ON WEAKLY REGULAR RINGS AND GENERALIZATIONS OF V-RINGS

Tikaram Subedi and A. M. Buhphang

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Abstract. In this paper, we have studied weakly regular rings and some generalizations of V-rings via GW-ideals. We have shown that: (1) If \( R \) is a left weakly regular ring whose maximal left (right) ideals are GW-ideals, then \( R \) is strongly regular; (2) If \( R \) is a right weakly regular ring whose maximal essential left ideals are GW-ideals, then \( R \) is ELT regular; (3) If \( R \) is a ring in which \( l(a) \) is a GW-ideal for all \( a \in R \), then \( R \) is left weakly regular if and only if \( R \) is right weakly regular; (4) A ring \( R \) is strongly regular if and only if \( R \) is a ZI left GP-V\(^{\prime}\)-ring whose maximal essential left (right) ideals are GW-ideals; (5) If \( R \) is a left (right) GP-V-ring such that \( l(a) \) is a GW-ideal for all \( a \in R \), then \( R \) is weakly regular.

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1. Introduction

Throughout this paper, \( R \) denotes an associative ring with identity and all our modules are unitary. The symbols \( J(R) \), \( Z(RR) \) and \( soc(RR) \) respectively stand for the Jacobson radical, left singular ideal and the left socle of \( R \). For any \( a \in R \), \( l(a) \) (\( r(a) \)) denotes the left (right) annihilator of \( a \). By an ideal, we mean a two sided ideal. As usual, a reduced ring is a ring without non-zero nilpotent elements. A ring \( R \) is a left (right) quasi duo ring if every maximal left (right) ideal of \( R \) is an ideal. A ring \( R \) is an ELT ring if every essential left ideal of \( R \) is an ideal. \( R \) is an MELT ring (MERT ring) if every maximal essential left (right) ideal of \( R \) is an ideal. The ring of \( 2 \times 2 \) matrices over a division ring is an MELT (MERT) ring but not left (right) quasi duo. \( R \) is (von Neumann) regular if for every \( a \in R \), there exists some \( b \in R \) such that \( a = aba \) and \( R \) is strongly regular if for every \( a \in R \), there exists some \( b \in R \) such that \( a = a^2b \). Over the last few decades regular rings and strongly regular rings are extensively studied (for example cf. \[1\]-\[12\]). Clearly, \( R \) is strongly regular if and only if \( R \) is a reduced regular ring. Following \[4\], \( R \)
is left (right) weakly regular if for every left (right) ideal \( I \) of \( R \), \( I = I^2 \) and \( R \) is weakly regular if it is both left and right weakly regular. Clearly, regular rings are weakly regular. But a weakly regular ring need not be regular (see [4]). A ring \( R \) is a left (right) V-ring if simple left (right) \( R \)-modules are injective ([5]). A left (right) \( R \)-module \( M \) is YJ-injective if for each \( 0 \neq a \in R \), there exists a positive integer \( n \) such that \( a^n \neq 0 \) and every left (right) \( R \)-homomorphism from \( Ra^n (a^n R) \) to \( M \) extends to a left (right) \( R \)-homomorphism from \( R \) to \( M \) ([2]). \( R \) is a left (right) GP-V-ring if every simple left (right) \( R \)-module is YJ-injective ([2]). \( R \) is a left (right) GP-V' ring if every simple singular left (right) \( R \)-module is YJ-injective ([2]). By ([6], Lemma 2), a regular ring is a left (right) GP-V-ring and hence a left (right) GP-V'-ring. But GP-V-rings are not necessarily regular (cf. [2]). \( R \) is a ZI ring if for every \( a \in R, b \in R, ab = 0 \) implies \( aRb = 0 \) ([2]). Clearly in a ZI ring \( R \), \( l(a) \) is an ideal for every \( a \in R \).

In this paper, some new properties of weakly regular rings, GP-V-rings and GP-V'-rings are given. Some of these results improve some known results over these classes of rings.

We first recall the following definition following [12].

**Definition 1.1.** A left ideal \( L \) of a ring \( R \) is a generalized weak ideal (GW-ideal) if for all \( a \in L \), there exists a positive integer \( n \) such that \( a^n R \subseteq L \). A right ideal \( K \) of \( R \) is defined similarly to be a GW-ideal.

**Example 1.2.** Let \( R = UT_2(\mathbb{Q}) \), the ring of upper triangular matrices over \( \mathbb{Q} \). Take \( L = \left\{ \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix} : a \in \mathbb{Q} \right\} \) and \( K = \left\{ \begin{pmatrix} 0 & 0 \\ 0 & a \end{pmatrix} : a \in \mathbb{Q} \right\} \), then it is easy to see that \( L \) is a left ideal, \( K \) is a right ideal of \( R \) but neither \( L \) nor \( K \) is a GW-ideal of \( R \).

**Example 1.3.** There exists a ring \( R \) in which every left (right) ideal of \( R \) is a GW-ideal, but not for every \( a \in R \), \( l(a) \) is an ideal.

**Proof.** Take \( R = \left\{ \begin{pmatrix} a & a_1 & a_2 & a_3 \\ 0 & a & a_4 & a_5 \\ 0 & 0 & a & a_6 \\ 0 & 0 & 0 & a \end{pmatrix} : a, a_i \in \mathbb{Z}_2, i = 1, 2, 3, 4, 5, 6 \right\} \). Then every non-unit of \( R \) is nilpotent and hence it follows that every left (right) ideal
of $R$ is a GW-ideal. Let $x = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$. Then it is easy to see that
\[
\begin{pmatrix} a & a_1 & a_2 & a_3 \\ 0 & a & a_4 & a_5 \\ 0 & 0 & a & a_6 \\ 0 & 0 & 0 & a \end{pmatrix} x = 0 \text{ if and only if } a = 0, a_4 = 0, a_1 = a_2. \quad \text{Thus}
\]
l$(x) = \begin{pmatrix} 0 & b_1 & b_2 \\ b_3 & 0 & 0 & b_4 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} : b_i \in \mathbb{Z}_2, i = 1, 2, 3, 4 \}$ \(= L \) (say).

Now $L$ is not an ideal as \[
\begin{pmatrix} 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \in L, \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \in R, \quad \text{but}
\]
\[
\begin{pmatrix} 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \in L, \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \notin L.
\]

**Example 1.4.** There exists a ring $R$ in which every right (left) ideal of $R$ is a GW-ideal, but not for every $a \in R$, $r(a)$ is an ideal.

**Proof.** Take $R = \begin{pmatrix} a & 0 & 0 & 0 \\ a_1 & a & 0 & 0 \\ a_2 & a_3 & a & 0 \\ a_4 & a_5 & a_6 & a \end{pmatrix} : a, a_i \in \mathbb{Z}_2, i = 1, 2, 3, 4, 5, 6 \}$. Here also every non-unit of $R$ is nilpotent so that every right (left) ideal of $R$ is a GW-ideal. Let $x = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 \end{pmatrix}$. Then it is easy to see that
\[
r(x) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ b_1 & 0 & 0 & 0 \\ b_1 & 0 & 0 & 0 \\ b_2 & b_3 & b_4 & 0 \end{pmatrix} : b_i \in \mathbb{Z}_2, i = 1, 2, 3, 4 \}$ \(= K \) (say).
Now \[
\begin{pmatrix}
0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{pmatrix}
\in K, \quad \begin{pmatrix}
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0
\end{pmatrix}
\in R
\] and
\[
\begin{pmatrix}
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0
\end{pmatrix}
\begin{pmatrix}
0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{pmatrix} = \begin{pmatrix}
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{pmatrix} \notin K.
\]
Hence \( K \) is not an ideal of \( R \).

For ease of reference, we quote the following results.

**Lemma 1.5.** ([4], Corollary 11) Let \( R \) be a reduced ring. Then \( R \) is left weakly regular if and only if \( R \) is right weakly regular.

**Lemma 1.6.** ([2], Lemma 1.1) Let \( R \) be a \( ZI \) right (left) \( GP-V' \)-ring, then \( R \) is a reduced ring.

From Lemma 4 (2) in [3], we have the following lemma.

**Lemma 1.7.** An ELT right weakly regular ring is regular.

Also from Lemma 7 in [7], the following lemma follows.

**Lemma 1.8.** If \( Z(RR) \neq 0 \), there exists some \( 0 \neq a \in Z(RR) \) such that \( a^2 = 0 \).

**Lemma 1.9.** ([5], Lemma 3.14) Let \( L \) be a left ideal of \( R \), then \( R/L \) is a flat left \( R \)-module if and only if for every \( a \in L \), there exists some \( b \in L \) such that \( a = ab \).

### 2. Weakly regular rings and GW-ideals

**Lemma 2.1.** Let \( R \) be a ring whose maximal left or right ideals are GW-ideals, then \( R/J(R) \) is reduced.

**Proof.** Let \( R \) be a ring whose maximal left ideals are GW-ideals. Suppose \( a^2 \in J(R) \) such that \( a \notin J(R) \), then \( a \notin M \) for some maximal left ideal \( M \) of \( R \). So \( M + Ra = R \) yielding \( Ma + Ra^2 = Ra \) and therefore \( xa + ya^2 = a \) for some \( x \in M, y \in R \). As \( M \) is a GW-ideal and \( x \in M \), there exists some positive integer \( n \) such that \( x^n a \in M \). We have,

\[
x^{n-1}a = x^{n-1}(xa + ya^2) = x^n a + x^{n-1}ya^2 \in M.
\]
Again since \( x^{n-1}a \in M \) and \( a^2 \in M \),
\[
x^{n-2}a = x^{n-2}(xa + ya^2) = x^{n-1}a + x^{n-2}ya^2 \in M.
\]
Continuing in this manner, we get \( a \in M \), a contradiction. Hence \( R/J(R) \) is reduced. We can similarly prove the lemma for a ring \( R \) whose maximal right ideals are GW-ideals.

**Corollary 2.2.** Let \( R \) be a semiprimitive ring whose maximal left or right ideals are GW-ideals, then \( R \) is reduced.

**Theorem 2.3.** Let \( R \) be a ring whose maximal left or right ideals are GW-ideals, then the following conditions are equivalent.

1. \( R \) is a strongly regular ring.
2. \( R \) is a left weakly regular ring.
3. \( R \) is a right weakly regular ring.

**Proof.** Let \( R \) be a ring whose maximal left ideals are GW-ideals. It is clear that (1) \( \Rightarrow \) (2) and (1) \( \Rightarrow \) (3).

(2) \( \Rightarrow \) (1). If \( a \in R \) such that \( l(a) + Ra \neq R \), let \( M \) be a maximal left ideal of \( R \) containing \( l(a) + Ra \). As \( R \) is left weakly regular, \( Ra = RaRa \) yielding \( a = \sum r_i a s_i a \) for some \( r_i \in R, s_i \in R \). Then \( 1 - \sum r_i a s_i \in l(a) \subseteq M \). Suppose \( \sum r_i a s_i \notin M \), then \( r_k a s_k \notin M \) for some \( k \) so that \( M + Rr_k a s_k = R \) which yields \( x + rrr_k a s_k = 1 \) for some \( x \in M, r \in R \). As \( M \) is a GW-ideal and \( s_k r r_k a \in M \), there exists a positive integer \( n \) such that \( (s_k r r_k a)^n s_k \in M \). Then
\[
(1 - x)^{n+1} = (rrr_k a s_k)^{n+1} = rrr_k a (s_k r r_k a)^n s_k \in M.
\]
This together with \( x \in M \) implies that \( 1 \in M \) which contradicts that \( M \neq R \). Thus \( \sum r_i a s_i \in M \) so that \( 1 \in M \) which is again a contradiction. Therefore \( l(a) + Ra = R \) for all \( a \in R \) and hence \( R \) is strongly regular.

(3) \( \Rightarrow \) (2). Since \( R \) is right weakly regular, it is semiprimitive and hence by hypothesis and Corollary 2.2, \( R \) is reduced so that by Lemma 1.5, \( R \) is left weakly regular. Similarly we can prove the result for a ring whose maximal right ideals are GW-ideals.

**Corollary 2.4.** ([5]) The following conditions are equivalent for a left (right) quasi duo ring \( R \).

1. \( R \) is a strongly regular ring.
2. \( R \) is a left weakly regular ring.
3. \( R \) is a right weakly regular ring.
Proposition 2.5. Let $R$ be a left (right) weakly regular ring whose maximal essential left ideals are GW-ideals, then $R$ is an ELT.

Proof. Let $R$ be a left weakly regular ring, then $R/soc(RR)$ is left weakly regular. Since every maximal essential left ideal of $R$ is a GW-ideal, it follows that every maximal left ideal of $R/soc(RR)$ is a GW-ideal. Thus by Theorem 2.3, $R/soc(RR)$ is strongly regular and hence is left duo. This implies that $R$ is an ELT ring.

The result can be proved similarly if $R$ is a right weakly regular ring whose maximal essential left ideals are GW-ideals.

Theorem 2.6. If $R$ is a right weakly regular ring whose maximal essential left ideals are GW-ideals, then $R$ is ELT regular.

Proof. By Proposition 2.5, $R$ is an ELT ring and so by hypothesis and lemma 1.7, $R$ is regular.

Corollary 2.7. A ring $R$ is ELT regular if and only if $R$ is an MELT right weakly regular ring.

Theorem 2.8. Let $R$ be a ring such that $l(a)$ is a GW-ideal for all $a \in R$, then $R$ is left weakly regular if and only if $R$ is right weakly regular.

Proof. Let $R$ be left weakly regular and $0 \neq a \in R$ such that $a^2 = 0$. Then $l(a) \neq R$ and so $l(a) \subseteq M$ for some maximal left ideal $M$ of $R$. As $R$ is left weakly regular, $Ra = RaRa$ and hence $a = \sum r_i a s_i$ for some $r_i \in R, s_i \in R$. This implies that $1 - \sum r_i a s_i \in l(a) \subseteq M$. If $\sum r_i a s_i \notin M$, then $r_k a s_k \notin M$ for some $k$ and so $M + r_k a s_k = R$ which implies that $x + r_k a s_k = 1$ for some $x \in M, r \in R$. Since $l(a)$ is a GW-ideal and $s_k r_k a \in l(a)$, there exists a positive integer $n$ such that $(s_k r_k a)^n s_k \in l(a) \subseteq M$. Arguing as in the proof of (2) $\Rightarrow$ (1) of Theorem 2.3, we get contradictions. So $R$ is reduced and thus by Lemma 1.5, $R$ is right weakly regular.

Conversely, assume that $R$ is right weakly regular. We prove that $R$ is reduced. Let $0 \neq a \in R$ such that $a^2 = 0$ and $K$ be a maximal right ideal of $R$ containing $r(a)$. Since $R$ is right weakly regular, $aR = aRaR$ which yields $a = \sum r_i a s_i$ for some $r_i \in R, s_i \in R$ so that $1 - \sum r_i a s_i \in r(a) \subseteq K$. If $\sum r_i a s_i \notin K$, then $r_k a s_k \notin K$ for some $k$. This implies that $K + r_k a s_k R = R$ yielding $x + r_k a s_k y = 1$ for some $x \in K, y \in R$. Now $l(a)$ is a GW-ideal and $s_k y r_k a \in l(a)$, so there exists some positive integer $n$ such that $(s_k y r_k a)^n s_k y r_k \in l(a)$. Hence $(1 - x)^{n+2} = (r_k a s_k y)^{n+2} = r_k a(s_k y r_k a)^n s_k y r_k a s_k y = 0$. Thus it follows that $1 \in K$ which is a contradiction.
The following conditions are equivalent for a ZI ring $R$.

1. $R$ is strongly regular.
2. $R$ is a left GP-V$^\prime$-ring whose maximal essential left ideals are GW-ideals.
3. $R$ is a left GP-V$^\prime$-ring whose maximal essential right ideals are GW-ideals.

Proof. (1) $\Rightarrow$ (2) and (1) $\Rightarrow$ (3) are well known.

(2) $\Rightarrow$ (1). $R$ is reduced by Lemma 1.6. If $l(a) + Ra \neq R$ for some $a \in R$, then it must be contained in a maximal left ideal $M$ of $R$. Since $R$ is a ZI ring, $M$ is an essential left ideal of $R$. Then $R/M$ is a simple singular left $R$-module and hence by hypothesis it is YJ-injective. Thus there exists a positive integer $n$ such that $a^n \neq 0$ and every left $R$-homomorphism from $Ra^n$ to $R/M$ extends to one from $R$ to $R/M$. Define $f : Ra^n \to R/M$ by $f(ra^n) = r + M$ for every $r \in R$. As $R$ is reduced, $l(a) = l(a^n)$. This yields $f$ is well-defined. It follows that there exists some $b \in R$ such that $1 - a^n b \in M$. Suppose $a^n b \notin M$, then $M + Ra^n b = R$ implying $x + ra^n b = 1$ for some $x \in M$, $r \in R$. Now, $M$ is a GW-ideal and $bra^n \in M$, so there exists a positive integer $k$ such that $(bra^n)^k b \in M$. Then

$$(1 - x)^{k+1} = (ra^n b)^{k+1} = ra^n (bra^n)^k b \in M,$$

so that $1 \in M$ which is a contradiction. So $a^n b \in M$ and then $1 \in M$, again a contradiction. Thus $l(a) + Ra = R$ for all $a \in R$ and $R$ is strongly regular.

(3) $\Rightarrow$ (1). By Lemma 1.6, $R$ is reduced so that $l(w) = r[w]$ for all $w \in R$. Suppose $l(a) + aR \neq R$ for some $a \in R$, then there exists a maximal right ideal $K$ of $R$ containing $l(a) + aR$. Suppose $RaR \notin K$, then $ras \notin K$ for some $r \in R$, $s \in R$ and so $K + rasR = R$ yielding $x + rast = 1$ for some $x \in K$, $t \in R$. Since $K$ is a GW-ideal and $astr \in K$, $r(ast r)^n \in K$ for some positive integer $n$. We therefore have

$$(1 - x)^{n+1} = (rast)^{n+1} = r(ast r)^n ast \in K,$$

whence $1 \in K$, a contradiction. Thus $RaR \subseteq K$ and hence $l(a) + RaR \subseteq L$ for some maximal left ideal $L$ of $R$. Since $R$ is a ZI ring, $L$ is an essential left ideal of $R$. Then $R/L$ is YJ-injective. Thus there exists a positive integer $m$ such that
If $R$ is strongly regular. Hence $l(a) + aR = R$ for all $a \in R$ proving that $R$ is regular. As $R$ is also reduced, $R$ is strongly regular.

Corollary 3.2. ([2], Theorem 2.3) If $R$ is a ZI ring, then the following statements are equivalent.

1. $R$ is a strongly regular ring.
2. $R$ is an MELT left GP-V-ring.
3. $R$ is an MERT right GP-V-ring.
4. $R$ is an MELT left GP-V'-ring.
5. $R$ is an MERT right GP-V'-ring.

Proposition 3.3. If $R$ is a ring such that every simple left $R$-module is either YJ-injective or flat and for every $a \in J(R)$, $l(a) = r(a)$, then $R$ is semiprimitive.

Proof. Suppose $0 \neq a \in J(R)$ such that $a^2 = 0$. Then $l(a) \neq R$ and so it is contained in a maximal left ideal $M$ of $R$. If $R/M$ is flat, then since $a \in l(a) \subseteq M$, by Lemma 1.9, there exists some $b \in M$ such that $a = ab$. Then $1 - b \in r(a) = l(a) \subseteq M$, whence $1 \in M$ which contradicts that $M \neq R$. If $R/M$ is YJ-injective, since $a^2 = 0$, every left $R$-homomorphism from $Ra$ to $R/M$ extends to one from $R$ to $R/M$. Define $f : Ra \rightarrow R/M$ by $f(ra) = r + M$ for every $r \in R$. As $l(a) \subseteq M$, $f$ is well-defined. Therefore $1 - ab \in M$ for some $b \in R$. But $ab \in J(R) \subseteq M$ which yields $1 \in M$, a contradiction. We have thus proved that for every $w \in J(R)$, $w^2 = 0$ implies $w = 0$ and thus $l(w) = l(w^m)$ for every positive integer $m$. Let $d \in J(R)$ and $T = l(d) + Rd$. If $T \neq R$, let $L$ be a maximal left ideal of $R$ containing $T$. If $R/L$ is flat, then by Lemma 1.9, there exists some $t \in L$ such that $d = dt$ which implies $1 - t \in r(d) = l(d) \subseteq L$ and so we get $1 \in L$, a contradiction. If $R/L$ is YJ-injective, there exists a positive integer $n$ such that $d^n \neq 0$ and every left $R$-homomorphism from $Rd^n$ to $R/L$ extends to one from $R$ to $R/L$. Define $f : Rd^n \rightarrow R/L$ by $f(rd^n) = r + L$ for each $r \in R$. Since $l(d) = l(d^n)$, $f$ is a well-defined left $R$-homomorphism. It follows that there exists some $s \in R$ such that $1 - d^n s \in L$. But $d^n s \in J(R) \subseteq L$. This yields $1 \in L$, a contradiction. This proves that for all $d \in J(R)$, $l(d) + Rd = R$ and so there exists some $x \in l(d)$, $y \in R$ such that $x + yd = 1$. Then $(yd - 1)d = 0$. As $d \in J(R)$, $yd - 1$ is invertible. Hence $d = 0$ so that $R$ is semiprimitive. □
Theorem 3.4. The following conditions are equivalent for a ring $R$.

(1) $R$ is strongly regular.

(2) $R$ is a ring whose maximal left ideals are GW-ideals, every simple left $R$-module is either YJ-injective or flat and for every $a \in J(R)$, $l(a) = r(a)$.

Proof. (1) $\Rightarrow$ (2) is clear.

(2) $\Rightarrow$ (1). By hypothesis, Corollary 2.2 and Proposition 3.3, $R$ is reduced so that $l(w) = r(w) = l(w^n)$ for every $w \in R$ and for every positive integer $n$. If $l(a) + Ra \neq R$ for some $a \in R$, then there exists a maximal left ideal $M$ of $R$ containing $l(a) + Ra$. If $R/M$ is flat, then $a \in Ra \subseteq M$ implies there exists some $b \in M$ such that $a = ab$, that is $1 - b \in r(a) = l(a) \subseteq M$ and so $1 \in M$, a contradiction. If $R/M$ is YJ-injective, arguing as in the proof of (2) $\Rightarrow$ (1) of Theorem 3.1, we get a contradiction. Therefore for all $a \in R$, $l(a) + Ra = R$. Thus $R$ is strongly regular.

Corollary 3.5. ([11], Proposition 6) The following conditions are equivalent for a ring $R$.

(1) $R$ is strongly regular.

(2) $R$ is a left quasi duo ring whose simple left modules are either YJ-injective or flat and for every $u \in J(R)$, $l(u) = r(u)$.

Theorem 3.6. The following conditions are equivalent for a ring $R$.

(1) $R$ is strongly regular.

(2) $R$ is a ring whose maximal right ideals are GW-ideals, every simple left $R$-module is either YJ-injective or flat and for every $a \in J(R)$, $l(a) = r(a)$.

Proof. It is clear that (1) $\Rightarrow$ (2).

(2) $\Rightarrow$ (1). $R$ is reduced by hypothesis, Corollary 2.2 and Proposition 3.3 and so $l(w) = r(w) = l(w^n)$ for every $w \in R$ and for every positive integer $n$. If $a \in R$ and $l(a) + aR \neq R$, arguing as in the proof of (3) $\Rightarrow$ (1) of Theorem 3.1, we get $l(a) + RaR \subseteq L$ for some maximal left ideal $L$ of $R$. If $R/L$ is flat, then since $a \in RaR \subseteq L$, there exists some $b \in L$ such that $a = ab$. This yields $1 - b \in r(a) = l(a) \subseteq L$, whence $1 \in L$ which is a contradiction. If $R/L$ is YJ-injective, proceeding as in the proof of (3) $\Rightarrow$ (1) of Theorem 3.1, we again get a contradiction. Therefore $l(a) + aR = R$ for all $a \in R$. This proves that $R$ is regular. Since $R$ is also reduced, $R$ is strongly regular.
Corollary 3.7. The following conditions are equivalent for a ring $R$.

1. $R$ is strongly regular.
2. $R$ is a right quasi duo ring whose simple left modules are either YJ-injective or flat and for every $u \in J(R)$, $l(u) = r(u)$.

Proposition 3.8. Let $R$ be a left GP-V-ring such that $l(a)$ is a GW-ideal for all $a \in R$, then $R$ is reduced.

Proof. Suppose $0 \neq a \in R$ such that $a^2 = 0$, then $l(a) \neq R$ and so there exists a maximal left ideal $M$ of $R$ containing $l(a)$. As $R$ is a left GP-V-ring and $a^2 = 0$, every left $R$-homomorphism from $Ra$ to $R/M$ extends to one from $R$ to $R/M$. Define $f : Ra \to R/M$ by $f(ra) = r + M$ for every $r \in R$. Then $f$ is a well-defined left $R$-homomorphism and hence it can be extended to one from $R$ to $R/M$. This implies that there exists some $b \in R$ such that $1 - ab \in M$. Since $l(a)$ is a GW-ideal and $ba \in l(a)$, there exists some positive integer $n$ such that $(ba)^n b \in l(a)$ and so $(ab)^{n+1} = a(ba)^n b \in l(a) \subseteq M$. Now

$$(ab)^n - (ab)^{n+1} = (ab)^n (1 - ab) \in M.$$  

This yields $(ab)^n \in M$. Again, as $1 - ab \in M$, we have

$$(ab)^{n-1} - (ab)^n = (ab)^{n-1} (1 - ab) \in M. $$

Since $(ab)^n \in M$, we get $(ab)^{n-1} \in M$. Proceeding in this manner we get $ab \in M$ and so $1 \in M$ contradicting that $M \neq R$. This proves that $R$ is reduced. 

Corollary 3.9. Let $R$ be a left GP-V-ring such that $l(a)$ is an ideal for all $a \in R$, then $R$ is reduced.

Theorem 3.10. Let $R$ be a left GP-V-ring such that $l(a)$ is a GW-ideal for all $a \in R$, then $R$ is weakly regular.

Proof. By Proposition 3.8, $R$ is reduced. Suppose $l(a) + RaR \neq R$ for some $a \in R$, then there exists a maximal left ideal $L$ of $R$ such that $l(a) + RaR \subseteq L$. Since $R$ is a left GP-V-ring, $R/L$ is YJ-injective. Arguing as in the proof of (3) $\Rightarrow$ (1) of Theorem 3.1, we get a contradiction. Thus $l(a) + RaR = R$ for all $a \in R$ and hence $R$ is left weakly regular. As $R$ is reduced, by Lemma 1.5, $R$ is right weakly regular. Therefore $R$ is weakly regular.

Proposition 3.11. Let $R$ be a right GP-V-ring such that $l(a)$ is a GW-ideal for all $a \in R$, then $R$ is reduced.
Proof. Let $0 \neq a \in R$ such that $a^2 = 0$. Since $r(a) \neq R$, there exists a maximal right ideal $K$ of $R$ such that $r(a) \subseteq K$. Since $a^2 = 0$ and $R$ is right GP-V-ring, every right $R$-homomorphism from $aR$ to $R/K$ extends to one from $R$ to $R/K$. Define $f : aR \to R/K$ by $f(ar) = r + K$ for every $r \in R$. Since $r(a) \subseteq K$, $f$ is well-defined. Thus we get $1 - ba \in K$ for some $b \in R$. As $l(a)$ is a GW-ideal and $ba \in l(a)$, there exists a positive integer $n$ such that $(ba)^n b \in l(a)$ yielding $(ab)^{n+1} \in l(a)$ so that $(ba)^{n+1} \in r(a) \subseteq K$. Since $1 - ba \in K$, we have $(ba)^n - (ba)^{n+1} = (1 - ba)(ba)^n \in K$, implying $(ba)^n \in K$. Again, as $1 - ba \in K$, $(ba)^{n-1} - (ba)^n = (1 - ba)(ba)^{n-1} \in K$. Since $(ba)^n \in K$, we get $(ba)^{n-1} \in K$. Proceeding in this manner, we get $ba \in K$. It follows that $1 \in K$ contradicting that $K \neq R$. This proves that $R$ is reduced.

Corollary 3.12. Let $R$ be a right GP-V-ring such that $l(a)$ is an ideal for all $a \in R$, then $R$ is reduced.

Theorem 3.13. If $R$ is a right GP-V-ring in which $l(a)$ is a GW-ideal for all $a \in R$, then $R$ is weakly regular.

Proof. $R$ is reduced by Proposition 3.11. If $r(a) + RaR \neq R$ for some $a \in R$, there exists a maximal right ideal $K$ of $R$ containing $r(a) + RaR$. Since $R$ is right GP-V-ring, there exists a positive integer $n$ such that $a^n \neq 0$ and every right $R$-homomorphism from $a^n R$ to $R/K$ extends to one from $R$ to $R/K$. Define $f : a^n R \to R/K$ by $f(a^n r) = r + K$. Since $R$ is reduced, $r(a^n) = r(a)$ so that $f$ is well-defined. It follows that there exists some $b \in R$ such that $1 - ba^n \in K$. But $ba^n \in RaR \subseteq K$, whence $1 \in K$, a contradiction. Therefore $r(a) + RaR = R$ for all $a \in R$. It follows that $R$ is weakly regular.

Proposition 3.14. Let $R$ be a left GP-V ring such that $l(a)$ is a GW-ideal for all $a \in R$, then $Z(RR) = 0$.

Proof. If $Z(RR) \neq 0$, by Lemma 1.8, there exists some $0 \neq a \in Z(RR)$ such that $a^2 = 0$. Let $M$ be a maximal left ideal of $R$ containing $l(a)$. Since $a \in Z(RR)$, $l(a)$ is an essential left ideal of $R$ and so $M$ is also an essential left ideal of $R$. Therefore $R/M$ is a simple singular left $R$-module and hence by hypothesis it is YJ-injective. As $a^2 = 0$, every left $R$-homomorphism from $Ra$ to $R/M$ can be extended to one from $R$ to $R/M$. Define $f : Ra \to R/M$ by $f(ra) = r + M$ for each $r \in R$. Arguing as in the proof of Proposition 3.8, we get a contradiction. This proves that $Z(RR) = 0$.

Corollary 3.15. Let $R$ be a left GP-V ring such that $l(a)$ is an ideal for all $a \in R$, then $Z(RR) = 0$. 

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References


Tikaram Subedi and A. M. Buhphang
Department of Mathematics
North Eastern Hill University
Permanent Campus
Shillong-793022, Meghalaya, India.
e-mails: tsubedi2010@gmail.com (Tikaram Subedi)
    ardeline17@gmail.com (A. M. Buhphang)