GENERALIZED SECOND CHANGE OF RINGS THEOREMS FOR HOMOLOGICAL DIMENSIONS

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Abstract. The goal of this paper is to generalize the second change of rings theorem for the injective dimension and the projective dimension.

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1. Introduction

Throughout this paper, $R$ denotes an associative ring with identity element. All modules, if not otherwise specified, are assumed to be left $R$-modules. If $x$ is a central element of $R$, when no confusion is likely, $R^x$ denotes the factor ring $R/xR$ and, for any $R$-module $A$, $Z(A)$ denotes the set of all zero-divisors of $A$.

Recall the following second change of rings theorems relative to the injective dimension and the projective dimension:

**Theorem A** [1, Theorem 205]. Let $x$ be a central non zero-divisor in $R$. If $M$ is an $R$-module such that $x$ is a non zero-divisor on $M$, then
\[ 1 + \text{id}_R\left( \frac{M}{xM} \right) \leq \text{id}_R(M) \]
except when $M$ is an injective $R$-module (in which case $M = xM$).

**Theorem B** [1, Exercise 1, page 155]. Let $x$ be a central element of $R$ such that $x \notin Z(R)$. Let $M$ be an $R$-module such that $M = xM$. Then
\[ 1 + \text{pd}_R(xM) \leq \text{pd}_R(M) \]
except when $M$ is projective over $R$ in which case $xM = 0$.

The aim of this paper is to generalize Theorem A and Theorem B dropping the hypothesis “$x \notin Z(M)$” and involving instead the submodule $xM := \{ z \in M : xz = 0 \}$ of $M$ annihilated by $x$. Our generalized second change of rings theorem for the injective dimension, Theorem 2.5, extends Theorem A by stating the following:
Let $x$ be a central element of $R$ such that $x \notin Z(R)$. Let $M$ be an $R$-module which is not an injective $R$-module. Then

$$1 + \text{id}_R\left(\frac{M}{xM}\right) \leq \text{id}_R(M) \text{ if and only if } \text{id}_R(xM) - 1 \leq \text{id}_R(M).$$

As for the projective dimension, via Theorem 2.7, we prove the following general version of Theorem B: Let $x$ be a central element of $R$ such that $x \notin Z(R)$. Let $M$ be an $R$-module which is not projective over $R$. Then

$$1 + \text{pd}_R(xM) \leq \text{pd}_R(M) \text{ if and only if } \text{pd}_R\left(\frac{M}{xM}\right) - 1 \leq \text{pd}_R(M).$$

2. Main results

This section aims at giving a general version of the second change of rings theorem for the homological dimensions, that is, Theorem A and Theorem B.

First, we establish the following results which will be useful in the sequel. Recall that for a central element $x$ of $R$, $R^*$ denotes the factor ring $R_{xR}$.

**Lemma 2.1.** Let $M$ be an $R$-module and $x$ a central element of $R$ such that $x \notin Z(R)$. Then $\text{Hom}_R(R^*, M) \cong xM$, $\text{Ext}_R^1(R^*, M) \cong \frac{M}{xM}$ and $\text{Tor}_R^1(R^*, M) \cong xM$.

**Proof.** As $x \notin Z(R)$, the following sequence is exact $0 \rightarrow R \xrightarrow{x} R \rightarrow R^* \rightarrow 0$. Applying the functor $\text{Hom}_R(-, M)$, we get the next exact sequence

$$0 \rightarrow \text{Hom}_R(R^*, M) \rightarrow \text{Hom}_R(R, M) \xrightarrow{x} \text{Hom}_R(R, M) \rightarrow \text{Ext}_R^1(R^*, M) \rightarrow 0.$$

Since $\text{Hom}_R(R, M) \cong M$, this latter sequence turns out to be the following exact one

$$0 \rightarrow \text{Hom}_R(R^*, M) \rightarrow M \xrightarrow{x} M \rightarrow \text{Ext}_R^1(R^*, M) \rightarrow 0.$$

Then the first two isomorphisms easily follows. Applying the functor $\otimes_R M$ instead of $\text{Hom}_R(-, M)$ to the initial exact sequence yields the last isomorphism. \hfill $\square$

**Lemma 2.2.** Let $x$ be a central element of $R$. Let $0 \rightarrow N \xrightarrow{\tau} P \xrightarrow{\alpha} M \rightarrow 0$ be an exact sequence of $R$-modules such that $P$ is a projective $R$-module. Then the natural sequence of $R^*$-modules

$$0 \rightarrow xM \xrightarrow{\tau} \frac{N}{xN} \xrightarrow{\pi} \frac{P}{xP} \xrightarrow{\pi} \frac{M}{xM} \rightarrow 0$$

is exact with $\frac{P}{xP}$ is a projective $R^*$-module.
Proof. Tensoring with $R^*$ the sequence $0 \rightarrow N \xrightarrow{i} P \xrightarrow{\alpha} M \rightarrow 0$ yields the exact sequence

$$0 \rightarrow \text{Tor}_1(R^*, M) \rightarrow \frac{N}{xM} \xrightarrow{i} \frac{P}{xP} \xrightarrow{\pi} \frac{M}{xM} \rightarrow 0.$$ 

As, by Lemma 2.1, $\text{Tor}_1(R^*, M) \cong \frac{M}{xM}$, we get the desired exact sequence. Moreover, as $P$ is $R$-projective, $\frac{P}{xP} \cong R^* \otimes_R P$ is an $R^*$-projective module. The proof is complete.

Lemma 2.3. Let $x$ be a central element of $R$ such that $x \notin Z(R)$. Let $0 \rightarrow M \xrightarrow{i} I \xrightarrow{\alpha} N \rightarrow 0$ be an exact sequence of $R$-modules such that $I$ is an injective $R$-module. Then the natural sequence of $R^*$-modules

$$0 \rightarrow xM \xrightarrow{x} xI \xrightarrow{x} xN \rightarrow \frac{M}{xM} \rightarrow 0$$ 

is exact with $xI$ is an injective $R^*$-module.

Proof. Applying the functor $\text{Hom}_R(R^*, -)$ to the sequence $0 \rightarrow M \xrightarrow{i} I \xrightarrow{\alpha} N \rightarrow 0$, we get the exact sequence

$$0 \rightarrow \text{Ext}_1(R^*, M) \rightarrow \frac{xM}{xM} \xrightarrow{x} \frac{xI}{xM} \xrightarrow{x} \frac{xN}{xM} \rightarrow \text{Ext}_1(R^*, M).$$ 

As, by Lemma 2.1, $\text{Ext}_1(R^*, M) \cong \frac{M}{xM}$, we obtain the desired exact sequence. Moreover, as $I$ is $R$-injective, by [2, Theorem 3.44], $xI : = \text{Hom}_R(R^*, I)$ is an injective $R^*$-module completing the proof. 

Next, through Theorem 2.4 and Theorem 2.5, we generalize the second change of rings theorem for the injective dimension, that is Theorem A. First, notice that if $M$ is an injective $R$-module and $x$ is a central element of $R$ such that $x \notin Z(R)$, then $M = xM$ and $xM$ is injective over $R^*$.

Theorem 2.4. Let $x$ be a central element of $R$ such that $x \notin Z(R)$. Let $M$ be an $R$-module which is not injective over $R$. Then

1) $1 + \text{id}_R\left(\frac{M}{xM}\right) \leq \max\{\text{id}_R(xM) - 1, \text{id}_R(M)\}$.

2) If $\text{id}_R(M) < +\infty$, then $\text{id}_R\left(\frac{M}{xM}\right)$ and $\text{id}_R(xM)$ are simultaneously finite.

3) Assume that $\text{id}_R(xM) > \text{id}_R(M)$. Then $\text{id}_R(xM) = 2 + \text{id}_R\left(\frac{M}{xM}\right)$.

Proof. 1) (and 2) If $\text{id}_R(M) = +\infty$, then we are done. Assume that $1 \leq \text{id}_R(M) = n < +\infty$. Let $0 \rightarrow M \rightarrow I \rightarrow A \rightarrow 0$ be an exact sequence of $R$-modules such that $I$ is injective over $R$. Note that, being a quotient of the injective $R$-module $I$, $A$ is a divisible $R$-module, so that, as $x \notin Z(R)$, $A = xA$. So, by [1, Theorem 204], $\text{id}_R(xA) \leq \text{id}_R(A) = n - 1$. On the other hand, applying the
functor $\text{Hom}_R(R^*, -)$ to the sequence (⋆) yields, by Lemma 2.3, the exact sequence of $R^*$-modules

$$0 \to xM \to xI \to xA \to \frac{M}{xM} \to 0.$$  

Let $K = \text{Im}(xI \to xA)$. Then we have the next two exact sequences of $R^*$-modules

$$\begin{align*}
0 &\to xM \to xI \to K \to 0 \ (\star \\&
0 &\to K \to xA \to \frac{M}{xM} \to 0 \ (\star \star).
\end{align*}$$

It is then easy to see that $\text{id}_{R^*}(\frac{M}{xM}) < +\infty$ if and only if $\text{id}_{R^*}(K) < +\infty$ if and only if $\text{id}_{R^*}(xM) < +\infty$ establishing (2). If $xM$ is injective over $R^*$, then $K$ is injective over $R^*$, so that, $xA \cong K \oplus \frac{M}{xM}$ and thus $\text{id}_{R^*}(\frac{M}{xM}) = \text{id}_{R^*}(xA) \leq n - 1$, as claimed. Next, suppose that $\text{id}_{R^*}(xM) \geq 1$. Consider the following portion of the long exact sequence associated to the sequence (⋆ ⋆)

$$\text{Ext}^j_{R^*}(B, \frac{M}{xM}) \to \text{Ext}^j_{R^*}(B, xA) \to \text{Ext}^{j+1}_{R^*}(B, K)$$

for each positive integer $j$ and each $R^*$-module $B$. Then $\text{id}_{R^*}(\frac{M}{xM}) \leq \max\{\text{id}_{R^*}(K) - 1, \text{id}_{R^*}(xA)\}$. Moreover, by the sequence (⋆ ⋆), as $\text{id}_{R^*}(xM) \geq 1$, we get $\text{id}_{R^*}(xM) = 1 + \text{id}_{R^*}(K)$. It follows that

$$\text{id}_{R^*}(\frac{M}{xM}) \leq \max\{\text{id}_{R^*}(xM) - 2, n - 1\}, \text{ as desired.}$$

3) Assume that $\text{id}_{R^*}(xM) > \text{id}_{R^*}(M)$. Then, applying (1), $1 + \text{id}_{R^*}(\frac{M}{xM}) \leq \text{id}_{R^*}(xM) - 1$, that is, $2 + \text{id}_{R^*}(\frac{M}{xM}) \leq \text{id}_{R^*}(xM)$. On the other hand, consider the above two exact sequences of $R^*$-modules

$$\begin{align*}
0 &\to xM \to xI \to K \to 0 \ (\star \star \\
0 &\to K \to xA \to \frac{M}{xM} \to 0 \ (\star \star \star).
\end{align*}$$

In view of the following portion of the long exact sequence associated to the sequence (⋆ ⋆ ⋆)

$$\text{Ext}^j_{R^*}(B, \frac{M}{xM}) \to \text{Ext}^{j+1}_{R^*}(B, K) \to \text{Ext}^{j+1}_{R^*}(B, xA)$$

for each positive integer $j$ and each $R^*$-module $B$, we get

$$\text{id}_{R^*}(K) \leq \max\{1 + \text{id}_{R^*}(\frac{M}{xM}), \text{id}_{R^*}(xA)\}.$$
It follows, using the sequence (**), that
\[ \text{id}_{R^*}(xM) \leq 1 + \text{id}_{R^*}(K) \leq 1 + \max \left\{ 1 + \text{id}_{R^*} \left( \frac{M}{xM} \right), \text{id}_{R^*}(xA) \right\} \leq 1 + \max \left\{ 1 + \text{id}_{R^*} \left( \frac{M}{xM} \right), \text{id}_{R}(M) - 1 \right\}. \]

As, by hypotheses, \( \text{id}_{R^*}(xM) > \text{id}_{R}(M) \), we get \( \text{id}_{R^*}(xM) \leq 2 + \text{id}_{R^*} \left( \frac{M}{xM} \right) \) yielding the desired equality and completing the proof. \( \square \)

Our next result represents the generalized version of Theorem A of the introduction.

**Theorem 2.5.** Let \( x \) be a central element of \( R \) such that \( x \not\in Z(R) \). Let \( M \) be an \( R \)-module which is not an injective \( R \)-module. Then
\[ 1 + \text{id}_{R^*} \left( \frac{M}{xM} \right) \leq \text{id}_{R}(M) \] if and only if \( \text{id}_{R^*}(xM) - 1 \leq \text{id}_{R}(M) \).

**Proof.** If \( \text{id}_{R^*}(xM) - 1 \leq \text{id}_{R}(M) \), then, by Theorem 2.4(1), \( 1 + \text{id}_{R^*} \left( \frac{M}{xM} \right) \leq \text{id}_{R}(M) \). Conversely, assume that \( 1 + \text{id}_{R^*} \left( \frac{M}{xM} \right) \leq \text{id}_{R}(M) \). Let us resume the notation of the proof of Theorem 2.4 and, thus, considering the above two exact sequences of \( R^* \)-modules
\[
\begin{align*}
&\begin{cases}
0 \to xM \to xI \to K \to 0 & (**) \\
0 \to K \to xA \to \frac{M}{xM} \to 0 & (***)
\end{cases}
\end{align*}
\]
we show in the proof of Theorem 2.4(3) that
\[ \text{id}_{R^*}(K) \leq \max \left\{ 1 + \text{id}_{R^*} \left( \frac{M}{xM} \right), \text{id}_{R^*}(xA) \right\}. \]
It follows that
\[ \text{id}_{R^*}(K) \leq \max \left\{ \text{id}(M), \text{id}(M) - 1 \right\} = \text{id}(M). \]
Hence, using the sequence (**), we get \( \text{id}_{R^*}(xM) \leq 1 + \text{id}_{R^*}(K) \leq 1 + \text{id}_{R}(M) \), as desired. \( \square \)

Next, through Theorem 2.6 and Theorem 2.7, we generalize the second change of rings theorem for the projective dimension, that is Theorem B. Note that if \( M \) is a projective \( R \)-module and \( x \) is a central element of \( R \) such that \( x \not\in Z(R) \), then \( \frac{M}{xM} \) is projective over \( R^* \) and \( xM = 0 \).
Theorem 2.6. Let $x$ be a central element of $R$ such that $x \not\in Z(R)$. Let $M$ be an $R$-module. Then

1) $\text{pd}_{R^*}(\frac{M}{xM}) \leq \max\{2 + \text{pd}_{R^*}(zM), \text{pd}_{R}(M)\}$.

2) If $\text{pd}_{R}(M) < +\infty$, then $\text{pd}_{R^*}(\frac{M}{xM})$ and $\text{pd}_{R^*}(zM)$ are simultaneously finite.

3) If $1 + \text{pd}_{R^*}(zM) > \text{pd}_{R}(M)$ and $M$ is not projective over $R$, then $\text{pd}_{R^*}(\frac{M}{xM}) = 2 + \text{pd}_{R^*}(zM)$.

Proof. 1) (and (2)) If $\text{pd}_{R}(M) = +\infty$, then we are done. Also, if $M$ is projective over $R$, then $\frac{M}{xM}$ is projective over $R^*$ and $xM = 0$. Then, assume that $1 \leq \text{pd}_{R}(M) = n < +\infty$. Let $0 \longrightarrow A \longrightarrow P \longrightarrow M \longrightarrow 0$ (*be an exact sequence of $R$-modules such that $P$ is projective over $R$. Note that as $A$ is a submodule of the projective module $P$ and $x \not\in Z(R)$, that is, $xA = 0$. So, by [3, Theorem 4.3.5], $\text{pd}_{R^*}(\frac{A}{xA}) \leq \text{pd}_{R}(A) \leq n - 1$. On the other hand, tensoring the sequence (*) with $R^*$ yields, by Lemma 2.2, the exact sequence of $R^*$-modules

$0 \longrightarrow xM \longrightarrow A_{xA} \longrightarrow P_{xP} \longrightarrow M_{xM} \longrightarrow 0$.

Now, let $H = \text{Im}(\frac{A}{xA} \to \frac{P}{xP})$. We have the next two exact sequences of $R^*$-modules

$$
\begin{cases}
0 \longrightarrow xM \longrightarrow A_{xA} \longrightarrow H \longrightarrow 0 (*) & \\
0 \longrightarrow H \longrightarrow P_{xP} \longrightarrow M_{xM} \longrightarrow 0 (**).
\end{cases}
$$

It is then clear that $\text{pd}_{R^*}(\frac{M}{xM}) < +\infty$ if and only if $\text{pd}_{R^*}(H) < +\infty$ if and only if $\text{pd}_{R^*}(zM) < +\infty$ establishing (2). Consider the following portion of the long exact sequence associated to the sequence (**)

$\text{Ext}_{R^*}^j(zM, B) \longrightarrow \text{Ext}_{R^*}^{j+1}(H, B) \longrightarrow \text{Ext}_{R^*}^{j+1}(\frac{A}{xA}, B)$

for each integer $j \geq 0$ and each $R^*$-module $B$. Then $\text{pd}_{R^*}(H) \leq \max\{1 + \text{pd}_{R^*}(zM), \text{pd}_{R^*}(\frac{A}{xA})\}$. Moreover, by the sequence (**), we have $\text{pd}_{R^*}(\frac{M}{xM}) \leq 1 + \text{pd}_{R^*}(H)$. It follows that

$\text{pd}_{R^*}(\frac{M}{xM}) \leq 1 + \max\{1 + \text{pd}_{R^*}(zM), n - 1\}$

$= \max\{2 + \text{pd}_{R^*}(zM), \text{pd}_{R}(M)\}$, as desired.

3) Assume that $1 + \text{pd}_{R^*}(zM) > \text{pd}_{R}(M) \geq 1$. Then, by (1), $\text{pd}_{R^*}(\frac{M}{xM}) \leq 2 + \text{pd}_{R^*}(zM)$. Conversely, proceeding as in (1), consider the above-mentioned two
exact sequences

\[
\begin{aligned}
0 & \rightarrow \ xM \rightarrow \frac{A}{xA} \rightarrow \ H \rightarrow \ 0 \ (**) \\
0 & \rightarrow \ H \rightarrow \frac{P}{xP} \rightarrow \frac{M}{xM} \rightarrow \ 0 \ (***)
\end{aligned}
\]

If \( \frac{M}{xM} \) is projective over \( R^* \), then \( H \) is projective over \( R^* \), and thus \( \text{pd}_{R^*}(\ xM \) = \( \text{pd}_{R^*}(\ \frac{A}{xA} \) \leq \( \text{pd}_{R^*}(M) - 1 \) which is contradictory to our initial assumption. Then \( \text{pd}_{R^*}(\ \frac{M}{xM} \) \geq 1, so that \( \text{pd}_{R^*}(H) = \text{pd}_{R^*}(\ \frac{M}{xM} \) - 1. Consider the following portion of the long exact sequence associated to the sequence (**) \[
\text{Ext}_j^R(\ \frac{A}{xA} \ ; \ B) \rightarrow \text{Ext}_j^R(\ \frac{M}{xM} \ ; \ B) \rightarrow \text{Ext}_{j+1}^R(H, B)
\]
for each integer \( j \geq 0 \) and each \( R^* \)-module \( B \). Hence
\[
\text{pd}_{R^*}(\ xM \) \leq \max\{\ \text{pd}_{R^*}(\ \frac{A}{xA} \), \ \text{pd}_{R^*}(H) - 1\} \leq \max\{\ \text{pd}_{R^*}(M) - 1, \ \text{pd}_{R^*}(\ \frac{M}{xM} \) - 2\}. \]
This ensures, as \( 1+\text{pd}_{R^*}(\ xM \) > \( \text{pd}_{R^*}(M) \), that \( \text{pd}_{R^*}(\ xM \) \leq \( \text{pd}_{R^*}(\ \frac{M}{xM} \) - 2.
It follows that \( \text{pd}_{R^*}(\ \frac{M}{xM} \) = 2+\text{pd}_{R^*}(\ xM \) establishing (3) and completing the proof.

The following stands for the dual result of Theorem 2.5 and it represents the generalized version of Theorem B of the introduction.

Theorem 2.7. Let \( x \) be a central element of \( R \) such that \( x \notin Z(R) \). Let \( M \) be an \( R \)-module which is not projective over \( R \). Then
\[
1 + \text{pd}_{R^*}(\ xM \) \leq \text{pd}_{R^*}(M) \text{ if and only if } \text{pd}_{R^*}(\ \frac{M}{xM} \) - 1 \leq \text{pd}_{R^*}(M).
\]

Proof. Let \( \text{pd}_{R^*}(\ \frac{M}{xM} \) - 1 \leq \text{pd}_{R^*}(M) \). If \( 1+\text{pd}_{R^*}(\ xM \) > \( \text{pd}_{R^*}(M) \), then, by Theorem 2.6(3),
\[
\text{pd}_{R^*}(\ \frac{M}{xM} \) = 2 + \text{pd}_{R^*}(\ xM \)
\geq 1 + \text{pd}_{R^*}(M) \text{ which is absurd.}
\]
It follows that \( 1+\text{pd}_{R^*}(\ xM \) \leq \( \text{pd}_{R^*}(M) \). Conversely, assume that \( 1+\text{pd}_{R^*}(\ xM \) \leq \( \text{pd}_{R^*}(M) \). Then, using Theorem 2.6(1),
\[
\text{pd}_{R^*}(\ \frac{M}{xM} \) \leq 1 + \max\{1 + \text{pd}_{R^*}(\ xM \), \ \text{pd}_{R^*}(M)\} \leq 1 + \text{pd}_{R^*}(M), \text{ as desired.}
\]
\( \square \)
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References


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