ON NH-EMBEDDED AND SS-QUASINORMAL SUBGROUPS OF
FINITE GROUPS

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Received: 27 February 2023; Revised: 24 March 2023; Accepted: 1 April 2023
Communicated by Abdullah Harmancı

Abstract. Let $G$ be a finite group. A subgroup $H$ is called $S$-semipermutable in $G$ if $HG_p = G_pH$ for any $G_p \in Syl_p(G)$ with $(|H|, p) = 1$, where $p$ is a prime number divisible $|G|$. Furthermore, $H$ is said to be $NH$-embedded in $G$ if there exists a normal subgroup $T$ of $G$ such that $HT$ is a Hall subgroup of $G$ and $H \cap T \leq H_{SG}$, where $H_{SG}$ is the largest $S$-semipermutable subgroup of $G$ contained in $H$, and $H$ is said to be $SS$-quasinormal in $G$ provided there is a supplement $B$ of $H$ to $G$ such that $H$ permutes with every Sylow subgroup of $B$. In this paper, we obtain some criteria for $p$-nilpotency and Supersolvability of a finite group and extend some known results concerning $NH$-embedded and $SS$-quasinormal subgroups.

Mathematics Subject Classification (2020): 20D10, 20D20
Keywords: $SS$-quasinormal, $NH$-embedded, $p$-nilpotent group, supersolvable

1. Introduction

Throughout, all groups considered in this paper will be finite. Let $G$ be a group, $H$ and $K$ of are subgroups of $G$, they are said to be permutable if $HK = KH$, i.e. $HK$ is also a subgroup of $G$. $H$ is a subgroup of $G$, it is said to be quasinormal in $G$ if $H$ permutes with all subgroups of $G$. Kegel [8] introduced the concept of $S$-quasinormal (or $S$-permutable), subgroup $H$ of $G$ said to be $S$-quasinormal if $H$ permutes with all Sylow subgroup of $G$. Recall that a supplement of $H$ to $G$ is a subgroup $B$ such that $G = HB$. As a generalization of $S$-quasinormal subgroup, Li [9] introduced the following definition:

W. Zheng is supported by Innovation Project of GUET Graduate Education (2023YCXSI12).
W. Meng is supported by National Natural Science Foundation of China (12161021), Guangxi Natural Science Foundation Program (2021JJA10003) and Guangxi Colleges and Universities Key Laboratory of Data Analysis and Computation. J.K. Lu is supported by National Natural Science Foundation of China (11861015) and Guangxi Natural Science Foundation Program (2020GXNSFAA238045).
Definition 1.1. [9] A subgroup $H$ of $G$ is said to be $SS$-quasinormal in $G$ provided there is a supplement $B$ of $H$ to $G$ such that $H$ permutes with every Sylow subgroup of $B$.

Li [9] investigated the $p$-nilpotency and supersolvability of finite groups by some $SS$-quasinormal subgroups of prime power order.

Recall that a subgroup $H$ is called $S$-semipermutable in $G$ if $HG_p = G_p H$ for any $G_p \in Syl_p(G)$ with $(|H|, p) = 1$, $p$ is a prime number divisible $G$(see [2]). Recently, Gao and Li [5] introduce the following concept:

Definition 1.2. [5] A subgroup $H$ of a group $G$ is said to be $NH$-embedded in $G$ if there exists a normal subgroup $T$ of $G$ such that $HT$ is a Hall subgroup of $G$ and $H \cap T \leq H_{S}\leq G$, where $H_{S}$ is the largest $S$-semipermutable subgroup of $G$ contained in $H$.

Gao and Li [5] showed that the finite group whose maximal subgroups of Sylow subgroups are $NH$-embedded in $G$ are supersolvable.

By the definition of $NH$-embedded and $SS$-quasinormal subgroups, it is obvious that all Hall subgroups, normal subgroups and $S$-semipermutable subgroups are $NH$-embedded subgroups. But the converse does not hold. Moreover, an $NH$-embedded subgroup need not be $SS$-quasinormal. Conversely, it easy to see that an $SS$-quasinormal subgroup need not be $NH$-embedded too.

In the light of above results, it is seem interesting to study the structure of finite groups assuming that maximal subgroups of Sylow subgroups are $SS$-quasinormal or $NH$-embedded in $G$. In this paper, we obtain some criteria for $p$-nilpotency and supersolvability of a finite group. The main results are as follows.

Theorem 1.3. Let $G$ be a finite group and $G_p$ a Sylow $p$-subgroup of $G$, where $p$ is the smallest prime dividing $|G|$. Assume that every maximal subgroup of $G_p$ is either $NH$-embedded or $SS$-quasinormal in $G$. Then $G$ is $p$-nilpotent.

Theorem 1.4. Let $G$ be a finite group. Suppose that every maximal subgroup of every Sylow subgroup of $G$ is either $NH$-embedded or $SS$-quasinormal in $G$. Then $G$ is supersolvable.

Theorem 1.5. Let $G$ be a finite group and $H$ a normal subgroup of $G$. Suppose that $G/H$ is supersolvable and every maximal subgroup of every Sylow subgroup of $H$ is either $NH$-embedded or $SS$-quasinormal in $G$. Then $G$ is supersolvable.

All unexplained notations and terminologies are standard and can be found in [4,6].
2. Preliminaries

In this section, we collect some results which will be used in the proof of main results.

**Lemma 2.1.** [9] Suppose that $H$ is SS-quasinormal in a group $G$, $K \leq G$, and $N$ is a normal subgroup of $G$. We have

1. if $H \leq K$, then $H$ is SS-quasinormal in $K$;
2. $HN/N$ is SS-quasinormal in $G/N$;
3. if $N \leq K$ and $K/N$ is SS-quasinormal in $G$, then $K$ is SS-quasinormal in $G$;
4. if $K$ is quasinormal in $G$, then $HK$ is SS-quasinormal in $G$.

**Lemma 2.2.** [9] Let $H$ be a $p$-subgroup of $G$. Then the following statements are equivalent:

1. $H$ is $S$-quasinormal in $G$;
2. $H \leq O_p(G)$, and $H$ is SS-quasinormal in $G$.

**Lemma 2.3.** [3] If $H$ is an $S$-quasinormal subgroup of the group $G$, then $H/H_G$ is nilpotent, where $H_G$ is the core of $H$ in $G$.

**Lemma 2.4.** [11] If $H$ is $S$-quasinormal in a group $G$ and $H$ is a $p$-group for some prime $p$, then $O^p(G) \leq N_G(H)$.

**Lemma 2.5.** [1] Let $H$ be a nilpotent subgroup of a group $G$. Then the following statements are equivalent:

1. $H$ is an $S$-quasinormal subgroup of $G$;
2. the Sylow subgroups of $H$ are $S$-quasinormal in $G$.

**Lemma 2.6.** [1] Let $P$ be a Sylow $p$-subgroup of a group $G$, and let $P_0$ be a maximal subgroup of $P$. Then the following statements are equivalent:

1. $P_0$ is normal in $G$;
2. $P_0$ is $S$-quasinormal in $G$.

**Lemma 2.7.** [5] Let $G$ be a group and $H \leq G$. Suppose that $H$ is NH-embedded in $G$. Let $N$ be a normal subgroup of $G$. Then

1. If $H \leq K \leq G$ and $K$ is subnormal in $G$, then $H$ is NH-embedded in $K$.
2. Suppose that $H$ is a $p$-group for some $p \in \pi(G)$. If $N \leq H$, then $H/N$ is NH-embedded in $G/N$.
3. Suppose that $H$ is a $p$-group for some $p \in \pi(G)$ and $N$ is a $p'$-subgroup of $G$. Then $HN/N$ is NH-embedded in $G/N$. 
Lemma 2.8. [12] Let \( G \) be a group and \( H \leq K \leq G \).

1. If \( H \) is \( S \)-semipermutable in \( G \), then \( H \) is \( S \)-semipermutable in \( K \);
2. Suppose that \( N \) is normal in \( G \) and \( H \) is a \( p \)-group. If \( H \) is \( S \)-semipermutable in \( G \), then \( HN/N \) is \( S \)-semipermutable in \( G/N \);
3. If \( H \) is \( S \)-semipermutable in \( G \) and \( H \leq O_p(G) \), then \( H \) is \( S \)-quasinormal in \( G \).

Lemma 2.9. [10] Let \( G \) be a group and \( H \) an \( S \)-semipermutable subgroup of \( G \). Suppose that \( H \) is a \( p \)-subgroup of \( G \) for some prime \( p \in \pi(G) \) and \( N \) is normal in \( G \). Then \( H \cap N \) is also an \( S \)-semipermutable subgroup of \( G \).

Lemma 2.10. [7] Let \( H \) be an \( S \)-semipermutable \( \pi \)-subgroup of \( G \). Then \( H^G \) contains a nilpotent \( \pi \)-complement, and all \( \pi \)-complements in \( H^G \) are conjugate. Also, if \( \pi \) consists of a single prime, then \( H^G \) is solvable.

Lemma 2.11. [4] Let \( U, V \) and \( W \) be subgroups of a group \( G \). Then the following statements are equivalent:

1. \( U \cap VW = (U \cap V)(U \cap W) \);
2. \( UV \cap UW = U(V \cap W) \).

3. Proof of Theorem

Proof of Theorem 1.3 Assume that the theorem is false and let \( G \) be a counterexample of minimal order. Let \( G_p \) be a Sylow \( p \)-subgroup of \( G \) and \( \mathcal{M}(G_p) = \{P_1, P_2, \ldots, P_m\} \) denote the set of all maximal subgroups of \( G_p \). By Theorem hypothesis, every member \( P_i \) of \( \mathcal{M}(G_p) \) is either \( NH \)-embedded or \( SS \)-quasinormal in \( G \). Without loss of generality, suppose that every member of the subset \( \mathcal{M}_1(G_p) = \{P_1, \ldots, P_k\} \) of \( \mathcal{M}(G_p) \) is \( NH \)-embedded in \( G \), and every member of the subset \( \mathcal{M}_2(G_p) = \{P_{k+1}, \ldots, P_m\} \) of \( \mathcal{M}(G_p) \) is \( SS \)-quasinormal in \( G \) for some \( 1 \leq k \leq m \).

The proof of the theorem will be divided into five steps as follows.

Step (1). \( G \) has a unique minimal normal subgroup \( N \) and \( G/N \) is \( p \)-nilpotent.

Let \( N \) is a minimal normal subgroup of \( G \). Then \( G_pN/N \) is a Sylow \( p \)-subgroup of \( G/N \). For any \( M/N \in \mathcal{M}(G_pN/N) \), let \( P = M \cap G_p \), then
\[
M = M \cap G_p N = (M \cap G_p)N = PN
\]
and
\[
P \cap N = (M \cap G_p) \cap N = PN \cap G_p \cap N = G_p \cap N.
\]
As \( |G_p : P| = |G_p : (G_p \cap M)| = |G_pM : M| = p \), we know that \( P \in \mathcal{M}(G_p) \). By Theorem hypothesis, \( P \) is either \( NH \)-embedded or \( SS \)-quasinormal in \( G \).
Suppose that $P$ is $SS$-quasinormal in $G$, then $M/N = PN/N$ is also $SS$-quasinormal in $G/N$ by Lemma 2.1(2).

Now, we assume that $P$ is $NH$-embedded in $G$, then there is a normal subgroup $T$ of $G$ such that $PT$ is a Hall subgroup of $G$ and $P \cap T \leq P_{SG}$. It is easy seen that $TN/N$ is normal in $G/N$ and $PN/N \cdot TN/N = PTN/N$ is a Hall subgroup of $G/N$. As $P \cap N = G_p \cap N$ is a Sylow $p$-subgroup of $N$, we have

$$|N \cap PT|_p = |N|_p = |N \cap P|_p = |(N \cap P)(N \cap T)|_p$$

and

$$|N \cap PT|_{p'} = \frac{|PT|_{p'} |N|_{p'}}{|NP_{p'}|_{p'} |NT|_{p'}} = |N \cap P|_{p'} = |(N \cap P)(N \cap T)|_{p'}.$$ 

This implies that $N \cap PT = (N \cap P)(N \cap T)$ and hence $PN \cap TN = (P \cap T)N$ by Lemma 2.11. Therefore,

$$PN/N \cap TN/N = (PN \cap TN)/N = (P \cap T)N/N \leq P_{SG}N/N.$$ 

As $P_{SG}$ is $S$-semipermuted in $G$, we get that $P_{SG}N/N$ is also $S$-semipermuted in $G/N$ by Lemma 2.8(2). This leads to $P_{SG}N/N \leq (PN/N)_{SG}/N$. So $M/N = PN/N$ is $NH$-embedded in $G/N$.

By above arguments, we know that $G/N$ satisfies the hypothesis of theorem. By the choice of $G$, we know that $G/N$ is $p$-nilpotent. Moreover, as the class of all $p$-nilpotent groups is saturated formation, we obtain that $N$ is the unique minimal normal subgroup of $G$.

Step (2). $O_{p'}(G) = 1$.

Suppose that $O_{p'}(G) > 1$, then $N \leq O_{p'}(G)$ by (1). As $G/N$ is $p$-nilpotent, we know that $G/O_{p'}(G)$ is $p$-nilpotent and hence $G$ is also $p$-nilpotent, a contradiction. Therefore, $O_{p'}(G) = 1$.

Step (3). $N \leq P$ for any $P_i \in M_1(G_p)$.

For any $H \in M_1(G_p)$, $H$ is $NH$-embedded in $G$, then there is a normal subgroup $T$ of $G$ such that $HT$ is a Hall subgroup of $G$ and $H \cap T \leq H_{SG}$. If $T = 1$, then $H$ is a Hall subgroup of $G$ and hence $H = 1$. This implies that $|G_p| = p$, as $P$ is a maximal subgroup of $G_p$. By Burnside theorem, $G$ is $p$-nilpotent, a contradiction. Hence $T \neq 1$ and $N \leq T$. If $H \cap N = 1$, then $|N|_p \leq p$. So $N$ is $p$-nilpotent by Burnside theorem. Let $U$ be a normal Hall $p'$-subgroup of $N$, then $U$ is normal in $G$. By minimality of $N$, we know that $U = 1$ and hence $N$ is a subgroup of order $p$. Consequently, the nilpotency of $G/N$ implies that $G$ is $p$-nilpotent, a contradiction. Therefore, we have $H \cap N \neq 1$. Since $H \cap N \leq H \cap T \leq H_{SG} \leq H$, we get that $H_{SG} \cap N \leq H \cap N$ and hence $H \cap N = H_{SG} \cap N$. By Lemma 2.9, $H \cap N$ is $S$-semipermuted in $G$. In the other hand, observe that $N \leq H_{SG}^G$, we know that $N$ is
solvable by Lemma 2.10. This implies that $N$ is a $p$-group and hence $N \leq O_p(G)$. In particular, $H \cap N \leq O_p(G)$. Applying Lemma 2.8(3), we get that $H \cap N$ is $S$-quasinormal in $G$. Thus, $O^p(G) \leq N_G(H \cap N)$ by Lemma 2.4. Noting that $H \cap N$ is normal in $G'$ which implies that $H \cap N$ is normal in $G$. Thus, $H \cap N = N$. This leads to $N \leq H$.

Step (4). For every $P_j \in M_2(G_p)$, there exists a normal subgroup $M_j$ of $G$ such that $N \leq M_j$.

For any $H \in M_2(G_p)$, we know that $H$ is $SS$-quasinormal in $G$. So there exists a subgroup $B$ of $G$ such that $G = HB$, and $HB_p = B_pH$ for every Sylow subgroup $B_p$ of $B$. So $|B : H \cap B|_p = |G : H|_p = p$ from $G = HB$. Thus, $B_p \not\subseteq H$ and $B_pH = HB_p$ is a Sylow $p$-subgroup of $G$. In view of $H \in M_2(G_p)$ and by comparison of orders, $|H \cap B|_p = H \cap B$. Therefore,

$$H \cap B = \bigcap_{b \in B} (B_p^b \cap H) = \bigcap_{b \in B} B_p^b = O_p(B).$$

From $|O_p(B) : B \cap H| = p$ or 1, we obtain $|B/O_p(B)|_p = 1$ or $p$. As $p$ is the smallest prime dividing $|G|$, by Burnside theorem, $B/O_p(B)$ is $p$-nilpotent. So $B$ is $p$-solvable. And there is a Hall $p'$-subgroup of $B$ from ([6], IV, 1.7). Let $K$ be a Hall $p'$-subgroup of $B$, $\pi(K) = \{q_2, \ldots, q_s\}$, $Q_i \in Syl_{q_i}(K)$. According to the definition of $SS$-quasinormal, $H$ and $<Q_2, \ldots, Q_s> = K$ are permutations. So $HK$ is a subgroup of $G$. Obviously, $K$ is a Hall $p'$-group of $G$, and $HK$ is a subgroup of index $p$ in $G$. As $p$ is the smallest prime dividing $|G|$, $HK \leq G$. If $HK = 1$, then $G$ is elementary commutative $p$-group, contradiction. So, $N \leq HK = M_j$.

Step (5). Final contradiction.

Set $V = \left( \bigcap_{i=1}^{k} P_i \right) \cap \left( \bigcap_{j=k+1}^{m} M_j \right)$. By above arguments, we know that $N \leq V$. Moreover, we have

$$N = N \cap G_p \leq V \cap G_p = \left( \bigcap_{i=1}^{k} P_i \right) \cap \left( \bigcap_{j=k+1}^{m} M_j \right) \cap G_p = \bigcap_{i=1}^{m} P_i = \Phi(G_p).$$

By ([6], III. 3.3), we know that $\Phi(G_p) \leq \Phi(G)$ and hence $N \leq \Phi(G)$. Since $G/N$ is $p$-nilpotent, we get that $G/\Phi(G)$ is $p$-nilpotent. As the class of all $p$-nilpotent is a statured formation, $G$ will be $p$-nilpotent. This is finally contradiction. The proof of theorem is complete. □

**Proof of Theorem 1.4** Assume that the theorem is false and let $G$ be a counterexample of minimal order. Let $p$ be the smallest prime dividing $|G|$. Then $G$ is $p$-nilpotent by Theorem 1.3. Let $U$ be a Hall normal $p'$-subgroup of $G$. It is easy seen that $U$ satisfies the theorem hypothesis by Lemma 2.1(1) and Lemma 2.7(1). By induction, $U$ is supersolvable and hence $G$ possesses Sylow tower property of supersolvable type. Let $q$ be the largest prime dividing $|G|$ and $Q$ is a Sylow $q$-subgroup of $G$. Then $Q$ is normal in $G$. By Lemmas 2.1(2) and 2.7(3), we know
that $G/Q$ satisfies the theorem hypothesis and hence $G/Q$ is supersolvable by the choice of $G$.

Let $N$ be a minimal normal subgroup of $G$. Similar to the proof of Theorem 1.3, $G/N$ satisfies the theorem hypothesis and hence $G/N$ is supersolvable by minimality of $G$. As the class of all supersolvable group is a saturated formation, $N$ will be a unique minimal normal subgroup of $G$. Therefore, we have $N \leq Q$.

We claim that $N \leq H$ for any $H \in \mathcal{M}(Q)$. By theorem hypothesis, we know that $H$ is either $NH$-embedded or $SS$-quasinormal in $G$. If $H$ is $SS$-quasinormal in $G$. As $Q$ is normal in $G$, we have $H \leq Q = O_q(G)$. Applying Lemma 2.2, $H$ is $S$-quasinormal in $G$. Moreover, $H$ is normal in $G$ by Lemma 2.6. So we have $N \leq H$.

Now, assume that $H$ is $NH$-embedded in $G$. Then there exists a normal subgroup $T$ of $G$ such that $HT$ is a Hall subgroup of $G$ and $H \cap T \leq H_{TG}$. If $T = 1$, then $H = HT$ is a Hall subgroup of $G$. This implies that $H = 1$ and hence $|Q| = q$. As $G/Q$ is supersolvable, we get that $G$ would be supersolvable, a contradiction. Therefore, $T \neq 1$ and hence $N \leq T$. Consequently, $G/T$ is supersolvable. If $H \cap T = 1$, then $|Q \cap T| = q$. This forces that $N = Q \cap T$ is a subgroup of order $q$. Therefore, $G$ is supersolvable, a contradiction. Consequently, $H \cap T \neq 1$. Observe that $H \cap T \leq H_{TG} \leq H$, we have $H \cap T = H_{TG} \cap T$ is $S$-semipermuted in $G$ by Lemma 2.9. In the other hand, $H \cap T \leq Q = O_q(G)$ which implies that $H \cap T$ is $S$-quasinormal in $G$ by Lemma 2.8(3). Applying Lemma 2.4, $O_q(G) \leq N_G(H \cap T)$. Furthermore, noting that $H \cap T$ is normal in $Q$. We can get that $H \cap T$ is normal in $G$. Therefore, $N \leq H \cap T \leq H$. The claim as desired.

By above arguments, we get that $N \leq \bigcap_{H \in \mathcal{M}(Q)} H = \Phi(Q)$. By ([6], III, 3.3), $\Phi(Q) \leq \Phi(G)$ and hence $N \leq \Phi(G)$. So $G/\Phi(G)$ is supersolvable and hence $G$ is supersolvable. This is a finally contradiction. The proof of theorem is complete. □

**Proof of Theorem 1.5** Assume that the result is false and let $G$ be a counterexample of minimal order. Applying Lemma 2.1(1) and Lemma 2.7(1), we know that every maximal subgroup of $H$ is either $NH$-embedded or $SS$-quasinormal in $H$. By Theorem 1.4, $H$ is supersolvable. Let $q$ be the largest prime dividing $|H|$ and $Q = O_q(G) \text{ char } Syl_q(H)$. Then $Q$ is normal in $H$ and so it is in $G$. Obviously, $(G/Q)/(H/Q) \cong G/H$ is supersolvable. By Lemmas 2.1(2) and 2.7(3), we know that every maximal subgroup of Sylow subgroups of $H/Q$ is either $NH$-embedded or $SS$-quasinormal in $G/Q$. Therefore, $G/Q$ satisfies the theorem hypothesis and hence $G/Q$ is supersolvable.

Let $N$ be a minimal normal subgroup of $G$ contained in $Q$. Similar to the proof of Theorem 1.3, we know that $G/N$ satisfies the theorem hypothesis and hence $G/N$
is supersolvable. As the class of all supersolvable group is a statured formation, $N$ will be a unique minimal normal subgroup of $G$ contained in $Q$.

In the following, similar to the proof of Theorem 1.4, we can get that $N \leq \Phi(Q)$ and hence $G/\Phi(Q)$ is supersolvable. By ([6], III, 3.3), $\Phi(Q) \leq \Phi(G)$. So $G/\Phi(G)$ is supersolvable. As the class of all supersolvable groups is a statured formation, we obtain that $G$ is supersolvable. This is a final contradiction. The proof of Theorem is complete. □

4. Some applications

As an immediate consequence of Theorem 1.3, we can get the corollaries as follows.

**Corollary 4.1.** ([5, Theorem 3.1]) Let $p$ be the smallest prime dividing $|G|$ and $G_p$ a Sylow $p$-subgroup of $G$. Suppose that every maximal subgroup of $G_p$ is NH-embedded in $G$. Then $G$ is $p$-nilpotent.

**Corollary 4.2.** ([1, Theorem 3.1]) Let $p$ be the smallest prime dividing $|G|$ and $G_p$ a Sylow $p$-subgroup of $G$. Suppose that every maximal subgroup of $G_p$ is SS-quasinormal in $G$. Then $G$ is $p$-nilpotent.

Theorem 1.5 immediately implies the following corollaries.

**Corollary 4.3.** ([9, Theorem 1]) Let $G$ be a finite group. If the maximal subgroups of the Sylow subgroups of $G$ are SS-quasinormal in $G$, then $G$ is supersolvable.

**Corollary 4.4.** ([5, Theorem 3.4]) Let $G$ be a group with a normal subgroup $E$ such that $G/E$ is supersolvable. If every maximal subgroup of every Sylow subgroup of $E$ is NH-embedded in $G$, then $G$ is supersolvable.

**Acknowledgement.** The authors would like to thank the referee for the valuable suggestions and comments. The corresponding author thanks the SUSTech International Center for Mathematics.

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