Theorem 3.2.11], since F is finitely presented. Consequently, the bottom row is also exact, completing the proof. □

Remark 2.27. The module in Example 2.21 is reg-projective but not projective.

3. On reg-Noetherian rings and reg-Noetherian modules

It is straightforward to see that every Noetherian ring is reg-Noetherian. The converse is not true in general, as shown in [4, Example 2.9].

To properly study the concept of reg-Noetherianity, we begin by introducing the notion of reg-Noetherianity for modules, which is defined as follows.

Definition 3.1. Let R be a ring and M be an R-module. Then M is said to be reg-Noetherian if M is a finitely generated R-module and every reg-submodule of M is a finitely generated R-module.

In particular, every Noetherian R-module is reg-Noetherian.

Remark 3.2. (1) Note that for a torsion R-module M, we have

$$M \text{ is reg-Noetherian} \Leftrightarrow M \text{ is Noetherian.}$$

(2) From Example 2.4, it is clear that a ring R is reg-Noetherian if and only if R itself is a reg-Noetherian R-module.

Theorem 3.3. *Let* $0 \to M' \xrightarrow{f} M \xrightarrow{g} M'' \to 0$ *be an exact sequence. If* M' *and* M'' *are reg-Noetherian modules, then so is* M*. Additionally, if* M' *is a reg-submodule of* M*, then the converse holds.*

Proof. Assume that M' and M'' are reg-Noetherian. Let N be a reg-submodule of M. Then $g(N)$ is a reg-submodule of M''. Indeed, for any $x \in M''$, there exists $m \in M$ such that $g(m) = x$. Since N is a reg-submodule, there exists $s \in R \setminus Z(R)$ such that $sm \in N$. Applying g, we obtain $sx \in g(N)$, proving that $g(N)$ is a reg-submodule of M''. Since M'' is reg-Noetherian, $g(N)$ is finitely generated, say

$$g(N) = \sum_{i=1}^{t} Rg(n_i),$$

where each $n_i \in N$. For any $n \in N$, we can write

$$g(n) = \sum_{i=1}^{t} r_i g(n_i), \quad \text{where } r_i \in R.$$

Rearranging, we get

$$n - \sum_{i=1}^{t} r_i n_i \in \ker(g) \cap N.$$

Thus, there exists $y \in \ker(g) \cap N$ such that

$$n = f(y) + \sum_{i=1}^{t} r_i n_i.$$

Since N is a reg-submodule of M, it follows that $\ker(g) \cap N$ is a reg-submodule of M'. As M' is reg-Noetherian, $\ker(g) \cap N$ is finitely generated, say

$$\ker(g) \cap N = \sum_{i=t+1}^{t+l} Rn_i,$$

for some $n_{t+1}, \dots, n_{t+l} \in \ker(g) \cap N$. Moreover, there exist $r_{t+1}, \dots, r_{t+l} \in R$ such that

$$f(y) = \sum_{i=t+1}^{t+l} r_i n_i.$$

Thus, we conclude that

$$n = \sum_{i=1}^{t+l} r_i n_i.$$

Therefore, N is finitely generated, proving that M is a reg-Noetherian module.

Now assume that M is reg-Noetherian, and consider a reg-submodule M' of M. Let X be a reg-submodule of M'. The exact sequence

$$0 \to M'/X \to M/X \to M'' \to 0$$

shows that both M'/X and M'' are torsion. It follows that X is a reg-submodule of M. Since M is reg-Noetherian, X is finitely generated, proving that M' is reg-Noetherian.

Next, let N be a submodule of M containing M' such that N/M' is a reg-submodule of $M'' \cong M/M'$. We claim that N is a reg-submodule of M. Indeed, for any $x \in M$, since M/M' is torsion, there exists $s \in R \setminus Z(R)$ such that

$$s(x + M') = M',$$

which implies that $sx \in M' \subset N$. Thus, N is a reg-submodule of M, and since M is reg-Noetherian, N is finitely generated. Consequently, N/M' is a finitely generated submodule of M'', showing that M'' is reg-Noetherian. □

Corollary 3.4. *Let R be a ring and M be a reg-Noetherian R-module. Then every reg-submodule of M is reg-Noetherian.*

Proof. This follows immediately from Theorem 3.3. □

Corollary 3.5. *Let R be a ring and $\{M_i\}_{1 \le i \le n}$ be a family of reg-Noetherian modules. Then $\bigoplus_{i=1}^n M_i$ is a reg-Noetherian module.*

Proof. We proceed by induction on n. Consider the exact sequence

$$0 \to M_1 \to \bigoplus_{i=1}^{n} M_i \to \bigoplus_{i=2}^{n} M_i \to 0,$$

and apply Theorem 3.3 to complete the proof. □

Corollary 3.6. *If R is a reg-Noetherian ring, then every finitely generated torsion module is reg-Noetherian (and so is Noetherian).*

Proof. If M is a finitely generated torsion R-module, then there exists an isomorphism $M \cong R^{(n)}/N$, where $n \in \mathbb{N}$ and N is a submodule of $R^{(n)}$. Since M is torsion, it follows that N is a reg-submodule of $R^{(n)}$. By considering the exact sequence

$$0 \to N \to R^{(n)} \to M \to 0,$$

and applying Theorem 3.3, we conclude that M is reg-Noetherian. □

Corollary 3.7. *Let R be a ring and I be a finitely generated regular ideal of R. Then R is a reg-Noetherian ring if and only if I and R/I are reg-Noetherian R-modules.*

Proof. This follows immediately from Theorem 3.3. □

Theorem 3.8. *Let R be a ring. If M is a reg-Noetherian R-module, then every factor module of M is reg-Noetherian.*

Proof. Let M be a reg-Noetherian module and N be a submodule of M. We claim that M/N is a reg-Noetherian module. Let P/N be a reg-submodule of M/N, where P is a submodule of M containing N. Since $\frac{M/N}{P/N} \cong \frac{M}{P}$ is a torsion R-module, P is finitely generated, and so P/N is a finitely generated submodule of M/N. Therefore, M/N is reg-Noetherian. □

Corollary 3.9. *If R is a reg-Noetherian ring and I is an ideal of R, then R/I is a reg-Noetherian R-module.*

Proof. This follows immediately from Theorem 3.8. □

Corollary 3.10. *Let R be a reg-Noetherian ring and M be an R-module. Then M is a reg-Noetherian module if and only if M is a finitely generated R-module.*

Proof. If M is a reg-Noetherian module, then it is easy to see that M is a finitely generated module.

Conversely, if M is a finitely generated module, then M is a factor of $R^{(n)}$, where $n \in \mathbb{N}$. Since $R^{(n)}$ is a reg-Noetherian module by Corollary 3.5, M is a reg-Noetherian module by Theorem 3.8. □

Corollary 3.11. *A ring R is reg-Noetherian if and only if every reg-submodule of a finitely generated R-module is finitely generated.*

Proof. Straightforward. □

Theorem 3.12 establishes that every finitely generated torsion module over a reg-Noetherian ring is finitely presented.

Theorem 3.12. *Let R be a reg-Noetherian ring and M be a finitely generated torsion R-module. Then M is finitely presented.*

Proof. Let M be a finitely generated torsion R-module. There exist $n \in \mathbb{N}$ and a sequence $0 \to N \to R^{(n)} \to M \to 0$. Since $R^{(n)}$ is a reg-Noetherian R-module by Corollary 3.5 and M is a torsion module, N is a finitely generated module. Therefore, M is a finitely presented module. □

Theorem 3.13 establishes that the class of reg-Noetherian modules is closed under localizations.

Theorem 3.13. *Let R be a ring and S be a multiplicative subset of R. If M is a reg-Noetherian R-module, then $S^{-1}M$ is a reg-Noetherian $(S^{-1}R)$-module.*

Proof. Let M be a reg-Noetherian R-module and $S^{-1}N$ be a reg-submodule of $S^{-1}M$, where N is a submodule of M. Then N is a reg-submodule of M, and so N is a finitely generated R-module. Thus, $S^{-1}N$ is a finitely generated $(S^{-1}R)$-module. Therefore, $S^{-1}M$ is a reg-Noetherian $(S^{-1}R)$-module. □

Corollary 3.14. *If R is a reg-Noetherian ring and S is a multiplicative subset of R, then $S^{-1}R$ is a reg-Noetherian ring.*

Proof. This follows immediately from Theorem 3.13. □

Let M be an R-module. We say that M satisfies the reg-condition if every reg-submodule N of M contains a finitely generated reg-submodule of M. Note that every ring R satisfies the reg-condition. Indeed, if I is a regular ideal of R, then there exists $s \in I \cap (R \setminus Z(R))$. Taking $J = sR \subset I$ makes J a finitely generated regular ideal of R and hence R satisfies the reg-condition.

Remark 3.15. It is easy to see from Theorem 3.3 that every reg-Noetherian module satisfies the reg-condition.

Definition 3.16. Let R be a ring and M be an R-module. An ascending chain of submodules of M: $M_1 \subseteq M_2 \subseteq \cdots \subseteq M_n \subseteq \cdots$ is said to be a reg-ascending chain if the following condition holds: M_i is a reg-submodule of M, for every positive integer i.

Based on Remark 3.15 above, we can characterize reg-Noetherian modules in a similar way to characterizing Noetherian modules.

Theorem 3.17. *Let R be a ring and M be an R-module. Then the following are equivalent:*

(1) *M is a reg-Noetherian module.*
(2) *M satisfies both the reg-condition and the reg-ascending chain condition (reg-ACC) on submodules of M. That is, if*

$$M_1 \subseteq M_2 \subseteq \cdots \subseteq M_n \subseteq \cdots$$

is a reg-ascending chain of submodules of M, then there exists a positive integer m such that if $n \geq m$, then $M_n = M_m$.
(3) *M satisfies both the reg-condition and the reg-maximal condition. That is, every nonempty set of reg-submodules of M possesses a maximal element.*

Proof. (1) ⇒ (2) First, observe by Remark 3.15 that M satisfies the reg-condition. Let $M_1 \subseteq M_2 \subseteq \cdots \subseteq M_n \subseteq \cdots$ be a reg-ascending chain of submodules of M. Set $N = \bigcup M_i$. By the hypothesis, N is finitely generated since it is a reg-submodule of M. Write $N = Rx_1 + \cdots + Rx_k$. Then, there exists m such that $x_i \in M_m$ for all i. This makes $N = M_m$ and hence $M_n = M_m$ for all $n \geq m$.

(2) ⇒ (3) Let Γ be a nonempty set of reg-submodules of M. Suppose that Γ has no maximal elements. If we take any $M_1 \in \Gamma$, then M_1 is not a maximal element and hence there is $M_2 \in \Gamma$ such that $M_1 \subset M_2$. Since M_2 is not a maximal element, we can find $M_3 \in \Gamma$ such that $M_2 \subset M_3$. As a result, we obtain a reg-ascending chain $M_1 \subset M_2 \subset \cdots \subset M_n \subset \cdots$ of submodules of M. This chain is not stationary, contradicting the reg-ACC condition.

(3) ⇒ (1) Let N be a reg-submodule of M. Set

$$\Gamma = \{A \subseteq N \mid A \text{ is a finitely generated reg-submodule of } M\}.$$

Observe that Γ is nonempty as M satisfies the reg-condition. Furthermore, Γ has a maximal element A. If $A \neq N$, then there exists $x \in N \setminus A$ such that the module $A + Rx$ is a finitely generated reg-submodule of M, which contradicts the maximality of A. Therefore, $N = A$ is finitely generated. □

We can now derive the following corollary, which recovers [6, Exercise 2.6.5].

Corollary 3.18. *Let R be a commutative ring. Then the following are equivalent:*

(1) *R is a reg-Noetherian ring.*
(2) *R satisfies the ascending chain condition on regular ideals.*
(3) *Every nonempty set of regular ideals in R has a maximal element.*

Proof. This follows immediately from Remark 3.15 and Theorem 3.17. □

4. On reg-coherent modules and reg-coherent rings

Following [16], a commutative ring R is called reg-coherent if every finitely generated regular ideal is finitely presented. It is obviously seen that every coherent ring is reg-coherent. But the converse is not true in general, as shown in [5, Example 2.9].

To properly study the concept of reg-coherence, we begin by introducing the notion of reg-coherent modules, defined as follows.

Definition 4.1. Let R be a ring. An R-module M is said to be reg-coherent if M is finitely generated and every finitely generated reg-submodule of M is finitely presented.

In particular, every coherent module is reg-coherent.

Remark 4.2. Note that for a torsion R-module M, we have

$$M \text{ is reg-coherent} \Leftrightarrow M \text{ is coherent.}$$

Now we are able to give a new characterization of reg-coherent rings.

Theorem 4.3. *The following are equivalent for a ring R:*

(1) *R is a reg-coherent ring.*
(2) *R is a reg-coherent R-module.*
(3) *Every finitely generated free R-module is reg-coherent.*
(4) *Every finitely presented module is reg-coherent.*
(5) *Every finitely generated reg-submodule of a finitely presented R-module is finitely presented.*
(6) *Every direct product of reg-flat R-modules is reg-flat.*
(7) *R^I is reg-flat for any index set I.*
(8) *For any injective R-module E, the module* $\mathrm{Hom}_R(R^+, E)$ *is reg-flat.*

Proof. (1) $\Leftrightarrow$ (2) This is straightforward.

(1) $\Rightarrow$ (6) Let $\{N_\alpha\}_{\alpha\in\Gamma}$ be a family of reg-flat modules, and let I be a finitely generated reg-ideal of R. By [3, Lemma 2.10], we have

$$\mathrm{Tor}_n^R\left(\prod_{\alpha\in\Gamma} N_\alpha, R/I\right) \cong \prod_{\alpha\in\Gamma} \mathrm{Tor}_n^R(N_\alpha, R/I) = 0$$

for every $n \geq 1$. Hence, $\prod_{\alpha\in\Gamma} N_\alpha$ is reg-flat, as required.

(6) $\Rightarrow$ (7) This is obvious.

(7) $\Rightarrow$ (1) Let I be a finitely generated regular ideal of R, and consider the short exact sequence

$$0 \to I \to R \to R/I \to 0.$$

By hypothesis, R^I is a reg-flat R-module. This gives rise to the following commutative diagram with exact rows:

$$\begin{array}{ccccccc} 0 & \longrightarrow & R^\Gamma \otimes_R I & \longrightarrow & R^\Gamma \otimes_R R & \longrightarrow & R^\Gamma \otimes_R \frac{R}{I} \\ & & \downarrow & & \downarrow \cong & & \downarrow \cong \\ 0 & \longrightarrow & I^\Gamma & \longrightarrow & R^\Gamma & \longrightarrow & \left(\frac{R}{I}\right)^\Gamma \end{array}$$

By [13, Lemma I.13.2], the right two vertical maps are isomorphisms, implying that the left vertical arrow is also an isomorphism. Furthermore, [13, Lemma I.13.2] ensures that I is finitely presented, as required. Therefore, R is a reg-coherent ring.

(4) $\Rightarrow$ (5) This follows directly.

(5) $\Rightarrow$ (1) Since every regular ideal of R is a reg-submodule of R, the result follows immediately.

(6) $\Rightarrow$ (3) Let F be a finitely generated free R-module, and let N be a finitely generated reg-submodule of F. Then F and F/N are finitely presented R-modules. Since R^I is a reg-flat module for any index set I, the following commutative diagram with exact rows holds:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & N \otimes_R R^I & \longrightarrow & F \otimes_R R^I & \longrightarrow & F/N \otimes_R R^I & \longrightarrow & 0 \\ & & \downarrow & & \downarrow \cong & & \downarrow \cong & & \\ 0 & \longrightarrow & N^I & \longrightarrow & F^I & \longrightarrow & (F/N)^I & \longrightarrow & 0. \end{array}$$

By [13, Lemma I.13.2], the two rightmost vertical arrows are isomorphisms, yielding $N \otimes_R R^I \cong N^I$. By [13, Lemma I.13.2], it follows that N is a finitely presented R-module. Hence, F is a reg-coherent R-module.

(3) $\Rightarrow$ (4) Let M be a finitely presented R-module. Then there exists an isomorphism $M \cong F/N$, where F is a finitely generated free R-module, and N is a finitely generated submodule of F. Let X be a finitely generated reg-submodule of M. Then $X \cong L/N$ for some finitely generated submodule L of F with $N \subset L$. Since F is a reg-coherent module and $M/X \cong F/L$ is torsion, it follows that L is a finitely presented R-module. By [9, (4.54) Lemma], X is finitely presented. Hence, M is a reg-coherent module.

(1) $\Leftrightarrow$ (8) See [16, Theorem 3.5]. $\square$

We remark that the equivalence of (1), (2), (6), and (7) of Theorem 4.3 are proved in [16, Theorem 3.2]. Additionally, reg-coherent rings are characterized in terms of 'reg-coflat modules' in [16, Theorem 3.5].

The following theorem characterizes when a finitely generated submodule of a reg-coherent module is reg-coherent.

Theorem 4.4. *Let R be a ring and M be a reg-coherent R-module. If N is a finitely generated reg-submodule of M, then N is a reg-coherent module.*

Proof. Let M be a reg-coherent R-module, and let N be a finitely generated reg-submodule of M. We claim that N is also a reg-coherent R-module.

Let X be a finitely generated reg-submodule of N. Consider the exact sequence

$$0 \to N/X \to M/X \to M/N \to 0.$$

Since M/N and N/X are torsion modules, it follows that M/X is also torsion. Consequently, X is finitely presented, which implies that N is a reg-coherent module, as required. □

Corollary 4.5. *If R is a reg-coherent ring, then every finitely generated regular ideal of R is a reg-coherent R-module.*

Proof. This follows from Theorem 4.4. □

Theorem 4.6. *Let R be a ring and $0 \to P \to N \to M \to 0$ be an exact sequence of R-modules and R-homomorphisms, where P is a finitely generated R-module. If N is a reg-coherent module, then so is M.*

Proof. We can set $M = N/P$. Let X/P be a finitely generated reg-submodule of M. Since N is a reg-coherent module and X is a finitely generated reg-submodule of N, it follows that X is finitely presented. We claim that X/P is a finitely presented R-module. Actually, it follows from [9, (4.54) Lemma] that X/P is finitely presented, and so M is a reg-coherent module. □

Corollary 4.7 is a consequence of Theorem 4.6.

Corollary 4.7. *Every factor module M/N of a reg-coherent module M by a finitely generated submodule N is also a reg-coherent module. In particular, every factor of a reg-coherent ring R by a finitely generated ideal I of R is a reg-coherent R-module.*

Proof. Straightforward. □

Corollary 4.8. *Let R be a ring and M, N be reg-coherent modules. Let $f : M \to N$ be an R-homomorphism.*

(1) *If* $\operatorname{im}(f)$ *is a torsion* R*-module and* $\ker(f)$ *is finitely generated, then* $\ker(f)$ *is a reg-coherent module.*
(2) *If* $\ker(f)$ *is finitely generated, then* $\operatorname{im}(f)$ *is a reg-coherent module.*
(3) *If* $\operatorname{coker}(f)$ *is a torsion* R*-module and* $\operatorname{im}(f)$ *is finitely generated, then* $\operatorname{im}(f)$ *is a reg-coherent module.*
(4) *If* $\operatorname{im}(f)$ *is finitely generated, then* $\operatorname{coker}(f)$ *is a reg-coherent module.*

Proof. Consider the exact sequences

$$0 \to \ker(f) \to M \to \operatorname{im}(f) \to 0$$

and

$$0 \to \operatorname{im}(f) \to N \to \operatorname{coker}(f) \to 0.$$

Applying Theorems 4.4 and 4.6 to these sequences completes the proof. □

Theorem 4.9. *Let* R *be a ring and* $0 \to P \xrightarrow{f} N \xrightarrow{g} M \to 0$ *be an exact sequence of* R*-modules and* R*-homomorphisms. If* P *and* M *are reg-coherent modules, then so is* N*.*

Proof. Let X be a finitely generated reg-submodule of N. Then we have the following commutative diagram with exact rows:

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \ker(g_{|X}) & \xrightarrow{f} & X & \xrightarrow{g} & g(X) & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle i} & & \downarrow{\scriptstyle j} & & \downarrow{\scriptstyle k} & & \\
0 & \longrightarrow & P & \xrightarrow{f} & N & \xrightarrow{g} & M & \longrightarrow & 0.
\end{array}$$

Since X is a finitely generated module, so is $g(X)$. Let $x \in M$. Then $g(n) = x$ for some $n \in N$. Since N/X is a torsion module, $sn \in X$ for some $s \in R \setminus Z(R)$, and so $sx \in g(X)$. Therefore, $M/g(X)$ is torsion. As M is reg-coherent, $g(X)$ is a finitely presented R-module. Therefore, $\ker(g_{|X})$ is a finitely generated R-module since X is finitely generated. Let $x \in P$. Then there exists $t \in R \setminus Z(R)$ such that $tf(x) \in X$, and so $tf(x) \in \ker(g_{|X})$ since $g(tf(x)) = 0$. We can consider f as an embedding, and so $P/\ker(g_{|X})$ is a torsion module. Then $\ker(g_{|X})$ is finitely presented since P is a reg-coherent module, and so X is a finitely presented R-module. Therefore, N is a reg-coherent module. □

Corollary 4.10. *Let* R *be a ring and* $\{M_i\}_{i=1}^{n}$ *be a family of reg-coherent modules. Then* $\bigoplus_{i=1}^{n} M_i$ *is a reg-coherent module.*

Proof. We proceed by induction on n. Consider the exact sequence

$$0 \to M_1 \to \bigoplus_{i=1}^{n} M_i \to \bigoplus_{i=2}^{n} M_i \to 0,$$

and apply Theorem 4.9 to complete the proof. □

Corollary 4.11. *Let R be a ring and let M and N be reg-coherent submodules of a reg-coherent R-module L. If $M+N$ is a torsion R-module and $M \cap N$ is finitely generated, then $M+N$ and $M \cap N$ are reg-coherent modules.*

Proof. Consider the exact sequence

$$0 \to M \cap N \to M \oplus N \to M+N \to 0.$$

Applying Theorems 4.4 and 4.6 completes the proof. □

Corollary 4.12. *Let R be a ring and I be a finitely generated regular ideal of R. Then R is a reg-coherent ring if and only if I and R/I are reg-coherent R-modules.*

Proof. Assume that R is a reg-coherent ring and let I be a finitely generated regular ideal of R. By Corollary 4.7, R/I is a reg-coherent R-module, and so I is a reg-coherent R-module by Theorem 4.4.

Conversely, assume that I and R/I are reg-coherent R-modules for any finitely generated regular ideal I of R. Then R is a reg-coherent ring by Theorem 4.9. □

Next, Theorem 4.13 gives an analog of the well-known behavior in [8, Theorem 2.2.6].

Theorem 4.13. *Let R be a ring and S be a multiplicative subset of R. If M is a reg-coherent R-module, then $S^{-1}M$ is a reg-coherent $(S^{-1}R)$-module.*

Proof. It is clear that $S^{-1}M$ is a finitely generated $(S^{-1}R)$-module. Let N be an R-module such that $S^{-1}N$ is a torsion $(S^{-1}R)$-module. Then N is a torsion R-module. Let X be a finitely generated $(S^{-1}R)$-submodule of $S^{-1}M$ such that $\frac{S^{-1}M}{X}$ is torsion. Then we can set $X = S^{-1}K$ where K is a finitely generated submodule of M. Therefore, $S^{-1}(M/K)$ is torsion, and so M/K is a torsion R-module. Hence, K is a finitely presented R-module. Thus, X is a finitely presented $(S^{-1}R)$-module. Therefore, $S^{-1}M$ is a reg-coherent $(S^{-1}R)$-module. □

Next, we turn our attention to the localization of reg-coherent rings. Using Theorem 4.13, we obtain immediately:

Corollary 4.14. *If R is a reg-coherent ring and S is a multiplicative subset of R, then $S^{-1}R$ is a reg-coherent ring.*

Proof. Straightforward. □

Theorem 4.15. *Let $f : R \to T$ be a finite surjective ring homomorphism (i.e., T is a finitely generated R-module). Let M be a finitely generated T-module which is a reg-coherent R-module. Then M is a reg-coherent T-module.*

Proof. Let X be a finitely generated T-submodule of M. Then X is a finitely generated R-module since f is finite. If M/X is a torsion T-module, then M/X is a torsion R-module, and so X is a finitely presented R-module. Therefore, X is a finitely presented T-module since $X \cong T \otimes_R X$. Hence, M is a reg-coherent T-module. □

Corollary 4.16. *Every factor of a reg-coherent ring by a finitely generated regular ideal is a reg-coherent ring.*

Proof. This follows immediately from Theorem 4.9, Corollary 4.12, and Theorem 4.15. □

To explore additional properties related to the classical notion of coherence, we recall a class of rings known as Marot rings. A ring R is called a Marot ring if every regular ideal of R is generated by its set of regular elements.

Definition 4.17. A ring R is called a divisible ring (or div-ring for short) if $sZ(R) = Z(R)$ for every $s \in R \setminus Z(R)$. We denote by $\mathscr{D}iv$ the set of all div-rings.

Remark 4.18. If $R \in \mathscr{D}iv$, then every finitely generated regular ideal is generated solely by regular elements. Indeed, let $I = \langle a_1, a_2, \dots, a_n \rangle$ be a regular ideal. Then, there exists some $1 \leq i \leq n$ such that a_i is regular in R. Since $a_i Z(R) = Z(R)$, every zero divisor in the generating set of I belongs to $\langle a_i \rangle$. Consequently, we can exclude all elements a_k that are zero divisors from the generating set of I, ensuring that I is generated only by regular elements.

Theorem 4.19. *The following are equivalent for a ring $R \in \mathscr{D}iv$:*

(1) *R is a reg-coherent ring.*
(2) *The intersection of two finitely generated regular ideals is finitely generated.*

Proof. Assume that R is a reg-coherent ring. Let I and J be finitely generated regular ideals of R. Then $I + J$ is also a finitely generated regular ideal and, hence, finitely presented. By [8, Theorem 2.1.2], it follows that $I \cap J$ is finitely generated.

Conversely, suppose I is a finitely generated regular ideal of R. Write $I = \langle x_1, \dots, x_r \rangle$. We prove the result by induction on r. By Remark 4.18, we may assume that each x_i is regular in R. For $r = 1$, we have $I = \langle x_1 \rangle$, which is a free ideal of R and hence finitely presented.

For $r \geq 2$, set $I' = \langle x_1, \ldots, x_{r-1} \rangle$, which is a finitely presented regular ideal. We now show that $I = I' + \langle x_r \rangle$ is finitely presented. Consider the exact sequence

$$0 \longrightarrow I' \cap \langle x_r \rangle \longrightarrow I' \oplus \langle x_r \rangle \longrightarrow I' + \langle x_r \rangle \longrightarrow 0.$$

Since $I' \cap \langle x_r \rangle$ is the intersection of two finitely generated regular ideals of R, it is finitely generated. By [8, Theorem 2.1.2], it follows that $I = I' + \langle x_r \rangle$ is a finitely presented regular ideal of R. □

Recall from [14, Theorem 4.3.6 (Krull's Intersection Theorem)] that if I is an ideal of a ring R and M is a Noetherian module, then $IB = B$, where

$$B = \bigcap_{n=1}^{\infty} I^n M.$$

If R is a Noetherian ring and M is a finitely generated R-module, it follows from the introduction of [1] that for every ideal I of R, we have

$$\bigcap_{n=1}^{\infty} I^n M = \{r \in R \mid (1-a)r = 0, \text{ for some } a \in I\}.$$

In particular, if R is a Noetherian domain, then $\bigcap_{n=1}^{\infty} I^n M = 0$.

Moreover, it is known from [14, Theorem 4.2.20] that any proper submodule of a Noetherian module admits a primary decomposition. The following theorem provides a regular-version of Krull's Intersection Theorem.

Theorem 4.20. (Reg-Krull's Intersection Theorem) *Let R be a reg-Noetherian ring and M be a finitely generated torsion R-module. If I is an ideal of R, then $IB = B$, where $B = \bigcap_{n=1}^{\infty} I^n M$.*

Proof. If $IB = M$, then it is clear that $B = IB$. Thus, we may assume that $IB \neq M$. However, M is a Noetherian R-module by Remark 3.2 (1) and Corollary 3.10, and so IB has a primary decomposition. Write

$$IB = Q_1 \cap Q_2 \cap \cdots \cap Q_s,$$

where Q_i is a $\mathfrak{p}_i$-primary submodule of M for some prime ideal $\mathfrak{p}_i$ of R. We will show that $B \subseteq Q_i$, and so $B \subseteq Q_1 \cap Q_2 \cap \cdots \cap Q_s = IB$. This makes $B = IB$.

If $I \subseteq \mathfrak{p}_i$, then, by [14, Theorem 4.2.17], there exists a positive integer m such that $\mathfrak{p}_i^m M \subseteq Q_i$. Thus,

$$B = \bigcap_{n=1}^{\infty} I^n M \subseteq \mathfrak{p}_i^m M \subseteq Q_i.$$

If $I \nsubseteq \mathfrak{p}_i$, then there exists an element $r \in I \setminus \mathfrak{p}_i$. Since $rB \subseteq IB \subseteq Q_i$ and Q_i is $\mathfrak{p}_i$-primary, it follows that $B \subseteq Q_i$. □

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References

[1] D. D. Anderson, *The Krull intersection theorem*, Pacific J. Math., 57(1) (1975), 11-14.

[2] S. U. Chase, *Direct products of modules*, Trans. Amer. Math. Soc., 97 (1960), 457-473.

[3] J. Chen and N. Ding, *On n-coherent rings*, Comm. Algebra, 24(10) (1996), 3211-3216.

[4] M. Chhiti and S. E. Mahdou, *Rings in which every regular ideal is finitely generated*, Moroc. J. Algebra Geom. Appl., 2(2) (2023), 218-225.

[5] M. Chhiti and S. E. Mahdou, *When every finitely generated regular ideal is finitely presented*, Commun. Korean Math. Soc., 39(2) (2024), 363-372.

[6] J. Elliott, Rings, Modules, and Closure Operations, Springer Monographs in Mathematics, Springer, Cham, 2019.

[7] E. E. Enochs and O. M. G. Jenda, Relative Homological Algebra, De Gruyter Expositions in Mathematics, 30, Walter de Gruyter & Co., Berlin, 2000.

[8] S. Glaz, Commutative Coherent Rings, Lecture Notes in Mathematics, 1371, Springer-Verlag, Berlin, 1989.

[9] T. Y. Lam, Lectures on Modules and Rings, Graduate Texts in Mathematics, 189, Springer-Verlag, New York, 1999.

[10] E. Matlis, *Commutative coherent rings*, Canadian J. Math., 34(6) (1982), 1240-1244.

[11] W. Qi and X. L. Zhang, *The homological properties of regular injective modules*, Commun. Korean Math. Soc., 39(1) (2024), 59-69.

[12] B. Stenström, *Coherent rings and FP-injective modules*, J. London Math. Soc., 2(2) (1970), 323-329.

[13] B. Stenström, Rings of Quotients, An introduction to methods of ring theory, Die Grundlehren der Mathematischen Wissenschaften, 217, Springer-Verlag, New York-Heidelberg, 1975.

[14] F. G. Wang and H. Kim, Foundations of Commutative Rings and Their Modules, Algebra and Applications, 22, Springer, Singapore, 2016.

[15] F. G. Wang and J. L. Liao, *S-injective modules and S-injective envelopes*, Acta Math. Sinica (Chinese Ser.), 54(2) (2011), 271-284.

[16] X. L. Xiao, F. G. Wang and S. Y. Lin, *The coherence study determined by regular ideals*, J. Sichuan Normal Univ., 45(1) (2022), 33-40.

[17] X. L. Zhang, *Characterizing S-flat modules and S-von Neumann regular rings by uniformity*, Bull. Korean Math. Soc., 59(3) (2022), 643-657.

[18] X. L. Zhang, *On τ_q-projectivity and τ_q-simplicity*, (2023), arXiv:2302.04560 [math.AC].

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