

ON THE CAPITULATION PROBLEM OF SOME PURE METACYCLIC FIELDS OF DEGREE 20 II

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ABSTRACT. Let n be a 5^{th} power-free natural number and $k_0 = \mathbb{Q}(\zeta_5)$ be the cyclotomic field generated by a primitive 5^{th} root of unity ζ_5 . Then $k = \mathbb{Q}(\sqrt[5]{n}, \zeta_5)$ is a pure metacyclic field of absolute degree 20. In the case that k possesses a 5-class group $C_{k,5}$ of type $(5, 5)$ and all the classes are ambiguous under the action of $Gal(k/k_0)$, the capitulation of 5-ideal classes of k in its unramified cyclic quintic extensions is determined.

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1. Introduction

Let k be a number field, and L be an unramified abelian extension of k . We say that an ideal \mathcal{I} of k or its class capitulates in L if \mathcal{I} becomes principal in L .

Let $\Gamma = \mathbb{Q}(\sqrt[5]{n})$ be a pure quintic field, where n is a 5^{th} power free natural number, and $k_0 = \mathbb{Q}(\zeta_5)$ be the cyclotomic field generated by a primitive 5^{th} root of unity ζ_5 . Then $k = \Gamma(\zeta_5)$ is the normal closure of Γ and a pure metacyclic field of absolute degree 20. Let $k_5^{(1)}$ be the Hilbert 5-class field of k , $C_{k,5}$ be the 5-ideal class group of k and $C_{k,5}^{(\sigma)}$ be the subgroup of ambiguous ideal classes under the action of $Gal(k/k_0) = \langle \sigma \rangle$.

In the case that $C_{k,5}$ is of type $(5, 5)$ and $\text{rank } C_{k,5}^{(\sigma)} = 1$, the capitulation of the 5-ideal classes of k in the six intermediate extensions of $k_5^{(1)}/k$ is determined in [2].

Let p and q be primes such that $p \equiv 1 \pmod{5}$ and $q \equiv \pm 2 \pmod{5}$. According to [1, Theorem 1.1], if $C_{k,5}$ is of type $(5, 5)$ and $\text{rank } C_{k,5}^{(\sigma)} = 2$, we have three forms of the radicand n as follows:

- $n = p^e$ with $e \in \{1, 2, 3, 4\}$ and $p \equiv 1 \pmod{25}$.
- $n = 5^{e_1} p^{e_2}$ with $e_1, e_2 \in \{1, 2, 3, 4\}$ and $p \not\equiv 1 \pmod{25}$.
- $n = p^{e_1} q^{e_2} \equiv \pm 1, \pm 7 \pmod{25}$ with $e_1, e_2 \in \{1, 2, 3, 4\}$, $p \not\equiv 1 \pmod{25}$ and $q \not\equiv \pm 7 \pmod{25}$.

In this paper, we investigate the capitulation of the 5-ideal classes of the pure metacyclic field k in the unramified cyclic quintic extensions of k within the Hilbert 5-class field $k_5^{(1)}$ of k , whenever $C_{k,5}$ is of type $(5, 5)$ and $\text{rank } C_{k,5}^{(\sigma)} = 2$, which means that all classes are ambiguous.

We will study the capitulation of $C_{k,5}$ in the six intermediate extensions K_1, \dots, K_6 of $k_5^{(1)}/k$ by distinguishing the three cases of the radicand n . Figure 1 illustrates the situation.

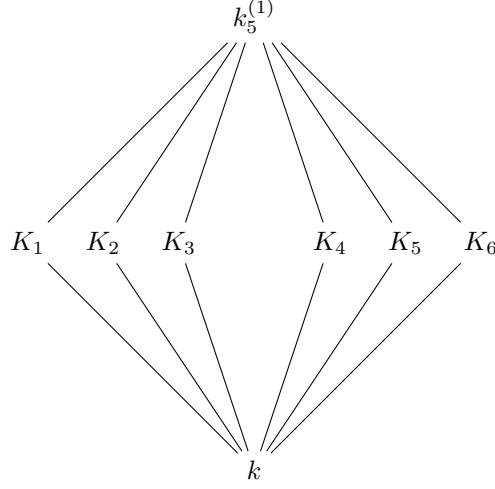


Figure 1: The unramified quintic sub-extensions of $k_5^{(1)}/k$

The theoretical results are underpinned by numerical examples obtained with the computational number theory system PARI/GP [6].

Notations.

Throughout this paper, we use the following notations:

- The lower case letters p and q denote a prime numbers such that, $p \equiv 1 \pmod{5}$ and $q \equiv \pm 2 \pmod{5}$.
- $\Gamma = \mathbb{Q}(\sqrt[5]{n})$: a pure quintic field, where $n \neq 1$ is a 5^{th} power-free natural number.
- $k_0 = \mathbb{Q}(\zeta_5)$: the cyclotomic field, where $\zeta_5 = e^{2i\pi/5}$ is a primitive 5^{th} root of unity.
- $k = \mathbb{Q}(\sqrt[5]{n}, \zeta_5)$: the normal closure of Γ , a quintic Kummer extension of k_0 .
- $\langle \tau \rangle = \text{Gal}(k/\Gamma)$ such that τ is identity on Γ , and sends ζ_5 to its square. Hence τ has order 4.
- $\langle \sigma \rangle = \text{Gal}(k/k_0)$ such that σ is identity on k_0 , and sends $\sqrt[5]{n}$ to $\zeta_5 \sqrt[5]{n}$. Hence σ has order 5.
- For a number field L , denote by:
 - \mathcal{O}_L : the ring of integers of L .
 - $C_L, h_L, C_{L,5}$: the class group, class number, and 5-class group of L .
 - $L_5^{(1)}, L^*$: the Hilbert 5-class field of L , and the absolute genus field of L .
 - $[\mathcal{I}]$: the class of a fractional ideal \mathcal{I} in the class group of L .
- $(\frac{a}{b})_5 = 1 \Leftrightarrow X^5 \equiv a \pmod{b}$ soluble in \mathcal{O}_{k_0} , where a, b are primes in \mathcal{O}_{k_0} .

2. Preliminaries

2.1. Decomposition laws in Kummer extension.

Since the pure quintic extensions of the 5^{th} cyclotomic field $k_0 = \mathbb{Q}(\zeta_5)$ and of $k = \mathbb{Q}(\sqrt[5]{n}, \zeta_5)$ are all Kummer's extensions, we recall the decomposition laws of ideals in these extensions.

Proposition 2.1. *Let L be a number field containing the l^{th} roots of unity, where l is prime, and θ be an element of L , such that $\theta \neq \mu^l$, for all $\mu \in L$. Therefore $L(\sqrt[l]{\theta})$ is a cyclic extension of*

degree l over L . We denote by ζ a primitive l^{th} root of unity.

(1) We assume that a prime ideal \mathcal{P} of L , divides θ exactly to the power \mathcal{P}^a .

- If $a = 0$ and \mathcal{P} does not divide l , then \mathcal{P} splits completely in $L(\sqrt[l]{\theta})$ when the congruence $\theta \equiv X^l \pmod{\mathcal{P}}$ has a solution in L .
- If $a = 0$ and \mathcal{P} does not divide l , then \mathcal{P} is inert in $L(\sqrt[l]{\theta})$ when the congruence $\theta \equiv X^l \pmod{\mathcal{P}}$ has no solution in L .
- If $l \nmid a$, then \mathcal{P} is totally ramified in $L(\sqrt[l]{\theta})$.

(2) Let \mathcal{B} be a prime factor of $1 - \zeta$ that divides $1 - \zeta$ exactly to the a^{th} power. Suppose that $\mathcal{B} \nmid \theta$, then \mathcal{B} splits completely in $L(\sqrt[l]{\theta})$ if the congruence

$$\theta \equiv X^l \pmod{\mathcal{B}^{al+1}} \quad (*)$$

has a solution in L . The ideal \mathcal{B} is inert in $L(\sqrt[l]{\theta})$ if the congruence

$$\theta \equiv X^l \pmod{\mathcal{B}^{al}} \quad (**)$$

has a solution in L , but $(*)$ has no solution. The ideal \mathcal{B} is totally ramified in L if the congruence $(**)$ has no solution.

Proof. See [3, Theorems 118, 119]. □

2.2. Relative genus field $(k/k_0)^*$ of k over k_0 .

Let $\Gamma = \mathbb{Q}(\sqrt[5]{n})$ be a pure quintic field, $k_0 = \mathbb{Q}(\zeta_5)$ the 5^{th} -cyclotomic field and $k = \Gamma(\zeta_5)$ be the normal closure of Γ . The relative genus field $(k/k_0)^*$ of k over k_0 is the maximal abelian extension of k_0 which is contained in the Hilbert 5-class field $k_5^{(1)}$ of k .

Let q^* be the exponent defined by $[N_{k/k_0}(k - \{0\}) : N_{k/k_0}(E_{k_0})] = 5^{q^*}$. Here N_{k/k_0} is the relative norm from k to k_0 , and E_{k_0} the group of units of k_0 . We note that $N_{k/k_0}(E_{k_0}) = E_{k_0}^5$ and $[E_{k_0} : E_{k_0}^5] = 5^2$, so we get that $q^* \in \{0, 1, 2\}$.

The group E_{k_0} is generated by ζ_5 and $\zeta_5 + 1$, then according to the definition of q^* , we see that:

$$q^* = \begin{cases} 2 & \text{if } \zeta, \zeta + 1 \in N_{k/k_0}(k - \{0\}), \\ 1 & \text{if } \zeta^i(\zeta + 1)^j \in N_{k/k_0}(k - \{0\}) \text{ for some } i \text{ and } j, \\ 0 & \text{if } \zeta^i(\zeta + 1)^j \notin N_{k/k_0}(k - \{0\}) \text{ for } 0 \leq i, j \leq 4 \text{ and } i + j \neq 0. \end{cases}$$

The relative genus field $(k/k_0)^*$ is given explicitly by the following proposition by means of the decomposition of n in k_0 and the value of q^* .

Proposition 2.2. Let $k = k_0(\sqrt[5]{n})$ such that $n = \mu \lambda^{e_\lambda} \pi_1^{e_1} \dots \pi_f^{e_f} \pi_{f+1}^{e_{f+1}} \dots \pi_g^{e_g}$ in k_0 , where μ is unity of \mathcal{O}_{k_0} , $\lambda = 1 - \zeta_5$ the unique prime above 5 in k_0 and each prime $\pi_i \equiv \pm 1, \pm 7 \pmod{\lambda^5}$ for $1 \leq i \leq f$ and $\pi_j \not\equiv \pm 1, \pm 7 \pmod{\lambda^5}$ for $f+1 \leq j \leq g$. Then we have:

- (i) There exists $h_i \in \{1, \dots, 4\}$ such that $\pi_{f+1} \pi_i^{h_i} \equiv \pm 1, \pm 7 \pmod{\lambda^5}$ for $f+2 \leq i \leq g$.
- (ii) If $n \not\equiv \pm 1, \pm 7 \pmod{\lambda^5}$ and $q^* = 1$, then the genus field $(k/k_0)^*$ is given as:

$$(k/k_0)^* = k \left(\sqrt[5]{\pi_1}, \dots, \sqrt[5]{\pi_f}, \sqrt[5]{\pi_{f+1} \pi_{f+2}^{h_{f+2}}}, \dots, \sqrt[5]{\pi_{f+1} \pi_g^{h_g}} \right)$$

where h_i is chosen as in (i).

- (iii) In the other cases of q^* and the congruence of n , the genus field $(k/k_0)^*$ is given by deleting an appropriate number of 5^{th} root from the right side of (ii).

Proof. See [4, Proposition 5.8]. □

3. Study of capitulation

Let Γ , k_0 and k as above. If $C_{k,5}$ is of type $(5, 5)$ and the subgroup of ambiguous classes $C_{k,5}^{(\sigma)}$ under the action of $\text{Gal}(k/k_0) = \langle \sigma \rangle$ has rank 2, we have $C_{k,5} = C_{k,5}^{(\sigma)}$.

By class field theory, the principal genus $C_{k,5}^{1-\sigma}$ corresponds to $(k/k_0)^*$, and since $C_{k,5} = C_{k,5}^{(\sigma)}$ we get that $C_{k,5}^{1-\sigma} = \{1\}$, whence $(k/k_0)^*$ coincides with the Hilbert 5-class field $k_5^{(1)}$ of k .

When $C_{k,5}$ is of type $(5, 5)$, it has 6 subgroups of order 5, denoted H_i , $1 \leq i \leq 6$. Let K_i be the intermediate extension of $k_5^{(1)}/k$ which corresponds by class field theory to H_i .

As each K_i is cyclic of order 5 over k , by Hilbert's theorem 94, there is at least one subgroup of order 5 of $C_{k,5}$, i.e. at least one H_l for some $l \in \{1, 2, 3, 4, 5, 6\}$, which capitulates in K_i .

Definition 3.1. Let \mathcal{S}_j be a generator of H_j ($1 \leq j \leq 6$) which corresponds to K_j . For $1 \leq j \leq 6$, let $i_j \in \{0, 1, 2, 3, 4, 5, 6\}$. We say that the capitulation is of type $(i_1, i_2, i_3, i_4, i_5, i_6)$ to mean the following:

- (1) when $i_j \in \{1, 2, 3, 4, 5, 6\}$, then only the class \mathcal{S}_{i_j} and its powers capitulate in K_j ;
- (2) when $i_j = 0$, then all the 5-classes capitulate in K_j .

We find ourselves in front of $7^6 = 117649$ possible types which need to be reduced.

It's easy to see that $C_{k,5} \simeq C_{k,5}^+ \times C_{k,5}^-$ such that $C_{k,5}^+ = \{\mathcal{A} \in C_{k,5} \mid \mathcal{A}^{\tau^2} = \mathcal{A}\}$ and $C_{k,5}^- = \{\mathcal{X} \in C_{k,5} \mid \mathcal{X}^{\tau^2} = \mathcal{X}^{-1}\}$, with $\text{Gal}(k/\Gamma) = \langle \tau \rangle$. We order the subgroups H_i of $C_{k,5}$ as follows:

$H_1 = C_{k,5}^+ = \langle \mathcal{A} \rangle$, $H_6 = C_{k,5}^- = \langle \mathcal{X} \rangle$, $H_2 = \langle \mathcal{A}\mathcal{X} \rangle$, $H_3 = \langle \mathcal{A}\mathcal{X}^2 \rangle$, $H_4 = \langle \mathcal{A}\mathcal{X}^3 \rangle$ and $H_5 = \langle \mathcal{A}\mathcal{X}^4 \rangle$.

By the action of $\text{Gal}(k/\mathbb{Q})$ on $C_{k,5}$, we can give the following proposition:

Proposition 3.1. *For all continuations of the automorphisms σ and τ we have:*

- (1) $K_i^\sigma = K_i$ ($i = 1, 2, 3, 4, 5, 6$), i.e σ sets all K_i .
- (2) $K_1^{\tau^2} = K_1$, $K_6^{\tau^2} = K_6$, $K_2^{\tau^2} = K_5$ and $K_3^{\tau^2} = K_4$. i.e τ^2 sets K_1 , K_6 and permutes K_2 with K_5 and K_3 with K_4 .

Proof. We will agree that for all $1 \leq i \leq 6$ and for all $w \in \text{Gal}(k/\mathbb{Q})$ we have $H_i^w = \{\mathcal{C}^w \mid \mathcal{C} \in H_i\}$.

(1) Since all classes are ambiguous because $C_{k,5} = C_{k,5}^{(\sigma)}$, σ sets all H_i .

(2) We have $H_1 = C_{k,5}^+ = \langle \mathcal{A} \rangle$ and $H_6 = C_{k,5}^- = \langle \mathcal{X} \rangle$, then $H_1^{\tau^2} = H_1$ and $H_6^{\tau^2} = H_6$.

- Since $(\mathcal{A}\mathcal{X})^{\tau^2} = \mathcal{A}^{\tau^2}\mathcal{X}^{\tau^2} = \mathcal{A}\mathcal{X}^{-1} = \mathcal{A}\mathcal{X}^4 \in H_5$, $H_2^{\tau^2} = H_5$.

- Since $(\mathcal{A}\mathcal{X}^2)^{\tau^2} = \mathcal{A}^{\tau^2}(\mathcal{X}^2)^{\tau^2} = \mathcal{A}\mathcal{X}^{-2} = \mathcal{A}\mathcal{X}^3 \in H_4$, $H_3^{\tau^2} = H_4$.

- Since $\tau^4 = 1$, we get that $H_5^{\tau^2} = H_2$ and $H_4^{\tau^2} = H_3$.

The relations between the fields K_i in (1) and (2) are nothing else than the translations of the corresponding relations for the subgroups H_i via class field theory. \square

To study the capitulation problem of k whenever $C_{k,5}$ is of type $(5, 5)$ and $C_{k,5} = C_{k,5}^{(\sigma)}$, we will investigate the three forms of the radicand n proved in [1, Theorem 1.1], and mentioned above.

3.1. The case $n = p^e$ where $p \equiv 1 \pmod{25}$.

Let $k = \Gamma(\zeta_5)$ be the normal closure of $\Gamma = \mathbb{Q}(\sqrt[5]{n})$, where $n = p^e$ such that $p \equiv 1 \pmod{25}$ and $e \in \{1, 2, 3, 4\}$. By [5, Theorem 2.13], since $p \equiv 1 \pmod{5}$ we have that p splits completely in $k_0 = \mathbb{Q}(\zeta_5)$ as $p = \pi_1\pi_2\pi_3\pi_4$, with π_i are primes in k_0 . As the discriminant of Γ/\mathbb{Q} is 5^3p^4 , we get that p is ramified in Γ , then the primes π_i are ramified in k .

If $\mathcal{P}_1, \mathcal{P}_2, \mathcal{P}_3$ and \mathcal{P}_4 are respectively the prime ideals of k above π_1, π_2, π_3 and π_4 , then $\mathcal{P}_i^5 = \pi_i \mathcal{O}_k$ ($i = 1, 2, 3, 4$). Since τ acts transitively on π_i , we have that τ^2 permutes π_1 with π_3 , hence τ^2 permutes \mathcal{P}_1 with \mathcal{P}_3 . Since $\pi_i^\sigma = \pi_i$, we have $\mathcal{P}_i^\sigma = \mathcal{P}_i$. In fact $[\mathcal{P}_i]$ ($i = 1, 2, 3, 4$) generate the subgroup of strong ambiguous ideal classes denoted $C_{k,s}^{(\sigma)}$ and defined by $C_{k,s}^{(\sigma)} = \{[\mathcal{P}] \in C_{k,5} \mid \mathcal{P}^\sigma = \mathcal{P}\}$.

The next theorem allow us to determine explicitly the intermediate extensions of $k_5^{(1)}/k$.

Theorem 3.1. *Let k and n as above. Let π_1, π_2, π_3 and π_4 be primes of k_0 congruent to 1 modulo λ^5 such that $p = \pi_1\pi_2\pi_3\pi_4$, then:*

- (1) $k_5^{(1)} = k(\sqrt[5]{\pi_1}, \sqrt[5]{\pi_3})$.
- (2) *The six intermediate extensions of $k_5^{(1)}/k$ are: $k(\sqrt[5]{\pi_1})$, $k(\sqrt[5]{\pi_3})$, $k(\sqrt[5]{\pi_1\pi_3})$, $k(\sqrt[5]{\pi_1\pi_3^2})$, $k(\sqrt[5]{\pi_1\pi_3^3})$ and $k(\sqrt[5]{\pi_1\pi_3^4})$. Furthermore τ^2 permutes $k(\sqrt[5]{\pi_1})$ with $k(\sqrt[5]{\pi_3})$ and $k(\sqrt[5]{\pi_1\pi_3^2})$ with $k(\sqrt[5]{\pi_1\pi_3^3})$, and sets $k(\sqrt[5]{\pi_1\pi_3})$, $k(\sqrt[5]{\pi_1\pi_3^4})$.*

Proof. (1) We have that $k_5^{(1)} = (k/k_0)^*$. Since $k = k_0(\sqrt[5]{n})$ with $n = p = \pi_1\pi_2\pi_3\pi_4$ in k_0 and $\pi_i \equiv 1 \pmod{\lambda^5}$ ($i = 1, 2, 3, 4$), by Proposition 2.2 we have $(k/k_0)^* = k(\sqrt[5]{\pi_1}, \sqrt[5]{\pi_3})$.

(2) If $k_5^{(1)} = k(\sqrt[5]{\pi_1}, \sqrt[5]{\pi_3})$, then the six intermediate extensions are: $k(\sqrt[5]{\pi_1})$, $k(\sqrt[5]{\pi_3})$, $k(\sqrt[5]{\pi_1\pi_3})$, $k(\sqrt[5]{\pi_1\pi_3^2})$, $k(\sqrt[5]{\pi_1\pi_3^3})$ and $k(\sqrt[5]{\pi_1\pi_3^4})$. We have $\tau^2(\pi_1) = \pi_3$, so it is easy to see that τ^2 sets the fields $k(\sqrt[5]{\pi_1\pi_3})$, $k(\sqrt[5]{\pi_1\pi_3^4})$.

Since $\tau^2(\pi_1) = \tau^2(\sqrt[5]{\pi_1^5}) = (\tau^2(\sqrt[5]{\pi_1}))^5 = \pi_3$, $\tau^2(\sqrt[5]{\pi_1})$ is 5th root of π_3 . Thus $k(\sqrt[5]{\pi_3}) = k(\tau^2(\sqrt[5]{\pi_1}))$, i.e. $k(\sqrt[5]{\pi_3}) = k(\sqrt[5]{\pi_1})^{\tau^2}$. By the same reasoning we prove that $k(\sqrt[5]{\pi_1}) = k(\sqrt[5]{\pi_3})^{\tau^2}$. Hence τ^2 permutes $k(\sqrt[5]{\pi_1})$ with $k(\sqrt[5]{\pi_3})$.

We have $\tau^2(\pi_1\pi_3^2) = \pi_1^2\pi_3$ then $\tau^2(\pi_1\pi_3^2) = \tau^2(\sqrt[5]{(\pi_1\pi_3^2)^5}) = (\tau^2(\sqrt[5]{\pi_1\pi_3^2}))^5 = \pi_1^2\pi_3$, then $\tau^2(\sqrt[5]{\pi_1\pi_3^2})$ is 5th root of $\pi_1^2\pi_3$. Thus $k(\sqrt[5]{\pi_1^2\pi_3}) = k(\tau^2(\sqrt[5]{\pi_1\pi_3^2}))$ i.e. $k(\sqrt[5]{\pi_1^2\pi_3}) = k(\sqrt[5]{\pi_1\pi_3^3}) = k(\sqrt[5]{\pi_1\pi_3^3})^{\tau^2}$. By the same reasoning we prove that $k(\sqrt[5]{\pi_1\pi_3^2}) = k(\sqrt[5]{\pi_1\pi_3^3})^{\tau^2}$. Hence τ^2 permutes $k(\sqrt[5]{\pi_1\pi_3^2})$ with $k(\sqrt[5]{\pi_1\pi_3^3})$. \square

The generators of $C_{k,5}$ when it is of type (5, 5) and the radicand n is as above are determined as follows:

Theorem 3.2. *Let k and n as above. Let π_1, π_2, π_3 and π_4 be primes of k_0 congruent to 1 modulo λ^5 such that $n = p = \pi_1\pi_2\pi_3\pi_4$. Let $\mathcal{P}_1, \mathcal{P}_2, \mathcal{P}_3$ and \mathcal{P}_4 be prime ideals of k such that $\mathcal{P}_i^5 = \pi_i \mathcal{O}_{k_0}$ ($i = 1, 2, 3, 4$). Then:*

$$C_{k,5} = \langle [\mathcal{P}_1\mathcal{P}_3], [\mathcal{P}_1\mathcal{P}_3^4] \rangle.$$

Proof. According to the proof of [1, Theorem 1.1], , for this case of the radicand n , we have that $\zeta_5^i(1 + \zeta_5)^j$ is norm of element in $k - \{0\}$ for some exponents i and j . By [4, Section 5.3], if ζ_5 is not norm of unit of k we have $C_{k,5} = C_{k,5}^{(\sigma)} \neq C_{k,s}^{(\sigma)}$, so $C_{k,s}^{(\sigma)}$ contained in $C_{k,5}^{(\sigma)}$. Hence we discuss two cases:

- 1st case: $C_{k,5} = C_{k,5}^{(\sigma)} \neq C_{k,s}^{(\sigma)}$.

We have that $C_{k,s}^{(\sigma)}$ is contained in $C_{k,5} = C_{k,5}^{(\sigma)}$, and by [4, Section 5.3] we have $C_{k,5}^{(\sigma)}/C_{k,s}^{(\sigma)} = C_{k,5}/C_{k,s}^{(\sigma)}$ is cyclic group of order 5. Since $C_{k,5}$ has order 25, $C_{k,s}^{(\sigma)}$ is cyclic of order 5.

We have that $C_{k,s}^{(\sigma)} = \langle [\mathcal{P}_1], [\mathcal{P}_2], [\mathcal{P}_3], [\mathcal{P}_4] \rangle$, $\mathcal{P}_1^{\tau^2} = \mathcal{P}_3$ and $\mathcal{P}_2^{\tau^2} = \mathcal{P}_4$, so \mathcal{P}_1 and \mathcal{P}_2 can not be both principals in k , otherwise $\mathcal{P}_3 = \mathcal{P}_1^{\tau^2}$ and $\mathcal{P}_4 = \mathcal{P}_2^{\tau^2}$ will be principals too, Thus $C_{k,s}^{(\sigma)} = \{1\}$, which is impossible. By the same reasoning we have that \mathcal{P}_3 and \mathcal{P}_4 can not be both principals in k .

Since $C_{k,s}^{(\sigma)}$ is cyclic of order 5 and without losing generality, we get that $C_{k,s}^{(\sigma)} = \langle [\mathcal{P}_1] \rangle$, so \mathcal{P}_1 and $\mathcal{P}_3 = \mathcal{P}_1^{\tau^2}$ are not principals. Since $C_{k,5} \simeq C_{k,5}^+ \times C_{k,5}^-$, it is sufficient to find generators of $C_{k,5}^+$ and $C_{k,5}^-$. As $[\mathcal{P}_1\mathcal{P}_3]^{\tau^2} = [(\mathcal{P}_1\mathcal{P}_3)^{\tau^2}] = [\mathcal{P}_1\mathcal{P}_3]$, then $C_{k,5}^+ = \langle [\mathcal{P}_1\mathcal{P}_3] \rangle$ and $[\mathcal{P}_1\mathcal{P}_3^4]^{\tau^2} = [(\mathcal{P}_1\mathcal{P}_3^4)^{\tau^2}] = [\mathcal{P}_1^4\mathcal{P}_3] = [\mathcal{P}_1\mathcal{P}_3^4]^{-1}$, then $C_{k,5}^- = \langle [\mathcal{P}_1\mathcal{P}_3^4] \rangle$. Hence $C_{k,5} = \langle [\mathcal{P}_1\mathcal{P}_3], [\mathcal{P}_1\mathcal{P}_3^4] \rangle$.

- 2nd case: $C_{k,5} = C_{k,5}^{(\sigma)} = C_{k,s}^{(\sigma)}$.

We apply the same reasoning as in the 1st case, because none of \mathcal{P}_i ($i = 1, 2, 3, 4$) is principal, otherwise $C_{k,5} = C_{k,s}^{(\sigma)} = \{1\}$, which is impossible. Hence $C_{k,5} = \langle [\mathcal{P}_1\mathcal{P}_3], [\mathcal{P}_1\mathcal{P}_3^4] \rangle$. \square

Now we are able to state the main theorem of capitulation in this case.

Theorem 3.3. *We keep the same assumptions as in Theorem 3.2. Then:*

- (1) If $(\frac{\pi_1}{\pi_3})_5 = 1$ we have $K_1 = k(\sqrt[5]{\pi_1\pi_3})$ or $k(\sqrt[5]{\pi_1\pi_3^4})$, $K_2 = k(\sqrt[5]{\pi_3})$, $K_3 = k(\sqrt[5]{\pi_1\pi_3^2})$ or $k(\sqrt[5]{\pi_1\pi_3^3})$, $K_4 = k(\sqrt[5]{\pi_1\pi_3^3})$ or $k(\sqrt[5]{\pi_1\pi_3^2})$, $K_5 = k(\sqrt[5]{\pi_1})$ and $K_6 = k(\sqrt[5]{\pi_1\pi_3^4})$ or $k(\sqrt[5]{\pi_1\pi_3})$. Otherwise we just permute K_2 and K_5 in equalities.
- (2) $[\mathcal{P}_1\mathcal{P}_3]$ capitulates in $k(\sqrt[5]{\pi_1\pi_3})$, $[\mathcal{P}_i]$ capitulates in $k(\sqrt[5]{\pi_i})$ ($i = 1, 3$), $[\mathcal{P}_1\mathcal{P}_3^2]$ capitulates in $k(\sqrt[5]{\pi_1\pi_3^2})$, $[\mathcal{P}_1\mathcal{P}_3^3]$ capitulates in $k(\sqrt[5]{\pi_1\pi_3^3})$ and $[\mathcal{P}_1\mathcal{P}_3^4]$ capitulates in $k(\sqrt[5]{\pi_1\pi_3^4})$.
- (3) (i) If $(\frac{\pi_1}{\pi_3})_5 = 1$ and $K_6 = k(\sqrt[5]{\pi_1\pi_3^4})$, then the possible types of capitulation are: $(0, 0, 0, 0, 0, 0)$, $(1, 0, 0, 0, 0, 0)$, $(0, 2, 0, 0, 5, 0)$, $(1, 2, 0, 0, 5, 0)$, $\{(0, 0, 3, 4, 0, 0) \text{ or } (0, 0, 4, 3, 0, 0)\}$, $\{(1, 0, 3, 4, 0, 0) \text{ or } (1, 0, 4, 3, 0, 0)\}$, $\{(0, 2, 3, 4, 5, 0) \text{ or } (0, 2, 4, 3, 5, 0)\}$, $\{(1, 2, 3, 4, 5, 0) \text{ or } (1, 2, 4, 3, 5, 0)\}$.
(ii) If $(\frac{\pi_1}{\pi_3})_5 = 1$ and $K_6 = k(\sqrt[5]{\pi_1\pi_3})$ then the same possible types of capitulation occur as in (i) with $i_6 = 0$ or 1 and $i_1 = 0$ or 6.
(iii) If $(\frac{\pi_1}{\pi_3})_5 \neq 1$ then the same possible types of capitulation occur as (i) and (ii) by permuting 2 and 5 in the given types of capitulation.

Proof. (1) According to Theorem 3.1, we have that τ^2 permutes $k(\sqrt[5]{\pi_1})$ with $k(\sqrt[5]{\pi_3})$ and $k(\sqrt[5]{\pi_1\pi_3^2})$ with $k(\sqrt[5]{\pi_1\pi_3^3})$, moreover τ^2 sets $k(\sqrt[5]{\pi_1\pi_3})$, $k(\sqrt[5]{\pi_1\pi_3^4})$.

By class field theory K_i corresponds to H_i ($i = 1, 2, 3, 4, 5, 6$). We determine explicitly the six subgroups H_i of $C_{k,5}$ as follows:

We have that $C_{k,5} = \langle \mathcal{A}, \mathcal{X} \rangle$, where $H_1 = C_{k,5}^+ = \langle \mathcal{A} \rangle$ and $H_6 = C_{k,5}^- = \langle \mathcal{X} \rangle$. By Theorem 3.2 we have $\mathcal{A} = [\mathcal{P}_1\mathcal{P}_3]$ and $\mathcal{X} = [\mathcal{P}_1\mathcal{P}_3^4]$, then $\mathcal{A}\mathcal{X} = [\mathcal{P}_1]^2$, $\mathcal{A}\mathcal{X}^2 = [\mathcal{P}_1\mathcal{P}_3^3]$, $\mathcal{A}\mathcal{X}^3 = [\mathcal{P}_1\mathcal{P}_3^2]^4$ and

$\mathcal{AX}^4 = [\mathcal{P}_3]^4$. Thus $H_2 = \langle [\mathcal{P}_1] \rangle$, $H_3 = \langle [\mathcal{P}_1 \mathcal{P}_3^3] \rangle$, $H_4 = \langle [\mathcal{P}_1 \mathcal{P}_3^2] \rangle$ and $H_5 = \langle [\mathcal{P}_3] \rangle$. Since τ^2 sets $k(\sqrt[5]{\pi_1 \pi_3})$ and $k(\sqrt[5]{\pi_1 \pi_3^4})$, if $K_1 = k(\sqrt[5]{\pi_1 \pi_3})$, then $K_6 = k(\sqrt[5]{\pi_1 \pi_3^4})$ and vice versa.

If $(\frac{\pi_1}{\pi_3})_5 = 1$ then $X^5 \equiv \pi_1 \pmod{\pi_3}$ is resolved in \mathcal{O}_{k_0} and by Proposition 2.1, we have that π_1 splits completely in $k_0(\sqrt[5]{\pi_3})$, which equivalent to say that \mathcal{P}_1 splits completely in $k(\sqrt[5]{\pi_3})$, so $K_2 = k(\sqrt[5]{\pi_3})$ and we get that $K_5 = k(\sqrt[5]{\pi_1})$. If $K_3 = k(\sqrt[5]{\pi_1 \pi_3^2})$, then $K_4 = k(\sqrt[5]{\pi_1 \pi_3^3})$ and vice versa. Since π_1 and π_3 divide $\pi_1 \pi_3$, $\pi_1 \pi_3^2$, $\pi_1 \pi_3^3$ and $\pi_1 \pi_3^4$, if $(\frac{\pi_1}{\pi_3})_5 \neq 1$, then $K_2 = k(\sqrt[5]{\pi_1})$ and $K_5 = k(\sqrt[5]{\pi_3})$.

(2) Since $\mathcal{P}_i^5 = \pi_i \mathcal{O}_k$ ($i = 1, 3$), we have $(\mathcal{P}_1 \mathcal{P}_3)^5 = \pi_1 \pi_3 \mathcal{O}_k$, then $(\mathcal{P}_1 \mathcal{P}_3)^5 = \pi_1 \pi_3 \mathcal{O}_{k(\sqrt[5]{\pi_1 \pi_3})}$ in $k(\sqrt[5]{\pi_1 \pi_3})$ and $\pi_1 \pi_3 \mathcal{O}_{k(\sqrt[5]{\pi_1 \pi_3})} = (\sqrt[5]{\pi_1 \pi_3} \mathcal{O}_{k(\sqrt[5]{\pi_1 \pi_3})})^5$, whence $\mathcal{P}_1 \mathcal{P}_3 \mathcal{O}_{k(\sqrt[5]{\pi_1 \pi_3})} = \sqrt[5]{\pi_1 \pi_3} \mathcal{O}_{k(\sqrt[5]{\pi_1 \pi_3})}$. Thus $\mathcal{P}_1 \mathcal{P}_3$ seen in $\mathcal{O}_{k(\sqrt[5]{\pi_1 \pi_3})}$ becomes principal, i.e $[\mathcal{P}_1 \mathcal{P}_3]$ capitulates in $k(\sqrt[5]{\pi_1 \pi_3})$. Since $(\mathcal{P}_1 \mathcal{P}_3^2)^5 = \pi_1 \pi_3^2 \mathcal{O}_k$, we have $(\mathcal{P}_1 \mathcal{P}_3^2)^5 = \pi_1 \pi_3^2 \mathcal{O}_{k(\sqrt[5]{\pi_1 \pi_3^2})}$ in $k(\sqrt[5]{\pi_1 \pi_3^2})$ and $\pi_1 \pi_3^2 \mathcal{O}_{k(\sqrt[5]{\pi_1 \pi_3^2})} = (\sqrt[5]{\pi_1 \pi_3^2} \mathcal{O}_{k(\sqrt[5]{\pi_1 \pi_3^2})})^5$, hence $\mathcal{P}_1 \mathcal{P}_3^2 \mathcal{O}_{k(\sqrt[5]{\pi_1 \pi_3^2})} = \sqrt[5]{\pi_1 \pi_3^2} \mathcal{O}_{k(\sqrt[5]{\pi_1 \pi_3^2})}$. Thus $\mathcal{P}_1 \mathcal{P}_3^2$ seen in $\mathcal{O}_{k(\sqrt[5]{\pi_1 \pi_3^2})}$ becomes principal, i.e $[\mathcal{P}_1 \mathcal{P}_3^2]$ capitulates in $k(\sqrt[5]{\pi_1 \pi_3^2})$.

By the same reasoning, we have $[\mathcal{P}_1 \mathcal{P}_3^3]$ capitulates in $k(\sqrt[5]{\pi_1 \pi_3^3})$ and $[\mathcal{P}_1 \mathcal{P}_3^4]$ capitulates in $k(\sqrt[5]{\pi_1 \pi_3^4})$.

We have $\mathcal{P}_1^5 = \pi_1 \mathcal{O}_k$, then $\mathcal{P}_1 \mathcal{O}_{k(\sqrt[5]{\pi_1})} = \sqrt[5]{\pi_1} \mathcal{O}_{k(\sqrt[5]{\pi_1})}$. Hence $[\mathcal{P}_1]$ capitulates in $k(\sqrt[5]{\pi_1})$. By the same reasoning, we have $[\mathcal{P}_3]$ capitulates in $k(\sqrt[5]{\pi_3})$.

(3) (i) If $(\frac{\pi_1}{\pi_3})_5 = 1$ and $K_6 = k(\sqrt[5]{\pi_1 \pi_3^4})$ we have $[\mathcal{P}_1 \mathcal{P}_3^4]$ capitulates in K_6 . According to [[4], Lemma 6.2], we have that $C_{k,5}^+ \simeq C_{\Gamma,5}$ and by class field theory $C_{\Gamma,5} \simeq \text{Gal}(\Gamma_5^{(1)}/\Gamma)$, then we obtain $C_{k,5}/C_{k,5}^- \simeq \text{Gal}(\Gamma_5^{(1)}/\Gamma) \simeq \text{Gal}(k\Gamma_5^{(1)}/k)$. Thus $k\Gamma_5^{(1)}$ is an unramified cyclic extension of k corresponds to $C_{k,5}^-$. We denote by $j_{k/\Gamma} : C_{\Gamma,5} \rightarrow C_{k,5}$ the homomorphism induced by extension of ideals of Γ in k . Since $C_{k,5}^+ = \langle [\mathcal{P}_1 \mathcal{P}_3] \rangle$ and $\mathcal{P}_1 \mathcal{P}_3 = j_{k/\Gamma}(\mathcal{J})$ such that $C_{\Gamma,5} = \langle \mathcal{J} \rangle$, $[\mathcal{P}_1 \mathcal{P}_3]$ capitulates in $K_6 = k\Gamma_5^{(1)}$. As $C_{k,5} = \langle [\mathcal{P}_1 \mathcal{P}_3], [\mathcal{P}_1 \mathcal{P}_3^4] \rangle$, then all classes capitulate in K_6 .

We determine possible types of capitulation $(i_1, i_2, i_3, i_4, i_5, i_6)$. We have that $i_6 = 0$, $K_2 = K_5^2$, $K_3 = K_4^2$, then the same number of classes capitulate in K_2 , K_5 and similarly for K_3 , K_4 .

If $i_1 \neq 0$ we have $i_1 = 1$, if $i_2 \neq 0$ we have $i_2 = 2$ and if $i_5 \neq 0$ we have $i_5 = 5$. i_3 and i_4 are both nulls or non nulls, so if i_3 and $i_4 \neq 0$, then $(i_3, i_4) = (3, 4)$ or $(4, 3)$. Thus the possible types of capitulation are:

$(0, 0, 0, 0, 0, 0)$, $(1, 0, 0, 0, 0, 0)$, $(0, 2, 0, 0, 5, 0)$, $(1, 2, 0, 0, 5, 0)$, $\{(0, 0, 3, 4, 0, 0) \text{ or } (0, 0, 4, 3, 0, 0)\}$, $\{(1, 0, 3, 4, 0, 0) \text{ or } (1, 0, 4, 3, 0, 0)\}$, $\{(0, 2, 3, 4, 5, 0) \text{ or } (0, 2, 4, 3, 5, 0)\}$, $\{(1, 2, 3, 4, 5, 0) \text{ or } (1, 2, 4, 3, 5, 0)\}$.

(ii) If $(\frac{\pi_1}{\pi_3})_5 = 1$ and $K_6 = k(\sqrt[5]{\pi_1 \pi_3})$ we have $[\mathcal{P}_1 \mathcal{P}_3]$ capitulates in K_6 , then if $i_6 \neq 0$ we have $i_6 = 1$. $[\mathcal{P}_1 \mathcal{P}_3^4]$ capitulates in K_1 , then if $i_1 \neq 0$ we have $i_1 = 6$, so the same possible types of capitulation occur as in (i) with $i_6 = 0$ or 1 and $i_1 = 0$ or 6.

(iii) If $(\frac{\pi_1}{\pi_3})_5 \neq 1$, by (1) we have $K_2 = k(\sqrt[5]{\pi_3})$ and $K_5 = k(\sqrt[5]{\pi_1})$ then the same possible types of capitulation occur as (i) and (ii) by permuting 2 and 5 in the given types of capitulation. \square

3.2. The case $n = p^{e_1} q^{e_2} \equiv \pm 1, \pm 7 \pmod{25}$ where $p \not\equiv 1 \pmod{25}$, $q \not\equiv \pm 7 \pmod{25}$.

Let $k = \Gamma(\zeta_5)$ be the normal closure of $\Gamma = \mathbb{Q}(\sqrt[5]{n})$, where $n = p^{e_1} q^{e_1} \equiv \pm 1, \pm 7 \pmod{25}$ such that $p \not\equiv 1 \pmod{25}$, $q \not\equiv \pm 7 \pmod{25}$ and $e_1, e_2 \in \{1, 2, 3, 4\}$. By [5, Theorem 2.13], since

$q \equiv \pm 2 \pmod{5}$ we have that q is inert in $k_0 = \mathbb{Q}(\zeta_5)$, so we set in the sequel $q = \pi_5$ as prime in k_0 .

By $\mathcal{P}_1, \mathcal{P}_2, \mathcal{P}_3, \mathcal{P}_4$ and \mathcal{P}_5 we denote respectively the prime ideals of k above $\pi_1, \pi_2, \pi_3, \pi_4$ and π_5 in k_0 , such that $\mathcal{P}_i^5 = \pi_i \mathcal{O}_k$ ($i = 1, 2, 3, 4, 5$). We have that τ^2 permutes π_1 with π_3 , then τ^2 permutes \mathcal{P}_1 with \mathcal{P}_3 , moreover τ^2 sets $q = \pi_5$ and also \mathcal{P}_5 .

The six intermediate extensions of $k_5^{(1)}/k$ are determined as follows:

Theorem 3.4. *Let $k, n, \pi_1, \pi_2, \pi_3, \pi_4$ and π_5 as above. Put $x_1 = \pi_1 \pi_5^{h_1}$ and $x_2 = \pi_1 \pi_3^4$ where $h_1 \in \{1, 2, 3, 4\}$ is chosen such that $x_1 \equiv x_2 \equiv 1 \pmod{\lambda^5}$, where $h_1 \in \{1, 2, 3, 4\}$. Then:*

- (1) $k_5^{(1)} = k(\sqrt[5]{x_1}, \sqrt[5]{x_2})$.
- (2) *The six intermediate extensions of $k_5^{(1)}/k$ are:*
 $k(\sqrt[5]{x_1}), k(\sqrt[5]{x_2}), k\left(\sqrt[5]{\pi_1 \pi_3 \pi_5^{2h_1}}\right), k\left(\sqrt[5]{\pi_1^2 \pi_3^4 \pi_5^{h_1}}\right), k\left(\sqrt[5]{\pi_1^4 \pi_3^2 \pi_5^{h_1}}\right)$ and $k\left(\sqrt[5]{\pi_3 \pi_5^{h_1}}\right)$.
Furthermore τ^2 permutes $k\left(\sqrt[5]{\pi_1^2 \pi_3^4 \pi_5^{h_1}}\right)$ with $k\left(\sqrt[5]{\pi_1^4 \pi_3^2 \pi_5^{h_1}}\right)$ and $k(\sqrt[5]{x_1})$ with $k\left(\sqrt[5]{\pi_3 \pi_5^{h_1}}\right)$, and sets $k(\sqrt[5]{x_2}), k\left(\sqrt[5]{\pi_1 \pi_3 \pi_5^{2h_1}}\right)$.

Proof. Since $k = k_0(\sqrt[5]{n})$ we can write n in k_0 as $n = \pi_1^e \pi_2^e \pi_3^e \pi_4^e \pi_5^e$, with π_i do not all verified $\pi_i \equiv 1 \pmod{\lambda^5}$, because we have $p \not\equiv 1 \pmod{25}$. By Proposition 2.2 there exist $h_1, h_2 \in \{1, \dots, 4\}$ such that $\pi_1 \pi_5^{h_1} \equiv \pm 1, \pm 7 \pmod{\lambda^5}$ and $\pi_1 \pi_3^{h_2} \equiv \pm 1, \pm 7 \pmod{\lambda^5}$. To investigate the correspondence between the six intermediate extensions of $k_5^{(1)}/k$ and the six subgroups of $C_{k,5}$, we assume that $h_2 = 4$. Put $x_1 = \pi_1 \pi_5^{h_1}$ and $x_2 = \pi_1 \pi_3^4$.

- (1) The fact that $k_5^{(1)} = k(\sqrt[5]{x_1}, \sqrt[5]{x_2})$ follows from Proposition 2.2.
- (2) The six intermediate extensions are: $k(\sqrt[5]{x_1}), k(\sqrt[5]{x_2}), k(\sqrt[5]{x_1 x_2}), k(\sqrt[5]{x_1 x_2^2}), k(\sqrt[5]{x_1 x_2^3})$ and $k(\sqrt[5]{x_1 x_2^4})$. Since $x_1 = \pi_1 \pi_5^{h_1}$ and $x_2 = \pi_1 \pi_3^4$, we have $k(\sqrt[5]{x_1 x_2}) = k\left(\sqrt[5]{\pi_1^2 \pi_3^4 \pi_5^{h_1}}\right)$, $k(\sqrt[5]{x_1 x_2^2}) = k\left(\sqrt[5]{\pi_1 \pi_3 \pi_5^{2h_1}}\right)$, $k(\sqrt[5]{x_1 x_2^3}) = k\left(\sqrt[5]{\pi_1^4 \pi_3^2 \pi_5^{h_1}}\right)$ and $k(\sqrt[5]{x_1 x_2^4}) = k\left(\sqrt[5]{\pi_3 \pi_5^{h_1}}\right)$. Since $\pi_1^{\tau^2} = \pi_3$, $\pi_3^{\tau^2} = \pi_1$ and $\pi_5^{\tau^2} = \pi_5$, and by the same reasoning as (2) of Theorem 3.1 we prove that τ^2 permutes $k\left(\sqrt[5]{\pi_1^2 \pi_3^4 \pi_5^{h_1}}\right)$ with $k\left(\sqrt[5]{\pi_1^4 \pi_3^2 \pi_5^{h_1}}\right)$ and $k(\sqrt[5]{x_1})$ with $k\left(\sqrt[5]{\pi_3 \pi_5^{h_1}}\right)$, and sets $k(\sqrt[5]{x_2}), k\left(\sqrt[5]{\pi_1 \pi_3 \pi_5^{2h_1}}\right)$. \square

The generators of $C_{k,5}$ in this case are determined as follows:

Theorem 3.5. *Let $k, n, \pi_1, \pi_2, \pi_3, \pi_4, \pi_5$ and h_1 as above. Let $\mathcal{P}_1, \mathcal{P}_2, \mathcal{P}_3, \mathcal{P}_4$ and \mathcal{P}_5 prime ideals of k such that $\mathcal{P}_i^5 = \pi_i \mathcal{O}_{k_0}$ ($i = 1, 2, 3, 4, 5$). Then:*

$$C_{k,5} = \langle [\mathcal{P}_1 \mathcal{P}_3 \mathcal{P}_5^{2h_1}], [\mathcal{P}_1 \mathcal{P}_3^4] \rangle$$

Proof. According to [1, Theorem 1.1], for this case of the radicand n , we have that $\zeta_5^i(1 + \zeta_5)^j$ is not norm of element in $k - \{0\}$ for any exponents i and j , then by [4, Section 5.3], we have $C_{k,5} = C_{k,5}^{(\sigma)} = C_{k,s}^{(\sigma)} = \langle [\mathcal{P}_1], [\mathcal{P}_2], [\mathcal{P}_3], [\mathcal{P}_4], [\mathcal{P}_5] \rangle$. Since $\mathcal{P}_1^{\tau^2} = \mathcal{P}_3$, $\mathcal{P}_2^{\tau^2} = \mathcal{P}_4$ and $\mathcal{P}_5^{\tau^2} = \mathcal{P}_5$, as the proof of Theorem 3.2 we have that $\mathcal{P}_1, \mathcal{P}_3$ and \mathcal{P}_5 are non principals. As $[\mathcal{P}_1 \mathcal{P}_3 \mathcal{P}_5^{2h_1}]^{\tau^2} = [(\mathcal{P}_1 \mathcal{P}_3 \mathcal{P}_5^{2h_1})^{\tau^2}] = [\mathcal{P}_3 \mathcal{P}_1 \mathcal{P}_5^{2h_1}] = [\mathcal{P}_1 \mathcal{P}_3 \mathcal{P}_5^{2h_1}]$ then $C_{k,5}^+ = \langle [\mathcal{P}_1 \mathcal{P}_3 \mathcal{P}_5^{2h_1}] \rangle$, and we have that $C_{k,5}^- = \langle [\mathcal{P}_1 \mathcal{P}_3^4] \rangle$. Hence $C_{k,5} = \langle [\mathcal{P}_1 \mathcal{P}_3 \mathcal{P}_5^{2h_1}], [\mathcal{P}_1 \mathcal{P}_3^4] \rangle$. \square

The main theorem of capitulation in this case is as follows:

Theorem 3.6. *We keep the same assumptions as Theorem 3.5. Then:*

(1) $K_1 = k\left(\sqrt[5]{\pi_1\pi_3^4}\right)$ or $k\left(\sqrt[5]{\pi_1\pi_3\pi_5^{2h_1}}\right)$, $K_2 = k\left(\sqrt[5]{\pi_1\pi_5^{h_1}}\right)$ or $k\left(\sqrt[5]{\pi_3\pi_5^{h_1}}\right)$, $K_3 = k\left(\sqrt[5]{\pi_1^2\pi_3^4\pi_5^{h_1}}\right)$ or $k\left(\sqrt[5]{\pi_1^4\pi_3^2\pi_5^{h_1}}\right)$, $K_4 = k\left(\sqrt[5]{\pi_1^4\pi_3^2\pi_5^{h_1}}\right)$ or $k\left(\sqrt[5]{\pi_1^2\pi_3^4\pi_5^{h_1}}\right)$, $K_5 = k\left(\sqrt[5]{\pi_3\pi_5^{h_1}}\right)$ or $k\left(\sqrt[5]{\pi_1\pi_5^{h_1}}\right)$ and $K_6 = k\left(\sqrt[5]{\pi_1\pi_3\pi_5^{2h_1}}\right)$ or $k\left(\sqrt[5]{\pi_1\pi_3^4}\right)$.

(2) $[\mathcal{P}_1\mathcal{P}_3\mathcal{P}_5^{2h_1}]$ capitulates in $k\left(\sqrt[5]{\pi_1\pi_3\pi_5^{2h_1}}\right)$, $[\mathcal{P}_1\mathcal{P}_5^{h_1}]$ capitulates in $k\left(\sqrt[5]{\pi_1\pi_5^{h_1}}\right)$, $[\mathcal{P}_1^2\mathcal{P}_3^4\mathcal{P}_5^{h_1}]$ capitulates in $k\left(\sqrt[5]{\pi_1^2\pi_3^4\pi_5^{h_1}}\right)$, $[\mathcal{P}_1^4\mathcal{P}_3^2\mathcal{P}_5^{h_1}]$ capitulates in $k\left(\sqrt[5]{\pi_1^4\pi_3^2\pi_5^{h_1}}\right)$, $[\mathcal{P}_3\mathcal{P}_5^{h_1}]$ capitulates in $k\left(\sqrt[5]{\pi_3\pi_5^{h_1}}\right)$ and $[\mathcal{P}_1\mathcal{P}_3^4]$ capitulates in $k\left(\sqrt[5]{\pi_1\pi_3^4}\right)$.

(3) If $K_1 = k\left(\sqrt[5]{\pi_1\pi_3\pi_5^{2h_1}}\right)$, then the possible types of capitulation are:

$(0, 0, 0, 0, 0, 0)$, $(1, 0, 0, 0, 0, 0)$, $\{(0, 5, 0, 0, 2, 0) \text{ or } (0, 2, 0, 0, 5, 0)\}$,
 $\{(1, 5, 0, 0, 2, 0) \text{ or } (1, 2, 0, 0, 5, 0)\}$, $\{(0, 5, 4, 3, 2, 0) \text{ or } (0, 2, 4, 3, 5, 0)\}$,
 $\{(1, 5, 4, 3, 2, 0) \text{ or } (1, 2, 4, 3, 5, 0)\}$,
 $\{(0, 5, 3, 4, 2, 0) \text{ or } (0, 2, 3, 4, 5, 0)\}$, $\{(1, 5, 3, 4, 2, 0) \text{ or } (1, 2, 3, 4, 5, 0)\}$,
 $\{(0, 0, 3, 4, 0, 0) \text{ or } (0, 0, 4, 3, 0, 0)\}$, $\{(1, 0, 3, 4, 0, 0) \text{ or } (1, 0, 4, 3, 0, 0)\}$.

If $K_1 = k\left(\sqrt[5]{\pi_1\pi_3^4}\right)$, then the same possible types occur, where i_6 takes the value 0 or 1.

Proof. (1) According to Theorem 3.4, we have that τ^2 permutes $k\left(\sqrt[5]{\pi_1^2\pi_3^4\pi_5^{h_1}}\right)$ with $k\left(\sqrt[5]{\pi_1^4\pi_3^2\pi_5^{h_1}}\right)$ and $k\left(\sqrt[5]{\pi_1\pi_3^4}\right)$ with $k\left(\sqrt[5]{\pi_1\pi_3\pi_5^{2h_1}}\right)$, and sets $k\left(\sqrt[5]{\pi_1\pi_5^{h_1}}\right)$, $k\left(\sqrt[5]{\pi_3\pi_5^{h_1}}\right)$. We determine first the six subgroups H_i of $C_{k,5}$. We have that $C_{k,5} = \langle \mathcal{A}, \mathcal{X} \rangle$, where $H_1 = C_{k,5}^+ = \langle \mathcal{A} \rangle$ and $H_6 = C_{k,5}^- = \langle \mathcal{X} \rangle$. By Theorem 3.5 we have $\mathcal{A} = [\mathcal{P}_1\mathcal{P}_3\mathcal{P}_5^{2h_1}]$ and $\mathcal{X} = [\mathcal{P}_1\mathcal{P}_3^4]$, then $\mathcal{A}\mathcal{X} = [\mathcal{P}_1\mathcal{P}_5^{h_1}]^2$, $\mathcal{A}\mathcal{X}^2 = [\mathcal{P}_1^2\mathcal{P}_3^4\mathcal{P}_5^{h_1}]^4$, $\mathcal{A}\mathcal{X}^3 = [\mathcal{P}_1^4\mathcal{P}_3^2\mathcal{P}_5^{h_1}]$ and $\mathcal{A}\mathcal{X}^4 = [\mathcal{P}_3\mathcal{P}_5^{h_1}]^3$. Hence $H_2 = \langle [\mathcal{P}_1\mathcal{P}_5^{h_1}] \rangle$, $H_3 = \langle [\mathcal{P}_1^2\mathcal{P}_3^4\mathcal{P}_5^{h_1}] \rangle$, $H_4 = \langle [\mathcal{P}_1^4\mathcal{P}_3^2\mathcal{P}_5^{h_1}] \rangle$ and $H_5 = \langle [\mathcal{P}_3\mathcal{P}_5^{h_1}] \rangle$. Since τ^2 sets $k\left(\sqrt[5]{\pi_1\pi_3\pi_5^{2h_1}}\right)$ and $k\left(\sqrt[5]{\pi_1\pi_3^4}\right)$, so if $K_1 = k\left(\sqrt[5]{\pi_1\pi_3\pi_5^{2h_1}}\right)$ then $K_6 = k\left(\sqrt[5]{\pi_1\pi_3^4}\right)$ and inversely.

By class field theory, the fact that H_i ($i = 2, 5$) corresponds to K_i ($i = 2, 5$) means that $\mathcal{P}_1\mathcal{P}_5^{h_1}$ splits completely in K_2 and $\mathcal{P}_3\mathcal{P}_5^{h_1}$ splits completely in K_5 . As $\pi_1\pi_5^{h_1}$ divides $\pi_1^2\pi_3^4\pi_5^{h_1}$ and $\pi_1^4\pi_3^2\pi_5^{h_1}$, by Proposition 2.1, $\pi_1\pi_5^{h_1}$ can not split in $k_0\left(\sqrt[5]{\pi_1^2\pi_3^4\pi_5^{h_1}}\right)$ and $k_0\left(\sqrt[5]{\pi_1^4\pi_3^2\pi_5^{h_1}}\right)$, this equivalent to say that $\mathcal{P}_1\mathcal{P}_5^{h_1}$ can not split completely in $k\left(\sqrt[5]{\pi_1^2\pi_3^4\pi_5^{h_1}}\right)$ and $k\left(\sqrt[5]{\pi_1^4\pi_3^2\pi_5^{h_1}}\right)$. By the same reasoning we have that $\mathcal{P}_3\mathcal{P}_5^{h_1}$ can not split completely in $k\sqrt[5]{\pi_1^2\pi_3^4\pi_5^{h_1}}$ and $k\sqrt[5]{\pi_1^4\pi_3^2\pi_5^{h_1}}$. Thus if $K_2 = k\left(\sqrt[5]{\pi_1\pi_5^{h_1}}\right)$ then $K_5 = k\left(\sqrt[5]{\pi_3\pi_5^{h_1}}\right)$ and inversely, which allow us to deduce that if $K_3 = k\left(\sqrt[5]{\pi_1^2\pi_3^4\pi_5^{h_1}}\right)$ then $K_5 = k\left(\sqrt[5]{\pi_1^4\pi_3^2\pi_5^{h_1}}\right)$ and inversely.

(2) We keep the same reasoning as the proof of (2) Theorem 3.3.

(3) If $K_1 = k\left(\sqrt[5]{\pi_1\pi_3\pi_5^{2h_1}}\right)$, then $K_6 = k\Gamma_5^{(1)} = k\left(\sqrt[5]{\pi_1\pi_3^4}\right)$ and we have that $[\mathcal{P}_1\mathcal{P}_3^4]$ capitulates in K_6 , moreover since $C_{k,5}^+ = \langle [\mathcal{P}_1\mathcal{P}_3\mathcal{P}_5^{2h_1}] \rangle \simeq C_{\Gamma,5}$ $\mathcal{P}_1\mathcal{P}_3\mathcal{P}_5^{2h_1} = j_{k/\Gamma}(\mathcal{J})$ such

that $C_{\Gamma,5} = \langle \mathcal{J} \rangle$, then $[\mathcal{P}_1 \mathcal{P}_3 \mathcal{P}_5^{2h_1}]$ capitulates in K_6 . As $C_{k,5} = \langle [\mathcal{P}_1 \mathcal{P}_3 \mathcal{P}_5^{2h_1}], [\mathcal{P}_1 \mathcal{P}_3^4] \rangle$, then all classes capitulate in $K_6 = k \left(\sqrt[5]{\pi_1 \pi_3^4} \right)$. We determine the possible types of capitulation $(i_1, i_2, i_3, i_4, i_5, i_6)$.

We have that $i_6 = 0$, $K_2 = K_5^{\tau^2}$, $K_3 = K_4^{\tau^2}$, then the same number of classes capitulate in K_2 , K_5 and similarly for K_3 , K_4 . If $i_1 \neq 0$ we have $i_1 = 1$. i_2 and i_5 are both nulls or non nulls, so if i_2 and $i_5 \neq 0$, then $(i_2, i_5) = (2, 5)$ or $(5, 2)$ depending on $\mathcal{P}_1 \mathcal{P}_5^{h_1}$ splits completely in $k \left(\sqrt[5]{\pi_1 \pi_5^{h_1}} \right)$ or in $k \left(\sqrt[5]{\pi_3 \pi_5^{h_1}} \right)$. Similarly if i_3 and $i_4 \neq 0$, then $(i_3, i_4) = (3, 4)$ or $(4, 3)$. Hence the possible types given are proved.

If $K_1 = k \left(\sqrt[5]{\pi_1 \pi_3^4} \right)$ then $K_6 = k\Gamma_5^{(1)} = k \left(\sqrt[5]{\pi_1 \pi_3 \pi_5^{2h_1}} \right)$ and we have $C_{k,5}^+ = \langle [\mathcal{P}_1 \mathcal{P}_3 \mathcal{P}_5^{2h_1}] \rangle$ capitulates in K_6 , the possible values of i_2, i_3, i_4, i_5 are as above, $(i_2, i_5) = (2, 5)$ or $(5, 2)$ if they are non nulls, $(i_3, i_4) = (3, 4)$ or $(4, 3)$ if they are non nulls. If $i_1 \neq 0$ then $i_1 = 6$ because $H_6 = \langle [\mathcal{P}_1 \mathcal{P}_3^4] \rangle$, and if $i_6 \neq 0$ then $i_1 = 1$ because $H_1 = \langle [\mathcal{P}_1 \mathcal{P}_3 \mathcal{P}_5^{2h_1}] \rangle$. Hence the possible types given are proved. \square

3.3. The case $n = 5^{e_1} p^{e_2}$ where $p \not\equiv 1 \pmod{25}$.

Let $k = \Gamma(\zeta_5)$ be the normal closure of $\Gamma = \mathbb{Q}(\sqrt[5]{n})$, where $n = 5^{e_1} p^{e_2}$ such that $p \not\equiv 1 \pmod{25}$ and $e_1, e_2 \in \{1, 2, 3, 4\}$. By [4, Lemma 5.1], since $n = 5^{e_1} p^{e_2} \not\equiv \pm 1, \pm 7 \pmod{25}$ we have $\lambda = 1 - \zeta_5$ is ramified in k/k_0 .

Let π_1, π_2, π_3 and π_4 primes of k_0 such that $p = \pi_1 \pi_2 \pi_3 \pi_4$. Let $\mathcal{P}_1, \mathcal{P}_2, \mathcal{P}_3, \mathcal{P}_4$ and \mathcal{I} prime ideals of k above $\pi_1, \pi_2, \pi_3, \pi_4$ and λ , we have $\mathcal{P}_i^5 = \pi_i \mathcal{O}_k$ and $\mathcal{I}^5 = \lambda \mathcal{O}_k$. According to [1, Theorem 1.1], for this case of the radicand n , we have that $\zeta_5^i (1 + \zeta_5)^j$ is not norm of element in $k - \{0\}$ for any exponents i and j , then we have $C_{k,5} = C_{k,5}^{(\sigma)} = C_{k,s}^{(\sigma)}$. Hence the results about the six intermediate extensions of $k_5^{(1)}/k$, the generators of $C_{k,5}$ and the capitulation problem in this case are the same as case 2 by substituting q by 5, π_5 by λ and \mathcal{P}_5 by \mathcal{I} .

4. Numerical examples

The task to determine the capitulation in a cyclic quintic extension of a base field of degree 20, that is, in a field of absolute degree 100, is definitely far beyond the reach of computational algebra systems like MAGMA and Pari/GP. For this reason we give examples of pure metacyclic fields $k = \mathbb{Q}(\sqrt[5]{n}, \zeta_5)$ such that $C_{k,5}$ is of type $(5, 5)$ and $C_{k,5} = C_{k,5}^{(\sigma)}$.

Table 1: $k = \mathbb{Q}(\sqrt[5]{n}, \zeta_5)$ with $C_{k,5}$ of type (5, 5) and $C_{k,5} = C_{k,5}^{(\sigma)}$.

No	n	Factorization	$n \pmod{25}$	Section	$C_{k,5}$	$C_{k,5}^{(\sigma)}$
1	55	5 . 11	+5	3.3	(5, 5)	2
2	82	2 . 41	+7	3.2	(5, 5)	2
3	93	3 . 31	-7	3.2	(5, 5)	2
4	99	3^2 . 11	-1	3.2	(5, 5)	2
5	124	2^2 . 31	-1	3.2	(5, 5)	2
6	143	11 . 13	-7	3.2	(5, 5)	2
7	151	151	+1	3.1	(5, 5)	2
8	176	2^4 . 11	+1	3.2	(5, 5)	2
9	205	5 . 41	+5	3.3	(5, 5)	2
10	251	251	+1	3.1	(5, 5)	2
11	355	5 . 71	+5	3.3	(5, 5)	2
12	382	2 . 191	+7	3.2	(5, 5)	2
13	393	3 . 131	-7	3.2	(5, 5)	2
14	407	11 . 37	+7	3.2	(5, 5)	2
15	524	2^2 . 131	-1	3.2	(5, 5)	2
16	543	3 . 181	-7	3.2	(5, 5)	2
17	568	2^3 . 71	-7	3.2	(5, 5)	2
18	601	601	+1	3.1	(5, 5)	2
19	605	5 . 11^2	+5	3.3	(5, 5)	2
20	655	5 . 131	+5	3.3	(5, 5)	2
21	724	2^2 . 181	-1	3.2	(5, 5)	2
22	905	5 . 181	+5	3.3	(5, 5)	2
23	943	23 . 41	-7	3.2	(5, 5)	2
24	976	2^4 . 61	+1	3.2	(5, 5)	2
25	982	2 . 491	+7	3.2	(5, 5)	2
26	993	3 . 331	-7	3.2	(5, 5)	2
27	1051	1051	+1	3.1	(5, 5)	2
28	1301	1301	+1	3.1	(5, 5)	2
29	1457	31 . 47	+7	3.2	(5, 5)	2
30	1555	5 . 311	+5	3.3	(5, 5)	2
31	1775	5^2 . 71	0	3.3	(5, 5)	2
32	1801	1801	+1	3.1	(5, 5)	2
33	1901	1901	+1	3.1	(5, 5)	2
34	2155	5 . 431	+5	3.3	(5, 5)	2
35	6943	53 . 131	-7	3.2	(5, 5)	2
36	8275	5^2 . 331	0	3.3	(5, 5)	2
37	8507	47 . 181	+7	3.2	(5, 5)	2
38	12707	97 . 131	+7	3.2	(5, 5)	2
39	30125	5^3 . 241	0	3.3	(5, 5)	2

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References

- [1] A. Azizi, F. Elmouhib and M. Talbi, *5-rank of ambiguous class groups of quintic Kummer extensions*, Proc. Indian Acad. Sci. Math. Sci., 132(12) (2022), 14 pp.
- [2] F. Elmouhib, M. Talbi and A. Azizi, *On the capitulation problem of some pure metacyclic fields of degree 20.*, Palest. J. Math., 11(1) (2022), 260-267.
- [3] E. Hecke, Lectures on the Theory of Algebraic Numbers, Graduate Texts in Mathematics, 77, Springer-Verlag, New York-Berlin, 1981.
- [4] M. Kulkarni, D. Majumdar and B. Sury, *l -class groups of cyclic extension of prime degree l* , J. Ramanujan Math. Soc., 30(4) (2015), 413-454.
- [5] L. C. Washington, Introduction to Cyclotomic Fields, Graduate Texts in Mathematics, 83, Springer-Verlag, New-York, 1982.
- [6] The PARI Group, PARI/GP, Version 2.4.9, Bordeaux, 2017, <http://pari.math.u-bordeaux.fr>

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