International Electronic Journal of Algebra
Volume 1 (2007) 51–54

A NOTE ON ABELIAN IDEALS OF A BOREL SUBALGEBRA

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Received: 9 October 2006; Revised: 18 December 2006

Communicated by A. Ç. Özcan

Abstract. In this paper we will construct a specific subset of a positive root system of a simple Lie algebra, and will show that our subset can yield an abelian ideal of a Borel subalgebra.

Mathematics Subject Classification (2000): 17B05
Keywords: Complex simple Lie algebra, affine root system, compatible set, abelian ideal.

1. Introduction

Let $\mathfrak{g}$ be a finite-dimensional complex simple Lie algebra with rank l, and $\mathfrak{h}$ a fixed Cartan subalgebra of $\mathfrak{g}$. Then we can have the root space decomposition $\mathfrak{g} = \mathfrak{h} \oplus (\bigoplus_{\alpha \in \Phi} \mathfrak{g}_\alpha)$ of $\mathfrak{g}$. We will denote by Φ^+ the set of positive roots in Φ, and will use the notation Δ to denote the basis in Φ^+. It is a basic fact in simple Lie algebra theory that $\mathfrak{b} = \mathfrak{h} \oplus (\bigoplus_{\alpha \in \Phi^+} \mathfrak{g}_\alpha)$ is a Borel subalgebra of $\mathfrak{g}$. Conversely, every Borel subalgebra containing the given Cartan subalgebra $\mathfrak{h}$ determines its corresponding positive root system in Φ. For a given *ad*-nilpotent ideal $\mathfrak{a}$ of the Borel subalgebra $\mathfrak{b}$, the ideal $\mathfrak{a}$ is stable under the $ad(\mathfrak{h})$-action, and it lies in $[\mathfrak{b}, \mathfrak{b}]$. Thus $\mathfrak{a}$ can have the decomposition $\mathfrak{a} = \bigoplus_{\mathfrak{a} \in \Psi} \mathfrak{g}_\alpha$ for some subset Ψ of Φ^+. Actually we can characterize *ad*-nilpotent ideals of the Borel subalgebra $\mathfrak{b}$ in terms of the subsets of Φ^+. Let us consider the collection of subsets of Φ^+:

$$S = \{\Psi \mid \Psi \subset \Phi^+ \text{ and } \Phi^+ \cap (\Psi + \Phi^+) \subset \Psi\}.$$

Then it is easy to check that every *ad*-nilpotent ideal of $\mathfrak{b}$ can be expressed by $\bigoplus_{\alpha \in \Psi} \mathfrak{g}_\alpha$ for some $\Psi \in S$. In [2], the authors defined the specific set of positive affine roots, which is called a *compatible set*, and they characterized abelian ideals of $\mathfrak{b}$ using a compatible set. In this paper, we will construct one subset of a positive root system Φ^+ of a simple Lie algebra $\mathfrak{g}$, and will apply Cellini and Papi's characterization to show that our subset of Φ^+ can yield an abelian ideal of $\mathfrak{b}$.

2. Affine root system

Throughout this paper, we will fix a complex simple Lie algebra $\mathfrak{g}$ of rank l, its Cartan subalgebra $\mathfrak{h}$ and Borel subalgebra $\mathfrak{b}$ containing $\mathfrak{h}$. For a given root system Φ of the simple Lie algebra $\mathfrak{g}$, let A be the corresponding fundamental alcove $A = \{\lambda \in {\mathfrak{h}_{\mathbb{R}}}^* \mid 0 \leq (\lambda, \alpha) \leq 1 \text{ for all } \alpha \in \Phi^+\}$.

Now suppose that $\mathfrak{a}$ is an *ad*-nilpotent ideal of $\mathfrak{b}$. Then an ideal $\mathfrak{a}$ of $\mathfrak{b}$ is *ad*-nilpotent if and only if $\mathfrak{a}$ is contained in $[\mathfrak{b}, \mathfrak{b}]$. Hence abelian ideals of $\mathfrak{b}$ are automatically *ad*-nilpotent ideals of $\mathfrak{b}$.

Let $\mathfrak{h}_{\mathbb{R}}$ be the real form of the Cartan subalgebra $\mathfrak{h}$, and $\mathfrak{h}_{\mathbb{R}}^*$ be the dual space of $\mathfrak{h}_{\mathbb{R}}$. Recall that $\mathfrak{h}_{\mathbb{R}}^*$ can be equipped with a positive-definite symmetric bilinear form $(\ ,\ )$ through the Killing form κ on $\mathfrak{h}_{\mathbb{R}}$. Now let W and $\widehat{W}$ be the Weyl group and the affine Weyl group over real vector space $\mathfrak{h}_{\mathbb{R}}^*$, respectively. For each $\alpha \in \Phi$ and each integer k, we define the hyperplane $H_{\alpha,k} = \{\lambda \in \mathfrak{h}_{\mathbb{R}}^* \mid (\lambda, \alpha) = k\}$. Then we can define the corresponding affine reflection

$$s_{\alpha,k}(\lambda) = \lambda - ((\lambda, \alpha) - k)\,\frac{2\alpha}{(\alpha, \alpha)}.$$

Recall that the affine Weyl group $\widehat{W}$ is defined by the group generated by all affine reflections $s_{\alpha,k}$, where $\alpha \in \Phi$ and $k \in \mathbb{Z}$. However there is another way to define the affine Weyl group due to V. Kac (See [4]). First we extend $\mathfrak{h}_{\mathbb{R}}^*$ to $\mathfrak{h}_{\mathbb{R}}^* \oplus \mathbb{R}\delta \oplus \mathbb{R}\Lambda_0$ by defining $(\delta, \delta) = (\delta, {\mathfrak{h}^*}_{\mathbb{R}}) = (\Lambda_0, {\mathfrak{h}^*}_{\mathbb{R}}) = (\Lambda_0, \Lambda_0) = 0$ and $(\delta, \Lambda_0) = 1$. The affine root system $\widehat{\Phi}$ corresponding to Φ is defined as $\Phi + \mathbb{Z}\delta = \{\alpha + k\delta \mid \alpha \in \Phi \text{ and } k \in \mathbb{Z}\}$. The set of positive affine roots is $\widehat{\Phi}^+ = (\Phi^+ + \mathbb{Z}_{\geq 0}\delta) \cup (-\Phi^+ + \mathbb{Z}_{>0}\delta)$. Let θ be the highest root of Φ. Since $\mathfrak{g}$ is a simple Lie algebra, Φ is an irreducible root system and there exits a unique highest root θ. Let $\alpha_0 = -\theta + \delta$ and $\widehat{\Delta} = \{\alpha_0, \cdots, \alpha_n\}$. Then for each $\alpha \in \widehat{\Phi}^+$, we define the corresponding reflection of $\mathfrak{h}_{\mathbb{R}}^* \oplus \mathbb{R}\delta \oplus \mathbb{R}\Lambda_0$ by $s_\alpha(\lambda) = \lambda - \frac{2(\alpha,\lambda)}{(\alpha,\alpha)}\alpha$ for $\lambda \in \mathfrak{h}_{\mathbb{R}}^* \oplus \mathbb{R}\delta \oplus \mathbb{R}\Lambda_0$. Now we define the affine Weyl group associated with Φ as the group W_{aff} generated by $\{s_\alpha \mid \alpha \in \widehat{\Delta}\}$. If we consider the projection $\pi : \mathfrak{h}_{\mathbb{R}}^* \oplus \mathbb{R}\delta + \Lambda_0 \longrightarrow {\mathfrak{h}_{\mathbb{R}}}^*$, then the map $w\mid_V \longrightarrow \pi \circ (w\mid_V)$ yields an isomorphism between W_{aff} and $\widehat{W}$, where $V = \mathfrak{h}_{\mathbb{R}}^* \oplus \mathbb{R}\delta + \Lambda_0$.

It is known that for given $\alpha \in \Delta^+$, $m \in \mathbb{Z}_{>0}$ and $n \in \mathbb{Z}_{\geq 0}$,

$$w^{-1}(-\alpha + m\delta) \in \widehat{\Phi}^- \text{ if and only if } H_{\alpha,m} \text{ separates } A \text{ and } w(A), \tag{2.1}$$

$$w^{-1}(\alpha + n\delta) \in \widehat{\Phi}^- \text{ if and only if } H_{\alpha,-n} \text{ separates } A \text{ and } w(A), \tag{2.2}$$

where $\widehat{\Phi}^- = -\widehat{\Phi}^+$ [2].

Now let us give the following fundamental definition.

Definition 2.1. [2] We call a subset L of $\widehat{\Phi}^+$ a *compatible subset* if it satisfies the following two conditions:

(1) $\lambda \in L, \mu \in L$ and $\lambda + \mu \in \widehat{\Phi}^+$ imply that $\lambda + \mu \in L$,
(2) $\lambda + \mu \in L, \lambda \in \widehat{\Phi}^+$ and $\mu \in \widehat{\Phi}^+$ imply that $\lambda \in L$ or $\mu \in L$.

Then we have the following theorem which connects the two concepts of compatible subsets and affine root systems.

Theorem 2.2. [6] *Suppose that $\widehat{\Phi}$ is an affine root system which is not $\widetilde{A_1}$-type. Then a finite subset L of $\widehat{\Phi}^+$ is compatible if and only if L is of the form $N(\widehat{w}) = \{\alpha \in \widehat{\Phi}^+ \mid w^{-1}(\alpha) \in \widehat{\Phi}^-\}$ for some $\widehat{w} \in \widehat{W}$.*

Now we can characterize the abelian ideals of $\mathfrak{b}$ in terms of a compatible set.

Theorem 2.3. [2] *The following are equivalent for $\Psi \subset \widehat{\Phi}^+$:*

(1) $\bigoplus_{\alpha\in\Psi} \mathfrak{g}_\alpha$ *is an abelian ideal.*
(2) $\{-\alpha + \delta \mid \alpha \in \Psi\}$ *is compatible.*

Proof. See [2, Proposition 2.8]. □

3. Main results

Let us start with the following key theorem.

Theorem 3.1. *Let $\widehat{w} \in \widehat{W}$ and $\widehat{w}(A) \subset 2A$, and let*

$$\Psi = \{\alpha \in \Phi^+ \mid \text{ the number of hyperplanes } H_{\alpha,k} \text{ separating } A \text{ and } \widehat{w}(A) \text{ is equal to } 1\}.$$

Then $\{-\alpha + \delta \mid \alpha \in \Psi\} = N(\widehat{w})$, where $N(\widehat{w}) = \{\alpha \in \widehat{\Phi}^+ \mid w^{-1}(\alpha) \in \widehat{\Phi}^-\}$.

Proof. For given $\widehat{w} \in \widehat{W}$ and $\widehat{w}(A) \subset 2A$, if there is exactly one hyperplane $H_{\alpha,k}$ separating A and $\widehat{w}(A)$, then the hyperplane $H_{\alpha,1}$ separates A and $\widehat{w}(A)$. This implies that $\widehat{w}^{-1}(-\alpha + \delta) \in \widehat{\Phi}^-$. Then we have that $-\alpha + \delta \in N(\widehat{w})$.

On the other hand, recall that $\widehat{\Phi}^+ = (\Phi^+ + \mathbb{Z}_{\geq 0}\delta) \cup (\Phi^- + \mathbb{Z}_{>0}\delta)$. If the element $\beta + k\delta \in N(\widehat{w})$ with the properties $\beta \in \Phi^+$ and $k \in \mathbb{Z}_{\geq 0}$ satisfies $\widehat{w}^{-1}(\beta + k\delta) \in \widehat{\Phi}^-$, then the hyperplane $H_{\beta,-k}$ separates A and $\widehat{w}(A)$, which contradicts the fact that both A and $\widehat{w}(A)$ belong to $2A$. So if a positive affine root in $\widehat{\Phi}^+$ is an element of $N(\widehat{w})$, then it can be written in the form $-\beta + k\delta$, where $\beta \in \Phi^+$ and $k \in \mathbb{Z}_{>0}$. From the fact that $\widehat{w}^{-1}(-\beta + k\delta) \in \widehat{\Phi}^-$, we know that the hyperplane $H_{\beta,k}$ separates A and $\widehat{w}(A)$. The possible value of k is 1 for the hyperpalnes $H_{\beta,k}$, because both A and $\widehat{w}(A)$ lie in $2A$. This implies that the affine root $-\beta + k\delta$ is actually of the form $-\beta+\delta$, and the number of hyperplanes $H_{\beta,k}$ separating A and $\widehat{w}(A)$ is equal to 1. $\square$

Corollary 3.2. *Let $\widehat{w} \in \widehat{W}$, $\widehat{w}(A) \subset 2A$. Let*

$$\Psi = \{\alpha \in \Phi^+ \mid \textit{ the number of hyperplanes } H_{\alpha,k} \textit{ separating } A \textit{ and } \widehat{w}(A) \textit{ is equal to } 1\}.$$

Then $\bigoplus_{\alpha\in\Psi} \mathfrak{g}_\alpha$ is an abelian ideal of the Borel subalgebra $\mathfrak{b}$.

Proof. A simple application of the Theorem 2.2, Theorem 2.3, and the Theorem 3.1. $\square$

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