AN APPROACH TO THE FAITH-MENAL CONJECTURE

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Abstract. The Faith-Menal conjecture is one of the three main open conjectures on QF rings. It says that every right noetherian and left FP-injective ring is QF. In this paper, it is proved that the conjecture is true if every nonzero complement left ideal of the ring $R$ is not small (or not singular). Several known results are then obtained as corollaries.

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1. Introduction

Throughout this paper rings are associative with identity. For a subset $X$ of a ring $R$, the left annihilator of $X$ in $R$ is $I(X) = \{r \in R : rx = 0 \text{ for all } x \in X\}$. For any $a \in R$, we write $I(a)$ for $I(\{a\})$. Right annihilators are defined analogously. We write $J$, $Z_l$, $Z_r$, $S_l$ and $S_r$ for the Jacobson radical, the left singular ideal, the right singular ideal, the left socle and the right socle of $R$, respectively. $I \subseteq_{ess} R_R$ means that $I$ is an essential right ideal of $R$. $f = c \cdot$ (resp., $f = \cdot c$) means that $f$ is a left (resp., right) multiplication map by the element $c \in R$.

Recall that a ring $R$ is quasi-Frobenius (QF) if $R$ is one-sided noetherian and one-sided self-injective. There are three famous conjectures on QF rings (see [11]). One of them is the Faith-Menal conjecture, which was raised by Faith and Menal in [4]. A ring $R$ is called right Johns if $R$ is right noetherian and every right ideal of $R$ is a right annihilator. $R$ is called strongly right Johns if the matrix ring $M_n(R)$ is right Johns for all $n \geq 1$. In [7], Johns used a false result of Kurshan [8, Theorem 3.3] to show that right Johns rings are right artinian. In [3], Faith and Menal gave a counter example to show that right Johns rings may not be right artinian. They characterized strongly right Johns rings as right noetherian and left FP-injective rings (see [4, Theorem 1.1]). Recall that a ring $R$ is called left FP-injective if every $R$-homomorphism from a submodule $N$ of a free left $R$-module $F$ to $R_R$ can be

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In this paper, we apply McCoy’s Lemma (see Lemma 2.1) to show that Faith-Menal conjecture is true if every nonzero complement left ideal of $R$ is not small (or not singular). A left ideal $I$ of $R$ is said to be a complement if it is maximal with respect to the property that $I \cap K = 0$ for some left ideal $K$ of $R$. A left ideal $I$ of $R$ is small if, for any proper left ideal $K$ of $R$, $I + K \neq R$.

2. Results

Let $R$ be a ring. An element $a \in R$ is called regular if there exists an element $b \in R$ such that $a = aba$. $R$ is called regular if every element of $R$ is regular.

Lemma 2.1. (McCoy’s Lemma) Let $R$ be a ring and $a, c \in R$. If $b = a - aca$ is a regular element of $R$, then $a$ is also a regular element of $R$.

Proof. This follows easily from the definition.

A ring $R$ is called left $P$-injective if every homomorphism from a principal left ideal $Rt$ to $R$ can be extended to one from $R$ to $R$.

Proposition 2.2. Suppose $R$ is a right noetherian and left $P$-injective ring. Then $J = Z_l$ is a nilpotent right annihilator, and $I(J)$ is essential both as a left and as a right ideal of $R$.

Proof. Since $R$ is left $P$-injective, $J = Z_l$ (see [11, Theorem 5.14]). By [5, Theorem 2.7], $J$ is nilpotent and $I(J)$ is essential both as a left and as a right ideal of $R$. Since $I(J)$ is an essential left ideal of $R$, $rl(J) \subseteq Z_l = J$. Thus $J = rl(J)$ is a right annihilator.

Lemma 2.3. [6, Lemma 5.8] If $R$ is a semiprime right Goldie ring, then $R$ has DCC on right annihilators.

Lemma 2.4. Suppose $R$ is a right noetherian and left $P$-injective ring such that every nonzero complement left ideal of $R$ is not small (or not singular). Then $R$ is right artinian.

Proof. First, we prove that $\overline{R} = R/J$ is a regular ring. Assume $a \notin J$. Then, since $J = rl(J) = Z_l$ by Proposition 2.2, there exists a nonzero complement left ideal $I$ of $R$ such that $l(a) \cap I = 0$. We claim that there exists some $b \in I$ such that $ba \notin J$. If not, then $Ia \subseteq J$. This implies that $l(J)Ia = 0$. Since $l(a) \cap I = 0$, $l(J)I \subseteq l(a) \cap I = 0$. Thus $I \subseteq rl(J) = J = Z_l$. So $I$ is a small and singular left ideal of $R$. This is a contradiction. Since $R$ is left $P$-injective, every homomorphism from
Rba to $R R$ is a right multiplication map by some $c \in R$. Define $f : Rba \to R$ by $f(rba) = rb$ for all $r \in R$. Then $f$ is well-defined and there exists $0 \neq c \in R$ such that $f = c$. So $b = bac$. This implies that $\overline{b} \in l_{R}(\overline{a} - \overline{aca})$, where $\overline{r}$ denotes $r + J$ in $R/J$ for any $r \in R$. Since $\overline{ba} \neq 0$, $l_{R}(\overline{a})$ is properly contained in $l_{R}(\overline{a} - \overline{aca})$. If $a - aca \in J$, then $a$ is a regular element of $R$. If not, let $a_1 = a - aca$. Then $l_{R}(a_1)$ is not essential in $R R$. In the same way, we get $a_2 = a_1 - a_1 c_1 a_1$ for some $c_1 \in R$ and $l_{R}(\overline{a_2})$ is properly contained in $l_{R}(\overline{a_1})$. If $a_2 \in J$, then $a_1$ is a regular element of $R$. Thus, by Lemma 2.1, $\overline{a}$ is a regular element of $R$. If $a_2 \notin J$, we have $a_3 = a_2 - a_2 c_2 a_2$ for some $c_2 \in R$ and $l_{R}(\overline{a_2})$ is properly contained in $l_{R}(\overline{a_3})$. Therefore we have such $a_k \in R$ step by step, $k = 1, 2, \cdots$. Since $R$ is right noetherian and $J(R) = 0$, $R$ is a semiprime and right Goldie ring. By Lemma 2.3, $R$ satisfies ACC on left annihilators. So there exists some positive integer $m$ such that $a_m \in J$ and $a_k = a_{k-1} - a_{k-1} c_{k-1} a_{k-1}$ for some $c_{k-1} \in R$, $k = 2, 3, \cdots, m$. By Lemma 2.1, $\overline{a}$ is a regular element of $R$. Since $\overline{a}$ is an arbitrary nonzero element of $R$, $R$ is a regular ring. Then $R$ is semisimple because $R$ is right noetherian. Moreover, by Proposition 2.2, $J$ is nilpotent and so $R$ is semiprimary. Thus $R$ is right artinian. \hfill $\square$

A ring $R$ is called left 2-injective if every left $R$-homomorphism from a 2-generated left ideal $I$ of $R$ to $R R$ can be extended to one from $R R$ to $R R$. It is clear that every left 2-injective ring is left $P$-injective, but the converse is not true (see [11, Example 2.5]).

Lemma 2.5. [12, Corollary 3] If $R$ is a left 2-injective ring satisfying ACC on left annihilators, then $R$ is QF.

Theorem 2.6. $R$ is QF if and only if $R$ is right noetherian, left 2-injective and every nonzero complement left ideal of $R$ is not small (or not singular).

Proof. “$\Rightarrow$”. This is obvious.

“$\Leftarrow$”. By Lemma 2.4, $R$ is right artinian. So $R$ has ACC on left annihilators. Thus Lemma 2.5 implies that $R$ is QF. \hfill $\square$

Since every left $FP$-injective ring is left 2-injective, we have

Corollary 2.7. The Faith-Menal conjecture is true if every nonzero complement left ideal of $R$ is not small (or not singular).
nor singular. But the converse is not true (see Example 2.11). $R$ is called left $C2$ if every left ideal that is isomorphic to a direct summand of $R R$ is also a direct summand of $R R$. Every left $P$-injective ring is left $C2$ (see [11, Proposition 5.10]). A left $CS$ and left $C2$ ring is called a left continuous ring.

**Theorem 2.8.** Suppose $R$ is a right noetherian, left $P$-injective, and left min-CS ring such that every nonzero complement left ideal is not small (or not singular). Then $R$ is QF.

**Proof.** By Lemma 2.4, $R$ is right artinian. So $R$ is a left GPF ring (i.e., $R$ is left $P$-injective, semiperfect, and $S_l \subseteq \text{ess} R R$). Then $R$ is left Kasch and $S_r = S_l$ by [11, Theorem 5.31]. Thus $\text{Soc}(Re)$ is simple for each local idempotent $e$ of $R$ (see [11, Lemma 4.5]). By [11, Lemma 3.1] the dual, $(eR/eJ)^* \cong l(J)e = S_r e = S_l e = \text{Soc}(Re)$ is simple for every local idempotent $e$ of $R$. Since $R$ is semiperfect, each simple right $R$-module is isomorphic to $eR/eJ$ for some local idempotent $e$ of $R$ (see [1, Theorem 27.10]). Thus $R$ is right mininjective by [11, Theorem 2.29]. Since a left $P$-injective ring is left mininjective, $R$ is a two-sided mininjective and right artinian ring. So $R$ is QF by [13, Theorem 2.5]. □

**Corollary 2.9.** [2, Theorem 2.21] If $R$ is right noetherian, left CS and left $P$-injective, then $R$ is QF.

**Corollary 2.10.** [10, Theorem 3.2] The following are equivalent for a ring $R$.

1. $R$ is QF.
2. $R$ is a right Johns and left CS ring.

**Example 2.11.** There is a left min-CS ring $S$ satisfying every nonzero complement left ideal of $S$ is neither small nor singular. But it is not left CS.

**Proof.** Let $k$ be a division ring and $V_k$ be a right $k$-vector space of infinite dimension. Take $R = \text{End}(V_k)$, then $R$ is regular but not left self-injective (see [9, Example 3.74B]). Let $S = M_{2 \times 2}(R)$, then $S$ is also regular. This implies $S$ is left $P$-injective and $J(S) = Z_l(S) = 0$. Thus $S$ is left $C2$ and every nonzero complement left ideal of $S$ is neither small nor singular. And every minimal left ideal of $S$ is a direct summand of $S S$. So $S$ is left min-CS. But $S$ is not left CS. For if $S$ is left CS, then $S$ is left continuous. Thus $R$ is left self-injective by [11, Theorem 1.35]. This is a contradiction. □

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References


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