RINGS WITH DIVISIBILITY ON ASCENDING CHAINS OF IDEALS

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Abstract. According to Dastanpour and Ghorbani, a ring $R$ is said to satisfy divisibility on ascending chains of right ideals ($\text{ACC}_d$) if, for every ascending chain of right ideals $I_1 \subseteq I_2 \subseteq I_3 \subseteq \ldots$ of $R$, there exists an integer $k \in \mathbb{N}$ such that for each $i \geq k$, there exists an element $a_i \in R$ such that $I_i = a_i I_{i+1}$.

In this paper, we examine the transfer of the $\text{ACC}_d$-condition on ideals to trivial ring extensions. Moreover, we investigate the connection between the $\text{ACC}_d$ on ideals and other ascending chain conditions. For example we will prove that if $R$ is a ring with $\text{ACC}_d$ on ideals, then $R$ has $\text{ACC}$ on prime ideals.

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1. Introduction

In [5], extending the notion of $\text{ACC}$ on right ideals (i.e. right noetherian rings), a ring $R$ is said to satisfy divisibility on ascending chains of right ideals ($\text{ACC}_d$ on right ideals, for short) if, for every ascending chain of right ideals $I_1 \subseteq I_2 \subseteq I_3 \subseteq I_4 \subseteq \ldots$ of $R$, there exists an integer $k \in \mathbb{N}$ such that for each $i \geq k$, there exists an element $a_i \in R$ such that $I_i = a_i I_{i+1}$. If $R$ is commutative and all the multiple factors $a_i$ are invertible, then $R$ is noetherian.

In [5], Dastanpour and Ghorbani investigated thoroughly the notion of $\text{ACC}_d$ on right ideals, highlighting some of its properties and obtaining several interesting results in the commutative case. For example, they prove that every commutative semilocal ring that satisfies $\text{ACC}_d$ on ideals has a finitely generated socle and has only finitely many minimal prime ideals.

In this paper we focus our attention on commutative rings and continue the investigation that was carried out by Dastanpour and Ghorbani in [5]. In particular, we provide sufficient conditions for the trivial extension of rings to satisfy the $\text{ACC}_d$ on ideals. Moreover, we consider the connection between $\text{ACC}_d$ and other ascending
chain conditions on ideals. For example, we prove that if \( R \) satisfies \( \text{ACC}_d \) on ideals, then \( R \) satisfies \( \text{ACC} \) on prime ideals.

Throughout this paper, all rings are commutative with identity, and all modules are unital. If \( R \) is a ring, we denote by \( \text{Nil}(R) \) to the set (ideal) of all nilpotent elements of \( R \). When \( A \) is a local ring with \( M \) as its unique maximal ideal, we will write and say \((A, M)\) is local.

2. Main results

Let \( A \) be a ring and \( E \) an \( A \)-module. Then \( A \wr E \), the trivial (ring) extension of \( A \) by \( E \), is the ring whose additive structure is that of the external direct sum \( A \oplus E \) and whose multiplication is defined by \((a, e)(b, f) := (ab, af + be)\) for all \( a, b \in A \) and all \( e, f \in E \). The basic properties of trivial ring extensions are summarized in [6] and [7]. Moreover, interesting examples and constructions of trivial ring extensions could be found in [1], [3] and [8].

In [5, Proposition 2.3], the authors proved that homomorphic images of rings with \( \text{ACC}_d \) on right ideals satisfy the \( \text{ACC}_d \) on right ideals. We will use this result to establish our next theorem on the transfer of the \( \text{ACC}_d \)-condition on ideals to trivial ring extensions.

**Theorem 2.1.** Let \( A \) be a ring, \( E \) a nonzero \( A \)-module, and \( R := A \wr E \) the trivial ring extension of \( A \) by \( E \). Then:

1. If \( R \) satisfies \( \text{ACC}_d \), then so is \( A \).
2. Assume that \((A, M)\) is local such that \( ME = 0 \):
   1. If \( R \) satisfies \( \text{ACC}_d \), then \( A \) is noetherian.
   2. If \( E \) is finitely generated, then \( R \) satisfies \( \text{ACC}_d \) if and only if \( A \) is noetherian, if and only if \( R \) is noetherian.

**Proof.** (1). Assume that \( R \) satisfies \( \text{ACC}_d \) on ideals. Inasmuch as \((0 \wr E)\) is an ideal of \( R \), we infer from [5, Proposition 2.3] that \( A \cong R/(0 \wr E) \) satisfies \( \text{ACC}_d \) on ideals.

2a. Assume that \( R \) satisfies \( \text{ACC}_d \). We need to show that \( A \) is noetherian. Let \( I_1 \subseteq I_2 \subseteq I_3 \subseteq I_4 \subseteq \ldots \) be an ascending chain of ideals of \( A \). Since \( I_1 \wr E \subseteq I_2 \wr E \subseteq I_3 \wr E \subseteq I_4 \wr E \subseteq \ldots \) is an ascending chain of ideals of the ring \( R \), there exists an integer \( k \in \mathbb{N} \) such that for each \( i \geq k \), there is an element \((a_i, e_i) \in R \) such that \( I_i \wr E = (a_i, e_i)(I_{i+1} \wr E) \). This means that if \( i \geq k \) and \((s_i, f_i) \in I_i \wr E \), then there exists an element \((s_{i+1}, f_{i+1}) \in I_{i+1} \wr E \) such that \((s_i, f_i) = (a_i, e_i)(s_{i+1}, f_{i+1})\), and so \((s_i, f_i) = (a_is_{i+1}, s_{i+1}e_i + a_if_{i+1})\). Inasmuch as \( ME = 0 \), \( s_{i+1} \in I_{i+1} \subseteq M \), and so \((s_i, f_i) = (a_is_{i+1}, a_if_{i+1})\). Therefore \((s_i, f_i) = \ldots \)
exists an integer \( A \) nonzero ideals of \( \text{as required.} \)

1. We only need to establish the forward implication. To see this, assume

Proof. Let \( \text{Theorem 2.4.} \)

Remark 2.3. Theorem 2.1 above shows that if \( \text{a ring,} \) \( E \) is a nonzero \( A \)

Example 2.2. If \( \text{the field of real numbers,} \) \( \mathbb{R}[[X]] \) is the ring of formal power

series, and \( \mathbb{Z}_2 := \{ \frac{a}{b} : a, b \in \mathbb{Z}, 2 \nmid b \} \) the localization of \( \mathbb{Z} \) at the prime ideal \( 2\mathbb{Z} \)

then \( A := \mathbb{Z}_2 + X\mathbb{R}[[X]] \) is a local ring with maximal ideal \( M = 2\mathbb{Z}_2 + X\mathbb{R}[[X]]. \)

Let \( E \) be an \( A/M \)-vector space and \( R := A \propto E \) be the trivial ring extension of \( A \)

by \( E. \) According to [4, Example 2.8], \( A \) is not noetherian and \( (R, M \propto E) \) is local

such that \( ME = 0. \) By Theorem 2.1, \( R \) does not satisfy the \( ACC_d \)-condition.

Remark 2.3. Theorem 2.1 above shows that if \( A \) is a ring, \( E \) is a nonzero \( A \)-module, and \( R := A \propto E \) has the \( ACC_d \), then so is \( A. \) However, Example 2.2 shows

that the converse need not be true. Moreover, we will construct below an example to show that if \( E \) is not finitely generated, then statement (2b) of Theorem 2.1 need not be true. But first, the next result will be needed to construct the example.

Theorem 2.4. Let \( A \) be a ring. Then:

1. If \( (A, M) \) is a local ring with \( M^2 = 0, \) then \( A \) satisfies \( ACC_d \) if and only if

   \( A \) is noetherian.

2. If \( P \) is a prime ideal of \( A \) such that \( P^2 = 0, \) then, \( A_P \) satisfies \( ACC_d \) if and only if \( A_P \) is noetherian, where \( A_P \) is the localization of \( A \) with respect to the prime ideal \( P. \)

Proof. (1) We only need to establish the forward implication. To see this, assume

that \( A \) satisfies \( ACC_d \) and let \( I_1 \subseteq I_2 \subseteq I_3 \subseteq I_4 \subseteq \ldots \) be an ascending chain of nonzero ideals of \( A. \) We will show that the chain is stationary. By the \( ACC_d, \) there exists an integer \( k \in \mathbb{N} \) such that for each \( i \geq k, \) there is an element \( a_i \in A \) such that \( I_i = a_iI_{i+1}. \) We claim that \( a_i \notin M \) for each \( i \geq k. \) Otherwise, if \( a_i \in M \) for some \( i \geq k, \) then \( I_i = a_iI_{i+1} = 0, \) a clear contradiction since \( M^2 = 0. \) Now, since \( a_i \notin M \) for each \( i \geq k, \) it follows that each \( a_i \) is invertible, and so \( I_i = a_iI_{i+1} = I_{i+1}, \) as required.
If $P$ is a prime ideal of $A$ with $P^2 = 0$, then $(A_P, (PA_P))$ is a local ring with $(PA_P)^2 = 0$. Now an application of part (1) above will yield the result; i.e. $A_P$ satisfies $\text{ACC}_d$ if and only if $A_P$ is noetherian. \hfill \Box

**Example 2.5.** Let $A$ be an integral domain, $K$ the quotient field of $A$, $E$ an infinite dimensional $K$-vector space, and $R := A \otimes E$ the trivial ring extension of $A$ by $E$. We claim that $R$ does not satisfy the $\text{ACC}_d$-condition. For, since $P := 0 \otimes E$ is a prime ideal of $R$, and $\dim_K E = \infty$, we infer from Theorem 2.1 that $R_P := K \otimes E$ is not noetherian. Now, by Theorem 2.4, $R_P$ does not satisfy the $\text{ACC}_d$-condition, and so $R$ also does not satisfy the $\text{ACC}_d$-condition. In particular, the trivial extension $\mathbb{Z} \otimes \mathbb{R}$ does not satisfy the $\text{ACC}_d$-condition.

**Example 2.6.** Let $(A, M)$ be a local noetherian integral domain, $E$ an infinite dimensional $A/M$-vector space, and $R := A \otimes E$ be an integral domain, $1 \in M$ is a prime ideal, 1 \leq n \leq \infty$, and so $(1 \in M)^n$ does not satisfy the $\text{ACC}_d$-condition. This shows that if $E$ is not finitely generated, then statement (2b) of Theorem 2.1 need not be true.

In the next theorem we establish one of the main results of this paper. More precisely, we will show that if $A$ is a ring satisfying the $\text{ACC}_d$-condition, then $A$ satisfies the $\text{ACC}$ on prime ideals. But first, we need to prove a couple of lemmas.

**Lemma 2.7.** Let $A$ be an integral domain, $I$ a proper ideal of $A$ and $P_1, P_2, \ldots, P_n$ a set of prime ideals of $A$ such that $\bigcap_{i=1}^n P_i = xI$ where $x$ is a non-zero element of $A$. Then there exists a subset $J$ of $\{1, 2, \ldots, n\}$ such that $\bigcap_{i \in J} P_i = I$.

**Proof.** Observe first that $x \notin \bigcap_{i=1}^n P_i$. Otherwise, the equation $\bigcap_{i=1}^n P_i = xI$ implies the existence of a non-zero element $y$ of $I$ such that $x = xy$, a contradiction since $A$ is an integral domain and $I \neq A$. With this observation in mind, we consider two cases:

Case 1: Suppose that $x \notin \bigcup_{i=1}^n P_i$. Since $x \notin P_i$, $xI \subseteq P_i$ and $P_i$ is a prime ideal, $1 \leq i \leq n$, we infer that $I \subseteq P_i$, $1 \leq i \leq n$. Therefore $I \subseteq \bigcap_{i=1}^n P_i$, and so $I = \bigcap_{i=1}^n P_i$.

Case 2: Suppose that $x \in \bigcup_{i=1}^n P_i$, and let $J$ be a subset of $\{1, 2, \ldots, n\}$ such that $x \in \bigcap_{i \in J} P_i$ and $x \notin \bigcup_{i \notin J} P_i$. Now for each $i \notin J$, since $P_i$ is a prime ideal, $xI \subseteq P_i$, and $x \notin P_i$, we infer that $I \subseteq P_i$ for each $i \notin J$. Therefore $xI \subseteq xP_i$ for each $i \notin J$, and so $(xI) \bigcap \bigcap_{i \notin J} P_i = xI$. Moreover, since $x \in P_j$, for each $j \in J$, it follows that $xP_j \bigcap P_i = xP_i$ for each $j \in J$. Furthermore, we have $(\bigcap_{i=1}^n P_i) \bigcap (\bigcap_{i \notin J} xP_i) = xI \bigcap \bigcap_{i \notin J} (xP_i) = xI$, and so $(\bigcap_{i \notin J} P_i) \bigcap (\bigcap_{i \notin J} xP_i) = xI$. Therefore, $(\bigcap_{i \notin J} P_i) \bigcap (\bigcap_{i \notin J} xP_i) = xI$. Since $xP_i \bigcap P_j = xP_i$
for each $j \in J$, it follows that $\bigcap_{i \notin J} xP_i = xI$, and so $x(\bigcap_{i \notin J} P_i) = xI$. Inasmuch as $A$ is an integral domain, we infer that $\bigcap_{i \notin J} P_i = I$, as required. □

**Lemma 2.8.** If $A$ is an integral domain satisfying the $\text{ACC}_d$-condition, then the following hold:

1. $A$ satisfies the $\text{ACC}$ on ideals each of which is an intersection of a finite number of prime ideals. In particular, $A$ satisfies the $\text{ACC}$ on prime ideals.

2. If whenever $I$ and $J$ are ideals of $A$, there exists a set of prime ideals $P_1, P_2, \ldots, P_n$ of $A$ such that $I \subseteq \bigcap_{i=1}^n P_i \subseteq J$, then $A$ is noetherian.

**Proof.** (1) Let $I_1 \subseteq I_2 \subseteq I_3 \subseteq I_4 \subseteq \ldots$ be an ascending chain of non-zero ideals of $A$ such that, for each $k \in \mathbb{N}$, $I_k = \bigcap_{i \in J_k} P_i$, an intersection of a finite number of prime ideals $P_i$. Since $A$ satisfies the $\text{ACC}_d$-condition, there exists an integer $k$ such that for each $n > k$, there exists an element $a_n$ of $A$ satisfying the relation $I_k = a_n I_n$. Thus, $\bigcap_{i \in J_k} P_i = a_n(\bigcap_{i \in J_n} P_i)$. By Lemma 2.7 above, there exists a subset $S_k \subseteq J_k$ such that $\bigcap_{i \in S_k} P_i = \bigcap_{i \in J_k} P_i = I_n$. Inasmuch as the sequence $\{|S_k|\}$ is bounded and decreasing, it is convergent, where $|S|$ denotes the number of elements of the set $S$. Therefore, the sequence $\{|S_k|\}$ is stationary. Let $K \in \mathbb{N}$ such that for each $i \geq K$, $S_i = S_K$. Since $I_K \subseteq I_i$, it follows that $I_K = I_i$ for each $i \geq K$. This shows that the chain $I_1 \subseteq I_2 \subseteq I_3 \subseteq I_4 \subseteq \ldots$ is stationary, as required.

(2) This claim follows easily from (1). □

**Theorem 2.9.** Let $A$ be a ring. If $A$ satisfies $\text{ACC}_d$, then $A$ satisfies $\text{ACC}$ on prime ideals.

**Proof.** We consider two cases:

Case 1: $A$ is an integral domain. In this case, apply Lemma 2.8.

Case 2: $A$ is not an integral domain. We need to show that $A$ satisfies the $\text{ACC}$ on prime ideals. To see this, let $P_1 \subseteq P_2 \subseteq P_3 \subseteq P_4 \subseteq P_5 \subseteq \ldots$ be an ascending chain of proper prime ideals of $A$. By [5, Proposition 2.3], since $P_1$ is a proper prime ideal of $A$, $A/P_1$ is an integral domain with the $\text{ACC}_d$-condition. Now, by Lemma 2.8, the ascending chain $P_2/P_1 \subseteq P_3/P_1 \subseteq P_4/P_1 \subseteq P_5/P_1 \subseteq \ldots$ is stationary, and so is the chain $P_1 \subseteq P_2 \subseteq P_3 \subseteq P_4 \subseteq P_5 \subseteq \ldots$, as required. □

The following example shows that the converse to Theorem 2.9 need not be true.

**Example 2.10.** Let $K$ be a field, $E$ an infinite dimensional $K$-vector space, and $R = K \otimes E$. Then $R$ satisfies the $\text{ACC}$ on prime ideals but $R$ does not satisfy the $\text{ACC}_d$-condition.
In the next result we highlight some of the interesting features of the rings that
satisfy the $ACC_d$-condition. But first, we need the following lemma.

**Lemma 2.11.** Let $A$ be a ring satisfying the $ACC_d$-condition and $I_1 \subseteq I_2 \subseteq I_3 \subseteq \ldots$ be a non-stationary ascending chain of ideals of $A$. Then for each $n \geq 1$, $I_n$ is strictly contained in a proper principal ideal of $A$.

**Proof.** Since $A$ satisfies the $ACC_d$-condition, there exists an integer $k$ such that for each $i \geq k$, $I_i = x_i I_{i+1}$ for some $x_i \in A$. Let $n$ be a nonzero integer, and consider the following two cases:

Case 1: If $n \geq k$, then $I_n = x_n I_{n+1} \subseteq x_n A$. As the chain is non-stationary, $x_n$ is not invertible, and so $I_n$ is properly contained in the principal ideal $x_n A$.

Case 2: If $n < k$, then we have $I_n \subseteq I_k \subseteq x_k A$, where $x_k A$ is a proper principal ideal by case 1 above. Thus $I_n$ is strictly contained in $x_k A$. □

**Theorem 2.12.** Let $A$ be a ring with the $ACC_d$-condition and $I$ be a proper ideal of $A$. Then:

1. Either $I$ is strictly contained in a proper principal ideal of $A$ or $A/I$ is noetherian.
2. If $\text{Nil}(A)$ is finitely generated that is not strictly contained in any proper principal ideal of $A$, then $A$ is noetherian.

**Proof.** (1) If every ascending chain of ideals containing $I$ is stationary, then $A/I$ is noetherian. Otherwise, there exists a non-stationary ascending chain of ideals of $A$ containing $I$. By Lemma 2.11, $I$ is strictly contained in a proper principal ideal of $A$.

(2) By (1), $A/\text{Nil}(A)$ is noetherian, and so every ideal of $A/\text{Nil}(A)$ is finitely generated. In particular, $P/\text{Nil}(A)$ is finitely generated, where $P$ is a prime ideal of $A$. Inasmuch as $\text{Nil}(A)$ is finitely generated, we infer that $P$ is finitely generated. This means that all prime ideals of $A$ are finitely generated, and so $A$ is noetherian. □

**Corollary 2.13.** Let $A$ be a ring with the $ACC_d$-condition and $I$ be an ideal of $A$ that is not strictly contained in any proper principal ideal of $A$. If every ideal contained in $I$ is finitely generated, then $A$ is noetherian.

**Proof.** Assume that $I$ is not strictly contained in any proper principal ideal. Then according to (1) of Theorem 2.12, $A/I$ is noetherian. Furthermore, every ideal contained in $I$ is finitely generated. So, $I$ is noetherian as an $A$-module. Since both $A/I$ and $I$ are noetherian as $A$-modules, $A$ is noetherian. □
In [2], a ring $A$ is called $Q$-noetherian if $A/P$ is a noetherian domain for every prime ideal $P$ of $A$.

**Corollary 2.14.** Let $A$ be a ring such that $\text{Nil}(A)$ is not strictly contained in any proper principal ideal of $A$. If $A$ satisfies the $\text{ACC}_d$-condition, then $A$ is $Q$-noetherian.

**Proof.** By Theorem 2.12, $A/\text{Nil}(A)$ is noetherian. Now if $P$ is a prime ideal of $A$, then $A/P$ is noetherian, and so $A$ is $Q$-noetherian. \hfill \Box

Finally, we end the paper with examples distinguishing the notion of a coherent ring from that of a ring with the $\text{ACC}_d$-condition. Recall first, a ring $R$ is called (left) coherent if every finitely generated (left) ideal of $R$ is finitely presented. Clearly every (left) noetherian ring is (left) coherent, but the converse need not be true in general. We will show in the next two examples that neither coherent implies $\text{ACC}_d$ nor $\text{ACC}_d$ implies coherent.

**Example 2.15.** Consider the ring extension $R = \mathbb{Z}_{(p)} \times \mathbb{Z}_{p^\infty}$, where $p$ is a prime number, $\mathbb{Z}_{(p)}$ is the ring of $p$-adic integers, and $\mathbb{Z}_{p^\infty}$ is the Pr"ufer $p$-group. As shown in [5, Example 2.2 (b)], $R$ satisfies the $\text{ACC}_d$-condition, and as shown in [9], $R$ is not a coherent ring.

**Example 2.16.** Let $(A, M)$ be a non noetherian local ring such that $M^2 = 0$ and $M$ is finitely generated. Then, by Theorem 2.4, $A$ does not satisfy the $\text{ACC}_d$-condition. However, since $M$ is finitely generated, $A$ is a coherent ring.

Next, we provide an example of a ring $R$ with $\text{ACC}$ on prime ideals that is not coherent.

**Example 2.17.** Let $K$ be a field, $E$ an infinite dimensional $K$-vector space, and $R = K \otimes E$. By Example 2.10, $R$ satisfies the $\text{ACC}$ on prime ideals. However, it is not difficult to see that $R$ is not coherent.

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