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THE STRUCTURE OF THE UNIT GROUP OF $\mathbb{F}_{3^k}(C_3 \times D_{2n})$ AND FINITE GROUP ALGEBRAS OF GROUPS OF ORDER 42

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Abstract. Let $\mathbb{F}_q$ be a finite field of characteristic $p > 0$ with $|\mathbb{F}_q| = q = p^k$ and $\mathcal{U}(\mathbb{F}_qG)$ be the unit group of the group algebra $\mathbb{F}_qG$ for some group G. There are 6 groups of order 42 up to isomorphism. In this paper, we provide a characterization of $\mathcal{U}(\mathbb{F}_{3^k}(C_3 \times D_{2n}))$ and establish the structures of the unit groups of some finite group algebras of groups of order 42.

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1. Introduction

Let $\mathbb{F}_qG$ be the group algebra of a group G over a finite field $\mathbb{F}_q$ of order $q = p^k$, for some prime p and a positive integer k. The group containing invertible elements of $\mathbb{F}_qG$ is denoted by $\mathcal{U}(\mathbb{F}_qG)$. Let $J(\mathbb{F}_qG)$ be the Jacobson radical of $\mathbb{F}_qG$ and $V = 1 + J(\mathbb{F}_qG)$. For $H \lhd G$, the canonical homomorphism $\omega : G \to G/H$ can be extended to form an epimorphism $\omega' : \mathbb{F}_qG \to \mathbb{F}_q(G/H)$ given by $\omega'(\sum_{g\in G} \alpha_g g) = \sum_{g\in G} \alpha_g \omega(g)$. Here, $Ker(\omega') = \Delta(G, H)$ is a two-sided ideal of $\mathbb{F}_qG$ generated by the set $\{h - 1 : h \in H\}$. For fundamental definitions and results utilized in this paper, see [16].

The structure of the unit groups of several finite group algebras have already been established in [1,4,5,11,14,19]. The dihedral group of order $2n$ is denoted by D_{2n}. Characterizations of $\mathcal{U}(\mathbb{Z}D_8)$ and $\mathcal{U}(\mathbb{Z}D_{12})$ are provided by the authors in [12]. Some general results describing $\mathcal{U}(\mathbb{F}_{3^k}(C_n \times D_6))$, $\mathcal{U}(\mathbb{F}_{2^k}D_{2p})$ for prime p and $\mathcal{U}(\mathbb{F}_{2^k}D_{2n})$ for odd integers n are given in [7,8,10]. In [2,3], the study of unitary subgroups of some group algebras have been undertaken.

The six non-isomorphic groups of order 42 are: D_{42}, C_{42}, $C_3 \times D_{14}$, $C_7 \times D_6$, $C_7 \rtimes C_6$ and $C_2 \times (C_7 \rtimes C_3)$. The structure of $\mathcal{U}(\mathbb{F}_q(D_{42}))$ is investigated by the authors in [13]. Section 3 of this paper describes the unit group $\mathcal{U}(\mathbb{F}_{3^k}(C_3 \times D_{2n}))$. Further, we characterize the structure of the unit groups of $\mathbb{F}_qC_{42}$, $\mathbb{F}_q(C_3 \times D_{14})$,

$\mathbb{F}_q(C_7 \times D_6)$ in Section 4. Additionally, the semisimple case is discussed for the two semidirect products $C_7 \rtimes C_6$ and $C_2 \times (C_7 \rtimes C_3)$ by configuring the corresponding Wedderburn decomposition.

2. Preliminaries

Some helpful results to explore the structure of $\mathbb{F}_qG/J(\mathbb{F}_qG)$ were given by Ferraz [6]. Let G be a finite group and e be the l.c.m. of the orders of all the p-regular elements in G. Let η be the primitive e-th root of unity over $\mathbb{F}_q$ and $\mathcal{B}$ be the set of integers t *mod* e for which $\eta \to \eta^t$ is an automorphism of $\mathbb{F}_q(\eta)$ over $\mathbb{F}_q$. If l is the multiplicative order of q *mod* e, then $\mathcal{B} = \{1, q, \dots, q^{l-1}\}$ *mod* e. For a p-regular element g, define β_g to be the sum of all conjugates of g. The cyclotomic $\mathbb{F}_q$-class of β_g is defined by

$$S(\beta_g) = \{\beta_{g^t} \mid t \in \mathcal{B}\}.$$

Lemma 2.1. [6, Proposition 1.2] *The number of cyclotomic $\mathbb{F}_q$-classes in G is equal to the number of simple components of $\mathbb{F}_qG/J(\mathbb{F}_qG)$.*

Lemma 2.2. [6, Theorem 1.3] *Let η be the same as defined above and d be the number of cyclotmic $\mathbb{F}_q$-classes in G. If $S_1, \dots, S_d$ are the cyclotomic $\mathbb{F}_q$-classes in G and $P_1, \dots, P_d$ are the simple components of the center of $\mathbb{F}_qG/J(\mathbb{F}_qG)$, then $|S_i| = [P_i : \mathbb{F}_q]$ for a suitable ordering of the indices.*

Let us recall a very useful result from [16, Proposition 3.6.11] which states that if $\mathbb{F}_qG$ is semisimple, then

$$\mathbb{F}_qG \cong \mathbb{F}_q(G/G') \bigoplus \Delta(G, G')$$

where $\mathbb{F}_q(G/G')$ is the sum of all the commutative simple components of $\mathbb{F}_qG$ and $\Delta(G, G')$ is the sum of all others.

Let I_p be the set of all p-elements including the identity element of G. Define a map $\theta : G \to \mathbb{F}_q$ such that $\theta(g) = 1$ if $g \in I_p$ and $\theta(g) = 0$ otherwise. We linearly extend θ from $\mathbb{F}_qG \mapsto \mathbb{F}_q$ such that $\theta(\alpha) = \sum_{g\in G} \alpha_g \theta(g) = \sum_{g\in I_p} \alpha_g$ for all $\alpha = \sum_{g\in G} \alpha_g g \in \mathbb{F}_qG$.

Lemma 2.3. [21, Lemma 2.2] *Let G be a finite group and θ be the map defined above. Then,*

1. $J(\mathbb{F}_qG) \subseteq Ker(\theta)$.
2. $Ker(\theta) = Ann(\widehat{I_p})$.
3. $J(\mathbb{F}_qG) \subseteq Ann(\widehat{I_p})$.

3. The structure of $\mathcal{U}(\mathbb{F}_{3^k}(C_3 \times D_{2n}))$

The group $C_3 \times D_{2n}$ of order $6n$, for $n \geq 1$, is represented as:

$$C_3 \times D_{2n} = \langle r, s, t \mid r^n = s^2 = t^3 = 1,\ rs = sr^{-1},\ rt = tr,\ st = ts \rangle.$$

Theorem 3.1. *Let $\mathbb{F}_q$ be a finite field of order $q = 3^k$. Then,*

$$\mathcal{U}(\mathbb{F}_q(C_3 \times D_{2n})) \cong ((\cdots(C_3^{3nk} \rtimes \underbrace{C_3^k) \rtimes C_3^k) \rtimes \cdots \rtimes C_3^k}_{n \ times}) \rtimes \mathcal{U}(\mathbb{F}_q D_{2n}).$$

Proof. Let $G = C_3 \times D_{2n}$, then $N = \langle t \rangle$ is a normal subgroup of G of order 3. Let $M = \langle r, s \rangle$, then the factor group $G/N \cong M \cong D_{2n}$. Now, define a map $\Psi : \mathbb{F}_q G \to \mathbb{F}_q M$ given by

$$\Psi(\sum_{j=0}^{n-1} \sum_{i=0}^{2} t^i r^j (a_{i+3j} + a_{i+3j+3n} s)) = \sum_{j=0}^{n-1} \sum_{i=0}^{2} r^j (a_{i+3j} + a_{i+3j+3n} s).$$

We obtain a group epimorphism $\Psi' : \mathcal{U}(\mathbb{F}_q G) \to \mathcal{U}(\mathbb{F}_q M)$ by restricting the ring epimorphism Ψ. Again, the restriction of the inclusion map from $\mathbb{F}_q M \to \mathbb{F}_q G$ results in a group monomorphism $\Phi : \mathcal{U}(\mathbb{F}_q M) \to \mathcal{U}(\mathbb{F}_q G)$ defined by

$$\Phi(\sum_{j=0}^{n-1} r^j (z_j + z_{j+n} s)) = \sum_{j=0}^{n-1} r^j (z_j + z_{j+n} s).$$

Let $\mathcal{K} = ker(\Psi')$. Since $\Psi' \circ \Phi = 1_{\mathcal{U}(\mathbb{F}_q M)}$, $\mathcal{U}(\mathbb{F}_q G) \cong \mathcal{K} \rtimes \mathcal{U}(\mathbb{F}_q M)$.

Consider $v = \sum_{j=0}^{n-1} \sum_{i=0}^{2} t^i r^j (a_{i+3j} + a_{i+3j+3n} s) \in \mathcal{K}$, i.e., $\Psi'(v) = 1$ and this results in the following equations:

$$a_0 = 1 - a_1 - a_2,\ a_{3d} = -a_{3d+1} - a_{3d+2} \text{ for } d = 1, \ldots, 2n-1.$$

Given this, an equivalent way of writing the set $\mathcal{K}$ is

$$\mathcal{K} = \{1 + \sum_{j=0}^{n-1} \sum_{i=1}^{2} (t^i - 1) r^j (b_{i+2j} + b_{i+2j+2n} s) \mid b_i \in \mathbb{F}_q\}.$$

It can be verified that $\mathcal{K}$ is a non-abelian group satisfying $\mathcal{K}^3 = 1$. The presumption $q = 3^k$ concludes $|\mathcal{K}| = 3^{4nk}$.

Consider some subgroups of $\mathcal{K}$ defined as:

$$S_d = \{1 + x_1 \hat{t} + x_2 (t + 2t^2) r^d s \mid x_i \in \mathbb{F}_q\} \qquad \text{for } d = 0, 1, \ldots, n-1,$$

and $U_n = \mathcal{K}$,

$$U_0 = \{1 + \sum_{j=0}^{n-1} \sum_{i=1}^{2} (t^i - 1) r^j y_{i+2j} + \hat{t} \sum_{i=0}^{n-1} r^i y_{i+2n+1} s \mid y_i \in \mathbb{F}_q\},$$

$$U_d = \{1+\sum_{i=1}^{2}(t^i-1)(\sum_{j=0}^{n-1} r^j y_{i+2j} + \sum_{j=0}^{d-1} r^j y_{i+2j+2n} s) + \hat{t}\sum_{i=d}^{n-1} r^i y_{i+d+2n+1} s \mid y_i \in \mathbb{F}_q\} \qquad \text{for } d=1,\ldots,n-1.$$

Here, S_d and U_d are subgroups of U_{d+1} and $I = S_d \cap U_d = \{1+x_1\hat{t} \mid x_1 \in \mathbb{F}_q\} \cong C_3^k$, for $d = 0,1,\ldots,n-1$. Moreover, S_d is an abelian group and therefore, for each $d = 0,1,\ldots,n-1$, there exists some subgroup R_d of S_d satisfying $S_d = I \times R_d$ with $R_d \cong C_3^k$. Considering some general elements

$$p_d = 1 + x_1\hat{t} + x_2(t+2t^2)r^d s \in S_d \qquad \text{for } d=0,\ldots,n-1,$$

$$q_0 = 1 + \sum_{j=0}^{n-1}\sum_{i=1}^{2}(t^i-1)r^j y_{i+2j} + \hat{t}\sum_{i=0}^{n-1} r^i y_{i+2n+1} s \in U_0,$$

$$q_d = 1+\sum_{i=1}^{2}(t^i-1)(\sum_{j=0}^{n-1} r^j y_{i+2j} + \sum_{j=0}^{d-1} r^j y_{i+2j+2n} s) + \hat{t}\sum_{i=d}^{n-1} r^i y_{i+d+2n+1} s \in U_d \qquad \text{for } d=1,\ldots,n-1.$$

Let us define:

$E_1 = \sum_{j=0}^{n-1}\sum_{i=1}^{2}(t^i-1)r^j y_{i+2j}$, $E_1' = \sum_{j=0}^{n-1}\sum_{i=1}^{2}(t^i-1)r^{-j} y_{i+2j}$,

$E_{2,0} = 0 = E_{2,0}'$, $E_{2,d} = \sum_{j=0}^{d-1}\sum_{i=1}^{2}(t^i-1)r^j y_{i+2j+2n}$ for $d=1,\ldots,n-1$,

$E_{2,d}' = \sum_{j=0}^{d-1}\sum_{i=1}^{2}(t^i-1)r^{-j} y_{i+2j+2n}$ for $d=1,\ldots,n-1$ and

$E_{3,d} = \hat{t}\sum_{i=d}^{n-1} r^i y_{i+d+2n+1}$ for $d=0,\ldots,n-1$.

Equivalently, q_d can be written as:

$$q_d = 1 + E_1 + E_{2,d}s + E_{3,d}s \in U_d \qquad \text{for } d=0,\ldots,n-1.$$

The fact that $S_d \subseteq \mathcal{K}$, gives us $S_d^3 = 1$. Thus, for $p_d \in S_d$,

$$p_d^{-1} = p_d^2 = 1 + (2x_1 + x_2^2)\hat{t} + 2x_2(t+2t^2)r^d s \qquad \text{for } d=0,\ldots,n-1.$$

The structure of $\mathcal{K}$ can be concluded by combining the information given so far along with the following steps.

Step 1: Let $q_0 \in U_0$ and $p_0 \in S_0$. Then,

$$\begin{aligned} q_0^{p_0} &= p_0^{-1} q_0 p_0 \\ &= q_0 + x_2(t + 2t^2)(E_1 - E_1')s \in U_0. \end{aligned}$$

By definition, S_0 normalizes U_0. Furthermore, U_0 is abelian and therefore $U_0 \cong C_3^{3nk}$. Clearly, $U_0 \cap R_0 = \{1\}$ and hence $U_1 \cong U_0 \rtimes R_0 \cong C_3^{3nk} \rtimes C_3^k$.

Step 2: Let $q_1 \in U_1$ and $p_1 \in S_1$. Then,

$$\begin{aligned} q_1^{p_1} &= p_1^{-1} q_1 p_1 \\ &= q_1 + x_2(t + 2t^2)(E_1 - E_1')rs + x_2(t + 2t^2)(E_{2,1}r^{-1} - E_{2,1}'r) \in U_1. \end{aligned}$$

Again, it follows that S_1 normalizes U_1 and as $U_1 \cap R_1 = \{1\}$, therefore $U_2 \cong U_1 \rtimes R_1 \cong (C_3^{3nk} \rtimes C_3^k) \rtimes C_3^k$.

In general,

$$q_d^{p_d} = q_d + x_2(t + 2t^2)(E_1 - E_1')r^d s + x_2(t + 2t^2)(E_{2,d}r^{-d} - E_{2,d}'r^d) \in U_d$$

for $d = 0, \ldots, n-1$.

Proceeding in a similar manner, it can be shown that S_d normalizes U_d and therefore $U_{d+1} \cong U_d \rtimes R_d$ for $d = 2, \ldots, n-1$. Consequently, $U_n \cong U_{n-1} \rtimes R_{n-1}$, that is

$$\mathcal{K} \cong ((\cdots (C_3^{3nk} \rtimes \underbrace{C_3^k) \rtimes C_3^k) \rtimes \ldots \rtimes C_3^k}_{n \text{ times}}).$$

Since $M \cong D_{2n}$, we get

$$\mathcal{U}(\mathbb{F}_q(C_3 \times D_{2n})) \cong ((\cdots (C_3^{3nk} \rtimes \underbrace{C_3^k) \rtimes C_3^k) \rtimes \ldots \rtimes C_3^k}_{n \text{ times}}) \rtimes \mathcal{U}(\mathbb{F}_q D_{2n}). \qquad \square$$

4. Some structures of $\mathcal{U}(\mathbb{F}_qG)$ for $|G| = 42$

The results pertaining to the structures of $\mathcal{U}(\mathbb{F}_qG)$ for the earlier mentioned groups G of order 42 are provided in this section.

Theorem 4.1. *Let $\mathbb{F}_q$ be a finite field of order $q = p^k$ with characteristic p and let $G = C_7 \times D_6$.*

(1) *If* **Char** $\mathbb{F}_q = 2$, *then* $\mathcal{U}(\mathbb{F}_qG) \cong C_2^{7k} \rtimes \mathcal{U}(\mathbb{F}_qG/J(\mathbb{F}_qG))$.

(2) *If* **Char** $\mathbb{F}_q = 3$, *then*

$$\mathcal{U}(\mathbb{F}_qG) \cong \begin{cases} (C_3^{21k} \rtimes C_3^k) \rtimes C_{q-1}^{14}, & \text{if } q \equiv 1 \text{ mod } 14; \\ (C_3^{21k} \rtimes C_3^k) \rtimes (C_{q-1}^2 \times C_{q^2-1}^6), & \text{if } q \equiv -1 \text{ mod } 14; \\ (C_3^{21k} \rtimes C_3^k) \rtimes (C_{q-1}^2 \times C_{q^6-1}^2), & \text{if } q \equiv 3, 5 \text{ mod } 14; \\ (C_3^{21k} \rtimes C_3^k) \rtimes (C_{q-1}^2 \times C_{q^3-1}^4), & \text{if } q \equiv -3, -5 \text{ mod } 14. \end{cases}$$

(3) *If* **Char** $\mathbb{F}_q = 7$, *then*

$$\mathcal{U}(\mathbb{F}_qG) \cong \mathcal{K} \rtimes (C_{q-1}^2 \times GL(2, \mathbb{F}_q)),$$

where $\mathcal{K}$ *is a non-abelian group of order* 7^{36k} *satisfying* $\mathcal{K}^7 = 1$.

(4) *If* **Char** $\mathbb{F}_q \neq 2, 3, 7$, *then* $\mathcal{U}(\mathbb{F}_qG)$ *is isomorphic to*

(a) $C_{q-1}^{14} \times GL(2, \mathbb{F}_q)^7$, *if* $q \equiv 1, 29$ *mod* 42.

(b) $C_{q-1}^2 \times C_{q^6-1}^2 \times GL(2, \mathbb{F}_q) \times GL(2, \mathbb{F}_{q^6})$, *if* $q \equiv 5, 17, 19, 31$ *mod* 42.

(c) $C_{q-1}^2 \times C_{q^3-1}^4 \times GL(2, \mathbb{F}_q) \times GL(2, \mathbb{F}_{q^3})^2$, *if* $q \equiv 11, 23, 25, 37$ *mod* 42.

(d) $C_{q-1}^2 \times C_{q^2-1}^6 \times GL(2, \mathbb{F}_q) \times GL(2, \mathbb{F}_{q^2})^3$, *if* $q \equiv 13, 41$ *mod* 42.

Proof. Let $G = \langle r, s, t \mid r^3 = s^2 = t^7 = 1,\ rs = sr^{-1},\ rt = tr,\ st = ts \rangle$.

1. Char $\mathbb{F}_q = 2$**:** For $p = 2$, we have $I_2 = \{1, s, rs, r^2s\}$. Then, $\widehat{I_2} = 1 + \hat{r}s$. Let us consider a general element $\gamma = \sum\limits_{k=0}^{1} \sum\limits_{j=0}^{6} \sum\limits_{i=0}^{2} h_{i+3j+21k} t^j r^i s^k \in \mathbb{F}_qG$ such that

$$\gamma(1 + \hat{r}s) = 0 \text{ i.e., } \gamma + \gamma\hat{r}s = 0.$$

After simplifying, we get the equations:

$$h_{3i} = h_{3i+m} = h_{3i+m+21} \text{ for } i = 0, 1, \ldots, 6, \text{ and } m = 0, 1, 2.$$

Consequently, $\gamma = \sum\limits_{i=0}^{6} q_i t^i \hat{r}(1 + s)$. Thus,

$$\text{Ann}(\widehat{I_2}) = \{\sum_{i=0}^{6} q_i t^i \hat{r}(1 + s) \mid q_i \in \mathbb{F}_q\}.$$

Observe that $t, \hat{r} \in Z(\mathbb{F}_qG)$ and $(1 + s)^2 = 0$, therefore $\text{Ann}(\widehat{I_2})^2 = (0)$. As a result, $\text{Ann}(\widehat{I_2}) \subseteq J(\mathbb{F}_qG)$. By using Lemma 2.3, we obtain $\text{Ann}(\widehat{I_2}) = J(\mathbb{F}_qG)$. It is trivial to see that $V^2 = 1$ and therefore $V \cong C_2^{7k}$. Hence, with the help of [9, Lemma 2.1], we conclude that

$$\mathcal{U}(\mathbb{F}_qG) \cong C_2^{7k} \rtimes \mathcal{U}(\mathbb{F}_qG/J(\mathbb{F}_qG)).$$

2. Char $\mathbb{F}_q = 3$**:** Using the result given by Gildea [7, Theorem 1.1] for the particular case of $n = 7$, we get

$$\mathcal{U}(\mathbb{F}_{3^k}(C_7 \times D_6)) \cong (C_3^{21k} \rtimes C_3^{7k}) \rtimes \mathcal{U}(\mathbb{F}_{3^k}(C_{14})).$$

Finally, we deduce our result by utilizing [18, Theorem 4.3].

3. Char $\mathbb{F}_q = 7$**:** Let $M = \langle r, s\rangle$ and $N = \langle t\rangle$. Since $N \lhd G$, $G/N \cong M \cong D_6$. Now, define a map $\Psi : \mathbb{F}_qG \to \mathbb{F}_qM$ given by

$$\Psi(\sum_{j=0}^{2}\sum_{i=0}^{6} t^i r^j (a_{i+7j} + a_{i+7j+21}s)) = \sum_{j=0}^{2}\sum_{i=0}^{6} r^j (a_{i+7j} + a_{i+7j+21}s).$$

We obtain a group epimorphism $\Psi' : \mathcal{U}(\mathbb{F}_qG) \to \mathcal{U}(\mathbb{F}_qM)$ by restricting the ring epimorphism Ψ. Again, the restriction of the inclusion map from $\mathbb{F}_qM \to \mathbb{F}_qG$ provides a group monomorphism $\Phi : \mathcal{U}(\mathbb{F}_qM) \to \mathcal{U}(\mathbb{F}_qG)$ defined by

$$\Phi(\sum_{j=0}^{2} r^j (z_j + z_{j+3}s)) = \sum_{j=0}^{2} r^j (z_j + z_{j+3}s).$$

Let $\mathcal{K} = ker(\Psi')$. Since $\Psi' \circ \Phi = 1_{\mathcal{U}(\mathbb{F}_qM)}$,

$$\mathcal{U}(\mathbb{F}_qG) \cong \mathcal{K} \rtimes \mathcal{U}(\mathbb{F}_qM) \cong \mathcal{K} \rtimes \mathcal{U}(\mathbb{F}_qD_6).$$

Consider $v = \sum_{j=0}^{2}\sum_{i=0}^{6} t^i r^j (a_{i+7j} + a_{i+7j+21}s) \in \mathcal{K}$ i.e., $\Psi'(v) = 1$, which leads to the following equations:

$$\sum_{i=0}^{6} a_i = 1, \ \sum_{i=0}^{6} a_{i+7d} = 0 \text{ for } d = 1, \dots, 5.$$

Thus,

$$\mathcal{K} = \{1 + \sum_{j=0}^{2}\sum_{i=1}^{6} (t^i - 1) r^j (b_{i+6j} + b_{i+6j+18}s) \mid b_i \in \mathbb{F}_q\}.$$

Consequently, it can be verified that $\mathcal{K}$ is a non-abelian group satisfying $\mathcal{K}^7 = 1$ with $|\mathcal{K}| = 7^{36k}$. Also, by [20, Theorem 2.3], we get $\mathcal{U}(\mathbb{F}_qD_6) \cong C_{q-1}^2 \times GL(2, \mathbb{F}_q)$. Hence,

$$\mathcal{U}(\mathbb{F}_qG) \cong \mathcal{K} \rtimes (C_{q-1}^2 \times GL(2, \mathbb{F}_q)).$$

4. Char $\mathbb{F}_q \neq 2, 3, 7$**:** Since $p \nmid |G|$, by Maschke's theorem $\mathbb{F}_qG$ is a semisimple group algebra and $J(\mathbb{F}_qG) = (0)$. Also,

$$\mathbb{F}_qG \cong \mathbb{F}_q(G/G') \bigoplus \Delta(G, G').$$

The conjugacy classes of G are:

$$\begin{aligned} [t^i] &= \{t^i\} && \text{for } i = 0, 1, \dots, 6; \\ [rt^i] &= \{rt^i, r^2t^i\} && \text{for } i = 0, 1, \dots, 6; \\ [st^i] &= \{st^i, rst^i, r^2st^i\} && \text{for } i = 0, 1, \dots, 6. \end{aligned}$$

Observe that $G/G' \cong C_{14}$, thus the Wedderburn decomposition is

$$\mathbb{F}_qG \cong \mathbb{F}_qC_{14} \bigoplus_{j=1}^{m} M(n_j, R_j)$$

where $n_j \geq 2$ and for each $j \in \{1, \ldots, m\}$, R_j represents a division algebra over $\mathbb{F}_q$. Since class sums form a basis for $Z(\mathbb{F}_qG)$, $dim(Z(\mathbb{F}_qG)) = 21$. This implies, $m \leq 7$. Also, for any choice of characteristic p, $e = 42$. The structure of $\mathbb{F}_qC_{14}$ has been given by [18, Theorem 4.3].

(**a**) If $q \equiv 1, 29 \ mod \ 42$, then $\mathcal{B} = \{1\} \ mod \ 42$ or $\mathcal{B} = \{1, 29\} \ mod \ 42$. This provides $|S(\beta_g)| = 1$ for all $g \in G$. Using Lemmas 2.1 and 2.2, we get

$$\mathbb{F}_qG \cong \mathbb{F}_q^{14} \bigoplus_{j=1}^{7} M(n_j, \mathbb{F}_q).$$

The equation generated by equating the dimensions of both sides is $\sum_{j=1}^{7} n_j^2 = 28$. The only possible solution is $n_j = 2$ for all $j \in \{1, \ldots, 7\}$. Hence,

$$\mathbb{F}_qG \cong \mathbb{F}_q^{14} \bigoplus M(2, \mathbb{F}_q)^7.$$

(**b**) If $q \equiv 5, 17, 19, 31 \ mod \ 42$, then $\mathcal{B} = \{1, 5, 17, 25, 37, 41\} \ mod \ 42$ or $\mathcal{B} = \{1, 13, 19, 25, 31, 37\} \ mod \ 42$. Thus, $|S(\beta_g)| = 1$ for $g = 1, \ r, \ s$, and $|S(\beta_g)| = 6$ for $g = t, \ rt, \ st$. Then,

$$\mathbb{F}_qG \cong \mathbb{F}_q^2 \bigoplus \mathbb{F}_{q^6}^2 \bigoplus M(n_1, \mathbb{F}_q) \bigoplus M(n_2, \mathbb{F}_{q^6}),$$

by using Lemmas 2.1 and 2.2. The necessary condition $n_1^2 + 6n_2^2 = 28$ is true only when $n_1 = n_2 = 2$. Hence,

$$\mathbb{F}_qG \cong \mathbb{F}_q^2 \bigoplus \mathbb{F}_{q^6}^2 \bigoplus M(2, \mathbb{F}_q) \bigoplus M(2, \mathbb{F}_{q^6}).$$

(**c**) If $q \equiv 11, 23, 25, 37 \ mod \ 42$, then $\mathcal{B} = \{1, 11, 23, 25, 29, 37\} \ mod \ 42$ or $\mathcal{B} = \{1, 25, 37\} \ mod \ 42$. This gives $|S(\beta_g)| = 1$ for $g = 1, \ r, \ s$ and $|S(\beta_g)| = 3$ for $g = t, \ t^3, \ rt, \ rt^3, \ st, \ st^3$. Using Lemmas 2.1 and 2.2,

$$\mathbb{F}_qG \cong \mathbb{F}_q^2 \bigoplus \mathbb{F}_{q^3}^4 \bigoplus M(n_1, \mathbb{F}_q) \bigoplus M(n_2, \mathbb{F}_{q^3}) \bigoplus M(n_3, \mathbb{F}_{q^3}),$$

with the restriction that $n_1^2 + 3n_2^2 + 3n_3^2 = 28$. The only possible solution of the equation is $n_1 = n_2 = n_3 = 2$. Hence,

$$\mathbb{F}_qG \cong \mathbb{F}_q^2 \bigoplus \mathbb{F}_{q^3}^4 \bigoplus M(2, \mathbb{F}_q) \bigoplus M(2, \mathbb{F}_{q^3})^2.$$

(**d**) If $q \equiv 13, 41 \ mod \ 42$, then $\mathcal{B} = \{1, 13\} \ mod \ 42$ or $\mathcal{B} = \{1, 41\} \ mod \ 42$. Thus, $|S(\beta_g)| = 1$ for $g = 1, \ r, \ s$, and $|S(\beta_g)| = 2$ for $g = t, \ t^2, \ t^3, rt, \ rt^2, \ rt^3, \ st, \ st^2, \ st^3$. Lemmas 2.1 and 2.2 provide us

$$\mathbb{F}_qG \cong \mathbb{F}_q^2 \bigoplus \mathbb{F}_{q^2}^6 \bigoplus M(n_1, \mathbb{F}_q) \bigoplus_{j=2}^{4} M(n_j, \mathbb{F}_{q^2}),$$

under the condition that $n_1^2 + \sum_{j=2}^{4} 2n_j^2 = 28$, which has the only solution as $n_j = 2$ for all $j \in \{1, \dots, 4\}$. Hence,

$$\mathbb{F}_qG \cong \mathbb{F}_q^2 \bigoplus \mathbb{F}_{q^2}^6 \bigoplus M(2, \mathbb{F}_q) \bigoplus M(2, \mathbb{F}_{q^2})^3. \qquad \square$$

Theorem 4.2. *Let $\mathbb{F}_q$ be a finite field of order $q = p^k$ with characteristic p and let $G = C_3 \times D_{14}$.*

(1) *If* **Char** $\mathbb{F}_q = 2$, *then*

$$\mathcal{U}(\mathbb{F}_qG) \cong C_2^{3k} \rtimes \mathcal{U}(\mathbb{F}_qG/J(\mathbb{F}_qG)).$$

(2) *If* **Char** $\mathbb{F}_q = 3$, *then*

$$\mathcal{U}(\mathbb{F}_qG) \cong \begin{cases} \mathcal{K} \rtimes (C_{q-1}^2 \times GL(2, \mathbb{F}_q)^3), & \text{if } q \equiv \pm 1 \text{ mod } 7; \\ \mathcal{K} \rtimes (C_{q-1}^2 \times GL(2, \mathbb{F}_{q^3})), & \text{if } q \equiv \pm 2, \pm 3 \text{ mod } 7, \end{cases}$$

where $\mathcal{K} \cong (((((((C_3^{21k} \rtimes C_3^k) \rtimes C_3^k) \rtimes C_3^k) \rtimes C_3^k) \rtimes C_3^k) \rtimes C_3^k) \rtimes C_3^k)$.

(3) *If* **Char** $\mathbb{F}_q = 7$, *then*

$$\mathcal{U}(\mathbb{F}_qG) \cong \mathcal{K} \rtimes C_{q-1}^6,$$

where $\mathcal{K}$ is a non-abelian group of order 7^{36k} satisfying $\mathcal{K}^7 = 1$.

(4) *If* **Char** $\mathbb{F}_q \neq 2, 3, 7$, *then $\mathcal{U}(\mathbb{F}_qG)$ is isomorphic to*
  (a) $C_{q-1}^6 \times GL(2, \mathbb{F}_q)^9$, *if* $q \equiv 1, 13$ *mod* 42.
  (b) $C_{q-1}^2 \times C_{q^2-1}^2 \times GL(2, \mathbb{F}_{q^3}) \times GL(2, \mathbb{F}_{q^6})$, *if* $q \equiv 5, 11, 17, 23$ *mod* 42.
  (c) $C_{q-1}^6 \times GL(2, \mathbb{F}_{q^3})^3$, *if* $q \equiv 19, 25, 31, 37$ *mod* 42.
  (d) $C_{q-1}^2 \times C_{q^2-1}^2 \times GL(2, \mathbb{F}_q)^3 \times GL(2, \mathbb{F}_{q^2})^3$, *if* $q \equiv 29, 41$ *mod* 42.

Proof. Let $G = \langle r, s, t \mid r^7 = s^2 = t^3 = 1,\ rs = sr^{-1},\ rt = tr,\ st = ts \rangle$.

1. Char $\mathbb{F}_q = 2$**:** For $p = 2$, we have $I_2 = \{1, s, rs, r^2s, \dots, r^6s\}$. Then, $\widehat{I_2} = 1 + \hat{r}s$. Let us consider a general element $\gamma = \sum_{k=0}^{1} \sum_{j=0}^{2} \sum_{i=0}^{6} h_{i+7j+21k} t^j r^i s^k \in \mathbb{F}_qG$ such that

$$\gamma(1 + \hat{r}s) = 0 \text{ i.e., } \gamma + \gamma\hat{r}s = 0.$$

After simplifying, we get the equations:

$$h_{7i} = h_{7i+m} = h_{7i+m+21} \text{ for } i = 0, 1, 2, \text{ and } m = 0, 1, \dots, 6.$$

Consequently, $\gamma = \sum_{i=0}^{2} q_i t^i \hat{r}(1 + s)$. Thus,

$$\text{Ann}(\widehat{I_2}) = \{\sum_{i=0}^{2} q_i t^i \hat{r}(1 + s) \mid q_i \in \mathbb{F}_q\}.$$

Observe that $t, \hat{r} \in Z(\mathbb{F}_qG)$ and $(1+s)^2 = 0$, therefore $\text{Ann}(\widehat{I_2})^2 = (0)$. As a result, $\text{Ann}(\widehat{I_2}) \subseteq J(\mathbb{F}_qG)$. By using Lemma 2.3, we obtain $\text{Ann}(\widehat{I_2}) = J(\mathbb{F}_qG)$. It is trivial to show that $V^2 = 1$ and therefore $V \cong C_2^{3k}$. Hence, with the help of [9, Lemma 2.1], we conclude that

$$\mathcal{U}(\mathbb{F}_qG) \cong C_2^{3k} \rtimes \mathcal{U}(\mathbb{F}_qG/J(\mathbb{F}_qG)).$$

2. Char $\mathbb{F}_q = 3$: Specifically, for $n = 7$, applying Theorem 3.1 provides

$$\mathcal{U}(\mathbb{F}_q(C_3 \times D_{14})) \cong \mathcal{K} \rtimes \mathcal{U}(\mathbb{F}_qD_{14})$$

where

$$\mathcal{K} \cong ((((((C_3^{21k} \rtimes C_3^k) \rtimes C_3^k) \rtimes C_3^k) \rtimes C_3^k) \rtimes C_3^k) \rtimes C_3^k) \rtimes C_3^k).$$

Furthermore, based on [17, Theorem 4.1], we find

$$\mathcal{U}(\mathbb{F}_qD_{14}) \cong \begin{cases} C_{q-1}^2 \times GL(2,\mathbb{F}_q)^3, & \text{if } q \equiv \pm 1 \ mod \ 7; \\ C_{q-1}^2 \times GL(2,\mathbb{F}_{q^3}), & \text{if } q \equiv \pm 2, \pm 3 \ mod \ 7. \end{cases}$$

3. Char $\mathbb{F}_q = 7$: Let $M = \langle s, t \rangle$ and $N = \langle r \rangle$. Since $N \lhd G$, $G/N \cong M \cong C_6$. Now, define a map $\Psi : \mathbb{F}_qG \to \mathbb{F}_qM$ given by

$$\Psi\Big(\sum_{j=0}^{2}\sum_{i=0}^{6} t^j r^i (a_{i+7j} + a_{i+7j+21}s)\Big) = \sum_{j=0}^{2}\sum_{i=0}^{6} t^j (a_{i+7j} + a_{i+7j+21}s).$$

We obtain a group epimorphism $\Psi' : \mathcal{U}(\mathbb{F}_qG) \to \mathcal{U}(\mathbb{F}_qM)$ by restricting the ring epimorphism Ψ. Again, the restriction of the inclusion map from $\mathbb{F}_qM \to \mathbb{F}_qG$ provides a group monomorphism $\Phi : \mathcal{U}(\mathbb{F}_qM) \to \mathcal{U}(\mathbb{F}_qG)$ defined by

$$\Phi\Big(\sum_{j=0}^{2} t^j (z_j + z_{j+3}s)\Big) = \sum_{j=0}^{2} t^j (z_j + z_{j+3}s).$$

Let $\mathcal{K} = ker(\Psi')$. Since $\Psi' \circ \Phi = 1_{\mathcal{U}(\mathbb{F}_qM)}$,

$$\mathcal{U}(\mathbb{F}_qG) \cong \mathcal{K} \rtimes \mathcal{U}(\mathbb{F}_qM) \cong \mathcal{K} \rtimes \mathcal{U}(\mathbb{F}_qC_6).$$

Consider $v = \sum_{j=0}^{2}\sum_{i=0}^{6} t^j r^i (a_{i+7j} + a_{i+7j+21}s) \in \mathcal{K}$ i.e., $\Psi'(v) = 1$, which leads to the following equations:

$$\sum_{i=0}^{6} a_i = 1, \ \sum_{i=0}^{6} a_{i+7d} = 0 \text{ for } d = 1, \ldots, 5.$$

Thus,

$$\mathcal{K} = \{1 + \sum_{j=0}^{2}\sum_{i=1}^{6} (r^i - 1)t^j (b_{i+6j} + b_{i+6j+18}s) \mid b_i \in \mathbb{F}_q\}.$$

Consequently, it can be verified that $\mathcal{K}$ is a non-abelian group satisfying $\mathcal{K}^7 = 1$ with $|\mathcal{K}| = 7^{36k}$. Also, by [18, Theorem 4.1], we get $\mathcal{U}(\mathbb{F}_qC_6) \cong C_{q-1}^6$. Hence,

$$\mathcal{U}(\mathbb{F}_qG) \cong \mathcal{K} \rtimes C_{q-1}^6.$$

4. Char $\mathbb{F}_q \neq 2, 3, 7$**:** Since $p \nmid |G|$, $\mathbb{F}_qG$ is a semisimple group algebra and $J(\mathbb{F}_qG) = (0)$. Also,

$$\mathbb{F}_qG \cong \mathbb{F}_q(G/G') \bigoplus \Delta(G, G').$$

The conjugacy classes of G are:

$$\begin{aligned}
&[t^i] = \{t^i\} && \text{for } i = 0, 1, 2;\\
&[rt^i] = \{rt^i, r^6t^i\} && \text{for } i = 0, 1, 2;\\
&[r^2t^i] = \{r^2t^i, r^5t^i\} && \text{for } i = 0, 1, 2;\\
&[r^3t^i] = \{r^3t^i, r^4t^i\} && \text{for } i = 0, 1, 2;\\
&[st^i] = \{st^i, rst^i, r^2st^i, \dots, r^6st^i\} && \text{for } i = 0, 1, 2.
\end{aligned}$$

Observe that $G/G' \cong C_6$, thus

$$\mathbb{F}_qG \cong \mathbb{F}_qC_6 \bigoplus_{j=1}^{m} M(n_j, R_j)$$

where $n_j \geq 2$ and for each $j \in \{1, \dots, m\}$, R_j represents a division algebra over $\mathbb{F}_q$. Since $dim(Z(\mathbb{F}_qG)) = 15$, $m \leq 9$. Also, for any value of characteristic p, $e = 42$. By [18, Theorem 4.1], the structure of $\mathbb{F}_qC_6$ has been determined.

(**a**) If $q \equiv 1, 13 \ mod\ 42$, then $\mathcal{B} = \{1\} \ mod\ 42$ or $\mathcal{B} = \{1, 13\} \ mod\ 42$. This gives $|S(\beta_g)| = 1$ for all $g \in G$. Using Lemmas 2.1 and 2.2, we get

$$\mathbb{F}_qG \cong \mathbb{F}_q^6 \bigoplus_{j=1}^{9} M(n_j, \mathbb{F}_q).$$

The equation $\sum_{j=1}^{9} n_j^2 = 36$ is obtained by equating the dimensions of both sides. The only possible solution is when $n_j = 2$ for all $j \in \{1, \dots, 9\}$. Hence,

$$\mathbb{F}_qG \cong \mathbb{F}_q^6 \bigoplus M(2, \mathbb{F}_q)^9.$$

(**b**) If $q \equiv 5, 11, 17, 23 \ mod\ 42$, then $\mathcal{B} = \{1, 5, 17, 25, 37, 41\} \ mod\ 42$ or $\mathcal{B} = \{1, 11, 23, 25, 29, 37\} \ mod\ 42$. Thus, $|S(\beta_g)| = 1$ for $g = 1,\ s$, $|S(\beta_g)| = 2$ for $g = t,\ st$, $|S(\beta_g)| = 3$ for $g = r$, and $|S(\beta_g)| = 6$ for $g = rt$. Then,

$$\mathbb{F}_qG \cong \mathbb{F}_q^2 \bigoplus \mathbb{F}_{q^2}^2 \bigoplus M(n_1, \mathbb{F}_{q^3}) \bigoplus M(n_2, \mathbb{F}_{q^6}),$$

by using Lemmas 2.1 and 2.2. The necessary condition $3n_1^2+6n_2^2=36$ is true only when $n_1=n_2=2$. Hence,

$$\mathbb{F}_qG \cong \mathbb{F}_q^2 \bigoplus \mathbb{F}_{q^2}^2 \bigoplus M(2,\mathbb{F}_{q^3}) \bigoplus M(2,\mathbb{F}_{q^6}).$$

(**c**) If $q \equiv 19,25,31,37$ *mod* 42, then $\mathcal{B}=\{1,13,19,25,31,37\}$ *mod* 42 or $\mathcal{B}=\{1,25,37\}$ *mod* 42. This provides $|S(\beta_g)|=1$ for $g=1,\ s,\ t,\ t^2,\ st,\ st^2$, and $|S(\beta_g)|=3$ for $g=r,\ rt,\ rt^2$. Using Lemmas 2.1 and 2.2,

$$\mathbb{F}_qG \cong \mathbb{F}_q^6 \bigoplus_{j=1}^{3} M(n_j,\mathbb{F}_{q^3}),$$

with the restriction that $3(n_1^2+n_2^2+n_3^2)=36$. The only possible solution of this equation is $n_1=n_2=n_3=2$. Hence,

$$\mathbb{F}_qG \cong \mathbb{F}_q^6 \bigoplus M(2,\mathbb{F}_{q^3})^3.$$

(**d**) If $q \equiv 29,41$ *mod* 42, then $\mathcal{B}=\{1,29\}$ *mod* 42 or $\mathcal{B}=\{1,41\}$ *mod* 42. Thus, $|S(\beta_g)|=1$ for $g=1,\ s,\ r,\ r^2,\ r^3$, and $|S(\beta_g)|=2$ for $g=t,\ st,\ rt,\ r^2t,\ r^3t$. Lemmas 2.1 and 2.2 provide us

$$\mathbb{F}_qG \cong \mathbb{F}_q^2 \bigoplus \mathbb{F}_{q^2}^2 \bigoplus_{j=1}^{3} M(n_j,\mathbb{F}_q) \bigoplus_{j=4}^{6} M(n_j,\mathbb{F}_{q^2}),$$

with the condition that $\sum_{j=1}^{3} n_j^2 + \sum_{j=4}^{6} 2n_j^2 = 36$, which only has the solution as $n_j=2$ for all $j \in \{1,\dots,6\}$. Hence,

$$\mathbb{F}_qG \cong \mathbb{F}_q^2 \bigoplus \mathbb{F}_{q^2}^2 \bigoplus M(2,\mathbb{F}_q)^3 \bigoplus M(2,\mathbb{F}_{q^2})^3. \qquad \square$$

Theorem 4.3. *Let $\mathbb{F}_q$ be a finite field of order $q=p^k$ with characteristic $p \neq 2,3,7$ and let $G=C_7 \rtimes C_6$. Then,*

$$\mathcal{U}(\mathbb{F}_qG) \cong \begin{cases} C_{q-1}^6 \times GL(6,\mathbb{F}_q), & \text{if } q \equiv 1,13,19,25,31,37 \ mod\ 42; \\ C_{q-1}^2 \times C_{q^2-1}^2 \times GL(6,\mathbb{F}_q), & \text{if } q \equiv 5,11,17,23,29,41 \ mod\ 42. \end{cases}$$

Proof. Let $G=\langle a,b \mid a^7=b^6=1,\ bab^{-1}=a^3\rangle$.
Since $p \nmid |G|$, $\mathbb{F}_qG$ is a semisimple group algebra and $J(\mathbb{F}_qG)=(0)$. Also,

$$\mathbb{F}_qG \cong \mathbb{F}_q(G/G') \bigoplus \Delta(G,G').$$

The conjugacy classes of G are:

$$\begin{aligned}
[1] &= \{1\},\\
[a] &= \{a,a^2,\dots,a^6\},\\
[b^i] &= \{b^i,ab^i,a^2b^i,\dots,a^6b^i\} \quad \text{for } i=1,\dots,5.
\end{aligned}$$

Observe that $G/G' \cong C_6$, thus

$$\mathbb{F}_qG \cong \mathbb{F}_qC_6 \bigoplus_{j=1}^{m} M(n_j, R_j)$$

where $n_j \geq 2$ and for each $j \in \{1, \dots, m\}$, R_j represents a division algebra over $\mathbb{F}_q$. Since $dim(Z(\mathbb{F}_qG)) = 7$, $m = 1$. Also, for any choice of characteristic p, $e = 42$. The structure of $\mathbb{F}_qC_6$ has been given by ([18], Theorem 4.1).

(**a**) If $q \equiv 1, 13, 19, 25, 31, 37$ mod 42, then $\mathcal{B} = \{1\}$ mod 42 or $\mathcal{B} = \{1, 13\}$ mod 42 or $\mathcal{B} = \{1, 13, 19, 25, 31, 37\}$ mod 42 or $\mathcal{B} = \{1, 25, 37\}$ mod 42. This gives $|S(\beta_g)| = 1$ for all $g \in G$. Using Lemmas 2.1 and 2.2, we get

$$\mathbb{F}_qG \cong \mathbb{F}_q^6 \bigoplus M(n_1, \mathbb{F}_q)$$

with the condition that $n_1^2 = 36$. Hence, $n_1 = 6$ and consequently

$$\mathbb{F}_qG \cong \mathbb{F}_q^6 \bigoplus M(6, \mathbb{F}_q).$$

(**b**) If $q \equiv 5, 11, 17, 23, 29, 41$ mod 42, then $\mathcal{B} = \{1, 5, 17, 25, 37, 41\}$ mod 42 or $\mathcal{B} = \{1, 11, 23, 25, 29, 37\}$ mod 42 or $\mathcal{B} = \{1, 29\}$ mod 42 or $\mathcal{B} = \{1, 41\}$ mod 42. This gives $|S(\beta_g)| = 1$ for $g = 1,\ a,\ b^3$, and $|S(\beta_g)| = 2$ for $g = b,\ b^2$. Using Lemmas 2.1 and 2.2, we get

$$\mathbb{F}_qG \cong \mathbb{F}_q^2 \bigoplus \mathbb{F}_{q^2}^2 \bigoplus M(n_1, \mathbb{F}_q)$$

with the restriction that $n_1^2 = 36$. Hence, $n_1 = 6$ and as a result

$$\mathbb{F}_qG \cong \mathbb{F}_q^2 \bigoplus \mathbb{F}_{q^2}^2 \bigoplus M(6, \mathbb{F}_q).$$

□

Theorem 4.4. *Let $\mathbb{F}_q$ be a finite field of order $q = p^k$ with characteristic $p \neq 2, 3, 7$ and let $G = C_2 \times (C_7 \rtimes C_3)$. Then,*

$$\mathcal{U}(\mathbb{F}_qG) \cong \begin{cases} C_{q-1}^6 \times GL(3, \mathbb{F}_q)^4, & \text{if } q \equiv 1, 25, 37 \ mod\ 42; \\ C_{q-1}^2 \times C_{q^2-1}^2 \times GL(3, \mathbb{F}_{q^2})^2, & \text{if } q \equiv 5, 17, 41 \ mod\ 42; \\ C_{q-1}^6 \times GL(3, \mathbb{F}_{q^2})^2, & \text{if } q \equiv 13, 19, 31 \ mod\ 42; \\ C_{q-1}^2 \times C_{q^2-1}^2 \times GL(3, \mathbb{F}_q)^4, & \text{if } q \equiv 11, 23, 29 \ mod\ 42. \end{cases}$$

Proof. Let $G = \langle a, b, c \mid a^7 = b^3 = c^2 = 1,\ bab^{-1} = a^2,\ ac = ca,\ bc = cb \rangle$. Since $p \nmid |G|$, by Maschke's theorem $\mathbb{F}_qG$ is a semisimple group algebra and $J(\mathbb{F}_qG) = (0)$. Also,

$$\mathbb{F}_qG \cong \mathbb{F}_q(G/G') \bigoplus \Delta(G, G').$$

The conjugacy classes of G are:

$$\begin{aligned} [1] &= \{1\}, \\ [c] &= \{c\}, \\ [ac^i] &= \{ac^i, a^2c^i, a^4c^i\} && \text{for } i = 0, 1; \\ [a^3c^i] &= \{a^3c^i, a^5c^i, a^6c^i\} && \text{for } i = 0, 1; \\ [bc^i] &= \{bc^i, abc^i, a^2bc^i, \dots, a^6bc^i\} && \text{for } i = 0, 1; \\ [b^2c^i] &= \{b^2c^i, ab^2c^i, a^2b^2c^i, \dots, a^6b^2c^i\} && \text{for } i = 0, 1. \end{aligned}$$

Observe that $G/G' \cong C_6$, thus

$$\mathbb{F}_qG \cong \mathbb{F}_qC_6 \bigoplus_{j=1}^{m} M(n_j, R_j)$$

where $n_j \geq 2$ and for each $j \in \{1, \dots, m\}$, R_j represents a division algebra over $\mathbb{F}_q$. Since $dim(Z(\mathbb{F}_qG)) = 10$, $m \leq 4$. Here, for any value of characteristic p, $e = 42$. By [18, Theorem 4.1], the structure of $\mathbb{F}_qC_6$ has been determined.

(**a**) If $q \equiv 1, 25, 37 \ mod\ 42$, then $\mathcal{B} = \{1\} \ mod\ 42$ or $\mathcal{B} = \{1, 25, 37\} \ mod\ 42$. This gives $|S(\beta_g)| = 1$ for all $g \in G$. Using Lemmas 2.1 and 2.2, we get

$$\mathbb{F}_qG \cong \mathbb{F}_q^6 \bigoplus_{j=1}^{4} M(n_j, \mathbb{F}_q).$$

The equation $\sum_{j=1}^{4} n_j^2 = 36$ is obtained by equating the dimensions of both the sides. The only possible solution is $n_j = 3$ for all $j \in \{1, \dots, 4\}$. Hence,

$$\mathbb{F}_qG \cong \mathbb{F}_q^6 \bigoplus M(3, \mathbb{F}_q)^4.$$

(**b**) If $q \equiv 5, 17, 41 \ mod\ 42$, then $\mathcal{B} = \{1, 5, 17, 25, 37, 41\} \ mod\ 42$ or $\mathcal{B} = \{1, 41\} \ mod\ 42$. Thus, $|S(\beta_g)| = 1$ for $g = 1,\ c$, and $|S(\beta_g)| = 2$ for $g = a,\ ac,\ b,\ bc$. Then,

$$\mathbb{F}_qG \cong \mathbb{F}_q^2 \bigoplus \mathbb{F}_{q^2}^2 \bigoplus M(n_1, \mathbb{F}_{q^2}) \bigoplus M(n_2, \mathbb{F}_{q^2}),$$

by using Lemmas 2.1 and 2.2. The necessary condition of $2(n_1^2 + n_2^2) = 36$ is satisfied only when $n_1 = n_2 = 3$. Hence,

$$\mathbb{F}_qG \cong \mathbb{F}_q^2 \bigoplus \mathbb{F}_{q^2}^2 \bigoplus M(3, \mathbb{F}_{q^2})^2.$$

(**c**) If $q \equiv 13, 19, 31 \ mod\ 42$, then $\mathcal{B} = \{1, 13, 19, 25, 31, 37\} \ mod\ 42$ or $\mathcal{B} = \{1, 13\} \ mod\ 42$. It follows from here that $|S(\beta_g)| = 1$ for $g = 1,\ c,\ b,\ b^2,\ bc,\ b^2c$, and $|S(\beta_g)| = 2$ for $g = a,\ ac$. Using Lemmas 2.1 and 2.2,

$$\mathbb{F}_qG \cong \mathbb{F}_q^6 \bigoplus M(n_1, \mathbb{F}_{q^2}) \bigoplus M(n_2, \mathbb{F}_{q^2}),$$

with the restriction that $2(n_1^2+n_2^2) = 36$. The only possible solution of the equation is $n_1 = n_2 = 3$. Hence,

$$\mathbb{F}_q G \cong \mathbb{F}_q^6 \bigoplus M(3, \mathbb{F}_{q^2})^2.$$

(**d**) If $q \equiv 11, 23, 29 \ mod\ 42$, then $\mathcal{B} = \{1, 11, 23, 25, 29, 37\} \ mod\ 42$ or $\mathcal{B} = \{1, 29\} \ mod\ 42$. Thus, $|S(\beta_g)| = 1$ for $g = 1,\ c,\ a,\ a^3,\ ac,\ a^3c$, and $|S(\beta_g)| = 2$ for $g = b,\ bc$. Lemmas 2.1 and 2.2 guide us

$$\mathbb{F}_q G \cong \mathbb{F}_q^2 \bigoplus \mathbb{F}_{q^2}^2 \bigoplus_{j=1}^{4} M(n_j, \mathbb{F}_q),$$

with the condition that $\sum_{j=1}^{4} n_j^2 = 36$, which is only possible when $n_j = 3$ for all $j \in \{1, \ldots, 4\}$. Hence,

$$\mathbb{F}_q G \cong \mathbb{F}_q^2 \bigoplus \mathbb{F}_{q^2}^2 \bigoplus M(3, \mathbb{F}_q)^4. \qquad \square$$

Theorem 4.5. *Let $\mathbb{F}_q$ be a finite field of order $q = p^k$ with characteristic p and let $G = C_{42}$.*

(1) *If* **Char** $\mathbb{F}_q \neq 2, 3, 7$, *then*

$$\mathcal{U}(\mathbb{F}_q G) \cong \begin{cases} C_{q-1}^{42}, & \text{if } q \equiv 1 \ mod\ 21; \\ C_{q-1}^2 \times C_{q^2-1}^2 \times C_{q^3-1}^4 \times C_{q^6-1}^4, & \text{if } q \equiv 2, 11 \ mod\ 21; \\ C_{q-1}^6 \times C_{q^3-1}^{12}, & \text{if } q \equiv 4, 16 \ mod\ 21; \\ C_{q-1}^2 \times C_{q^2-1}^2 \times C_{q^6-1}^6, & \text{if } q \equiv 5, 17 \ mod\ 21; \\ C_{q-1}^{14} \times C_{q^2-1}^{14}, & \text{if } q \equiv 8 \ mod\ 21; \\ C_{q-1}^6 \times C_{q^6-1}^6, & \text{if } q \equiv 10, 19 \ mod\ 21; \\ C_{q-1}^6 \times C_{q^2-1}^{18}, & \text{if } q \equiv 13 \ mod\ 21; \\ C_{q-1}^2 \times C_{q^2-1}^{20}, & \text{if } q \equiv 20 \ mod\ 21. \end{cases}$$

(2) *If* **Char** $\mathbb{F}_q = 2$, *then*

$$\mathcal{U}(\mathbb{F}_q G) \cong \begin{cases} C_2^{21k} \times C_{q-1}^{21}, & \text{if } q \equiv 1 \ mod\ 21; \\ C_2^{21k} \times C_{q-1} \times C_{q^2-1} \times C_{q^3-1}^2 \times C_{q^6-1}^2, & \text{if } q \equiv 2, 11 \ mod\ 21; \\ C_2^{21k} \times C_{q-1}^3 \times C_{q^3-1}^6, & \text{if } q \equiv 4, 16 \ mod\ 21; \\ C_2^{21k} \times C_{q-1}^7 \times C_{q^2-1}^7, & \text{if } q \equiv 8 \ mod\ 21. \end{cases}$$

(3) *If* **Char** $\mathbb{F}_q = 3$, *then*

$$\mathcal{U}(\mathbb{F}_q G) \cong \begin{cases} C_3^{28k} \times C_{q-1}^{14}, & \textit{if } q \equiv 1 \textit{ mod } 14; \\ C_3^{28k} \times C_{q-1}^{2} \times C_{q^6-1}^{2}, & \textit{if } q \equiv 3, 5 \textit{ mod } 14; \\ C_3^{28k} \times C_{q-1}^{2} \times C_{q^3-1}^{4}, & \textit{if } q \equiv 9, 11 \textit{ mod } 14; \\ C_3^{28k} \times C_{q-1}^{2} \times C_{q^2-1}^{6}, & \textit{if } q \equiv 13 \textit{ mod } 14. \end{cases}$$

(4) *If* **Char** $\mathbb{F}_q = 7$, *then* $\mathcal{U}(\mathbb{F}_q G) \cong C_7^{36k} \times C_{q-1}^{6}$.

Proof. Let $C_{42} = \langle a \mid a^{42} = 1 \rangle$.

1. Char $\mathbb{F}_q \neq 2, 3, 7$**:** By Maschke's theorem $\mathbb{F}_q G$ is a semisimple group algebra. Consider,

$$\mathbb{F}_q G \cong \mathbb{F}_q(C_2 \times C_{21}) \cong (\mathbb{F}_q C_2) C_{21} \cong (\mathbb{F}_q \bigoplus \mathbb{F}_q) C_{21} \cong (\mathbb{F}_q C_{21})^2.$$

Let $P = C_{21} = \langle b \mid b^{21} = 1 \rangle$. Now, our goal is to determine the structure of $\mathbb{F}_q P$. In view of this, all the conjugacy classes of P are p-regular and $e = 21$.
If $q \equiv 1$ *mod* 21, then $\mathcal{B} = \{1\}$ *mod* 21 and thus $|S(\beta_g)| = 1$ for all $g \in P$. By Lemmas 2.1 and 2.2, we get

$$\mathbb{F}_q P \cong \mathbb{F}_q^{21}.$$

If $q \equiv 2, 11$ *mod* 21, then $\mathcal{B} = \{1, 2, 4, 8, 11, 16\}$ *mod* 21, which implies $|S(\beta_g)| = 1$ for $g = 1$, $|S(\beta_g)| = 2$ for $g = b^7$, $|S(\beta_g)| = 3$ for $g = b^3,\ b^9$, and $|S(\beta_g)| = 6$ for $g = b,\ b^5$. By Lemmas 2.1 and 2.2,

$$\mathbb{F}_q P \cong \mathbb{F}_q \bigoplus \mathbb{F}_{q^2} \bigoplus \mathbb{F}_{q^3}^2 \bigoplus \mathbb{F}_{q^6}^2.$$

If $q \equiv 4, 16$ *mod* 21, then $\mathcal{B} = \{1, 4, 16\}$ *mod* 21. Thus, $|S(\beta_g)| = 1$ for $g = 1,\ b^7,\ b^{14}$, and $|S(\beta_g)| = 3$ for $g = b,\ b^2,\ b^3,\ b^5,\ b^9,\ b^{10}$. Using Lemmas 2.1 and 2.2,

$$\mathbb{F}_q P \cong \mathbb{F}_q^3 \bigoplus \mathbb{F}_{q^3}^6.$$

If $q \equiv 5, 17$ *mod* 21, then $\mathcal{B} = \{1, 4, 5, 16, 17, 20\}$ *mod* 21. Thus, $|S(\beta_g)| = 1$ for $g = 1$, $|S(\beta_g)| = 2$ for $g = b^7$, and $|S(\beta_g)| = 6$ for $g = b,\ b^2,\ b^3$. Using Lemmas 2.1 and 2.2,

$$\mathbb{F}_q P \cong \mathbb{F}_q \bigoplus \mathbb{F}_{q^2} \bigoplus \mathbb{F}_{q^6}^3.$$

If $q \equiv 8$ *mod* 21, then $\mathcal{B} = \{1, 8\}$ *mod* 21. This leads us to $|S(\beta_g)| = 1$ for $g = 1,\ b^3,\ b^6,\ b^9,\ b^{12},\ b^{15},\ b^{18}$, and $|S(\beta_g)| = 2$ for $g = b,\ b^2,\ b^4,\ b^5,\ b^7,\ b^{10},\ b^{13}$. Lemmas 2.1 and 2.2 imply

$$\mathbb{F}_q P \cong \mathbb{F}_q^7 \bigoplus \mathbb{F}_{q^2}^7.$$

If $q \equiv 10, 19$ *mod* 21, then $\mathcal{B} = \{1, 4, 10, 13, 16, 19\}$ *mod* 21. Thus $|S(\beta_g)| = 1$ for $g = 1,\ b^7,\ b^{14}$, and $|S(\beta_g)| = 6$ for $g = b,\ b^2,\ b^3$. By Lemmas 2.1 and 2.2,

$$\mathbb{F}_q P \cong \mathbb{F}_q^3 \bigoplus \mathbb{F}_{q^6}^3.$$

If $q \equiv 13$ *mod* 21, then $\mathcal{B} = \{1, 13\}$ *mod* 21. Hence $|S(\beta_g)| = 1$ for $g = 1,\ b^7,\ b^{14}$, and $|S(\beta_g)| = 2$ for $g = b,\ b^2,\ b^3,\ b^4,\ b^6,\ b^8,\ b^9,\ b^{11},\ b^{16}$. By Lemmas 2.1 and 2.2,

$$\mathbb{F}_q P \cong \mathbb{F}_q^3 \bigoplus \mathbb{F}_{q^2}^9.$$

If $q \equiv 20$ *mod* 21, then $\mathcal{B} = \{1, 20\}$ *mod* 21. Thus, $|S(\beta_g)| = 1$ for $g = 1$, and $|S(\beta_g)| = 2$ for $g = b^m$ for $m = 1, 2, \ldots, 10$. Lemmas 2.1 and 2.2 imply that

$$\mathbb{F}_q P \cong \mathbb{F}_q \bigoplus \mathbb{F}_{q^2}^{10}.$$

2. Char $\mathbb{F}_q = 2$**:** Let $H = \langle a^{21} \rangle$. Then, $[G : H] = 21 \neq 0$ in $\mathbb{F}_q$. By [15, Theorem 7.2.7 and Lemma 8.1.17], $J(\mathbb{F}_q G) = \Delta(G, H)$ and therefore $\mathbb{F}_q G / J(\mathbb{F}_q G) \cong \mathbb{F}_q(G/H) \cong \mathbb{F}_q C_{21}$. Since $dim(J(\mathbb{F}_q G)) = 21$ and $J(\mathbb{F}_q G)^2 = (0)$, $V^2 = 1$ and $V \cong C_2^{21k}$. Thus,

$$\mathcal{U}(\mathbb{F}_q G) \cong C_2^{21k} \times \mathcal{U}(\mathbb{F}_q C_{21}).$$

The structure of $\mathcal{U}(\mathbb{F}_q C_{21})$ has been derived in part 1.

3. Char $\mathbb{F}_q = 3$**:** Let $N = \langle a^{14} \rangle$. Then, $[G : N] = 14 \neq 0$ in $\mathbb{F}_q$. By [15, Theorem 7.2.7 and Lemma 8.1.17], $J(\mathbb{F}_q G) = \Delta(G, N)$ and $\mathbb{F}_q G / J(\mathbb{F}_q G) \cong \mathbb{F}_q C_{14}$. Since $dim(J(\mathbb{F}_q G)) = 28$ and $J(\mathbb{F}_q G)^3 = (0)$, $V^3 = 1$ and $V \cong C_3^{28k}$. Thus,

$$\mathcal{U}(\mathbb{F}_q G) \cong C_3^{28k} \times \mathcal{U}(\mathbb{F}_q C_{14}).$$

The rest follows by [18, Theorem 4.3].

4. Char $\mathbb{F}_q = 7$**:** Let $K = \langle a^6 \rangle$. Then, $[G : K] = 6 \neq 0$ in $\mathbb{F}_q$. Thus, by [15, Theorem 7.2.7 and Lemma 8.1.17], $J(\mathbb{F}_q G) = \Delta(G, K)$ and $\mathbb{F}_q G / J(\mathbb{F}_q G) \cong \mathbb{F}_q C_6$. Since $dim(J(\mathbb{F}_q G)) = 36$ and $J(\mathbb{F}_q G)^7 = (0)$, $V^7 = 1$ and $V \cong C_7^{36k}$. Hence,

$$\mathcal{U}(\mathbb{F}_q G) \cong C_7^{36k} \times \mathcal{U}(\mathbb{F}_q C_6).$$

By [18, Theorem 4.1],

$$\mathcal{U}(\mathbb{F}_q G) \cong C_7^{36k} \times C_{q-1}^6. \qquad \square$$

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