International Electronic Journal of Algebra
Volume 11 (2012) 82-95

SOME RESULTS ON COFINITE MODULES

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Received: 08 May 2011; Revised: 02 October 2011
Communicated by Sait Halıcıoğlu

Abstract. Let R be a Noetherian ring and $\mathfrak{a}$ be a proper ideal of R. We generalize the Rees characterization of grade for $\mathfrak{a}$-cofinite modules and as a consequence, we extend Grothendieck's Non-vanishing Theorem. We also generalize the classical Auslander-Buchsbaum and Bass formulas.

Mathematics Subject Classification (2010): 13D05, 13D45, 13E10, 13C11
Keywords: Auslander-Buchsbaum formula, Bass formula, local cohomology, cofinite module

1. Introduction

Throughout this paper, R is a commutative Noetherian ring, $\mathfrak{a} \subseteq \mathfrak{b}$ are two proper ideals of R and M is an R-module. Furthermore, if R is a local ring with maximal ideal $\mathfrak{m}$ and residue class field k we will refer to R as $(R, \mathfrak{m})$ or if we need the residue class field of R as $(R, \mathfrak{m}, k)$. The undefined terminology is the same as that in [4] and [5].

We say M is $\mathfrak{a}$-*cofinite* if $\mathrm{Supp}M \subseteq V(\mathfrak{a})$ and $\mathrm{Ext}^i_R(R/\mathfrak{a}, M)$ is a finitely generated R-module for all $i \geq 0$. The notion of $\mathfrak{a}$-cofinite module was first introduced in [10] and recently has been studied extensively by many authors; see, for example, [1, 2, 8, 13, 17, 27]. It is a well-known result that if $(R, \mathfrak{m})$ is a complete local ring, then the R-module M is Artinian if and only if $\mathrm{Supp}M \subseteq V(\mathfrak{m})$ and $\mathrm{Ext}^i_R(R/\mathfrak{m}, M)$ is finitely generated for all $i \geq 0$ (see [10, Proposition 1.1]). In view of this fact, the following conjecture was made by Grothendieck (see [9, Expose XIII, Conjecture 1.2]).

Grothendieck's conjecture. Let M be a finitely generated R-module. Then the module $\mathrm{Hom}(R/\mathfrak{a}, H^i_{\mathfrak{a}}(M))$ is finitely generated for all $i \geq 0$.

Hartshorne later refined this conjecture and proposed the following.

Hartshorne's conjecture. Let M be a finitely generated R-module. Then the local cohomology module $H^i_{\mathfrak{a}}(M)$ is $\mathfrak{a}$-cofinite for all $i \geq 0$.

This work was partially supported by Shahrekord University.

Hartshorne showed that, in general, this conjecture is false, even if R is a regular local ring. In the positive direction, the best well known result is that when either $\mathfrak{a}$ is principal or R is local and $\dim R/\mathfrak{a} = 1$, then the module $H^i_{\mathfrak{a}}(M)$ is $\mathfrak{a}$-cofinite for all $i \geq 0$. These results are proved in [17] and [8], respectively.

Mafi [19] used spectral sequences to show that, for a finitely generated module M, the module $H^n_{\mathfrak{a}}(M)$ is $\mathfrak{a}$-cofinite whenever the modules $H^i_{\mathfrak{a}}(M)$ are $\mathfrak{a}$-cofinite for all $i < n$ and $H^n_{\mathfrak{a}}(M)$ is Artinian. In Section 2, we give a generalization of Mafi's result without using spectral sequences.

In Section 3, we extend the Rees characterization of grade (Theorem 3.2) to $\mathfrak{a}$-cofinite modules and then generalize the concept of grade and depth for $\mathfrak{a}$-cofinite modules and as an application, we extend the Grothendieck's Non-vanishing Theorem.

One of the basic problems concerning local cohomology is to find when the set of associated primes of $H^n_{\mathfrak{a}}(M)$ is finite. The question of finiteness of associated primes of the local cohomology $H^n_{\mathfrak{a}}(R)$ when R is a regular local ring was first raised by Huneke [12]. In this direction, Huneke and Sharp [14] (when R is regular and contains a field of positive characteristic) and Lyubeznik [18] (when R is regular and contains a field of zero characteristic or is of mixed characteristic and unramified) gave an affirmative answer to this question. On the other hand it is not true in general, in view of the non local (respectively local) example given by Singh [25] (respectively Katzman [16]). In Section 4, we shall show that if M is an $\mathfrak{a}$-cofinite module and $\mathfrak{c}$ is an ideal of R such that $M \neq \mathfrak{c}M$, then the first non-vanishing local cohomology $H^t_{\mathfrak{c}}(M)$, where $t = \text{grade}(\mathfrak{c}, M)$, has only finitely many associated primes.

Notation. For modules M and N over the ring R, set

$$\begin{aligned} \text{fd}_R(M,N) &= \sup\{i | \text{Tor}_i^R(M,N) \neq 0\}, \\ \text{id}_R(M,N) &= \sup\{i | \text{Ext}_R^i(M,N) \neq 0\}. \end{aligned}$$

Recall that, if M is a module over a local ring $(R, \mathfrak{m}, k)$, then

$$\begin{aligned} \text{depth} M &= \inf\{i | \text{Ext}_R^i(k,M) \neq 0\}, \\ \text{width} M &= \inf\{i | \text{Tor}_i^R(k,M) \neq 0\}. \end{aligned}$$

For an R-module M, we say that $\text{flat}\dim_{\mathfrak{a}} M \leq n$ (respectively $\text{inj}\dim_{\mathfrak{a}} M \leq n$) if $\text{Tor}_k^R(R/\mathfrak{p}, M) = 0$ (respectively $\text{Ext}_R^k(R/\mathfrak{p}, M) = 0$) for all $\mathfrak{p} \in V(\mathfrak{a})$ and all $k \geq n+1$. Here $V(\mathfrak{a})$ denotes the set of prime ideals of R which contain $\mathfrak{a}$. We call $\text{flat}\dim_{\mathfrak{a}} M$ (respectively $\text{inj}\dim_{\mathfrak{a}} M$) the $\mathfrak{a}$-*relative flat (respectively injective)* *dimension* of M. In view of [5, Corollary 3.1.12] and its dual, if $\mathfrak{a}$ is the zero ideal of R, then $\operatorname{inj}\dim_{\mathfrak{a}} M = \operatorname{inj}\dim M$ and $\operatorname{flat}\dim_{\mathfrak{a}} M = \operatorname{fl}\dim M$.

The classical Auslander-Buchsbaum formula asserts that if a nonzero finite module M over a local ring R has finite projective dimension, then $\operatorname{proj}\dim M + \operatorname{depth} M = \operatorname{depth} R$. In Section 5, we generalize this formula by proving the following. Let $(R, \mathfrak{m}, k)$ be a local ring and let M be a nonzero $\mathfrak{a}$-cofinite R-module of finite flat dimension. Then $\operatorname{flat}\dim_{\mathfrak{a}} M = \operatorname{depth} R - \operatorname{depth} M$. As a consequence, for any nonzero R-module N,

$$p = \operatorname{fd}_R(M, N) \geq \operatorname{depth} R - \operatorname{depth} M - \operatorname{depth} N$$

with equality if and only if $\operatorname{depth}\operatorname{Tor}_p^R(M, N) = 0$.

The classical Bass formula asserts that if a nonzero finite module M over a local ring R has finite injective dimension, then $\operatorname{inj}\dim M = \operatorname{depth} R$. In Section 6, we generalize this formula by proving the following. Let $(R, \mathfrak{m}, k)$ be a local ring and let M be a nonzero $\mathfrak{a}$-cofinite R-module. Then $\operatorname{inj}\dim_{\mathfrak{a}} M = \operatorname{depth} R - \operatorname{width} M$. As a consequence, we have the following statement which is a generalization of Ischebeck's result: let $(R, \mathfrak{m}, k)$ be a local ring and M be an $\mathfrak{a}$-cofinite R-module of finite injective dimension and N be an arbitrary R-module. Then

$$q = \operatorname{id}_R(N, M) \geq \operatorname{depth} R - \operatorname{width} M - \operatorname{depth} N$$

with equality if and only if $\operatorname{width}\operatorname{Ext}_R^q(N, M) = 0$. In particular, the equality holds for any finite R-module N.

2. Cofiniteness of Local Cohomology

Theorem 2.1. *Let n be a non-negative integer such that $\operatorname{Ext}_R^n(R/\mathfrak{a}, M)$ is a finitely generated R-module. If $\operatorname{Ext}_R^i(R/\mathfrak{a}, H_{\mathfrak{a}}^j(M))$ is finitely generated for all $i \leq n+1$ and $j < n$, then $\operatorname{Hom}_R(R/\mathfrak{a}, H_{\mathfrak{a}}^n(M))$ is finitely generated. In particular, $\operatorname{Ass}(H_{\mathfrak{a}}^n(M))$ is finite.*

Proof. We prove the theorem by induction on $n(\geq 0)$. If $n = 0$, then

$$\operatorname{Hom}(R/\mathfrak{a}, \Gamma_{\mathfrak{a}}(M)) \cong \operatorname{Hom}(R/\mathfrak{a}, M)$$

is finitely generated. Suppose, inductively, that $n > 0$ and the result has been proved for $n - 1$. Since $\operatorname{Ext}_R^i(R/\mathfrak{a}, \Gamma_{\mathfrak{a}}(M))$ is finitely generated for all $i \leq n + 1$, by using the exact sequence $0 \to \Gamma_{\mathfrak{a}}(M) \to M \to M/\Gamma_{\mathfrak{a}}(M) \to 0$ we get that $\operatorname{Ext}_R^n(R/\mathfrak{a}, (M/\Gamma_{\mathfrak{a}}(M))$ is finitely generated. On the other hand, $H_{\mathfrak{a}}^0(M/\Gamma_{\mathfrak{a}}(M)) = 0$ and $H_{\mathfrak{a}}^i(M/\Gamma_{\mathfrak{a}}(M)) \cong H_{\mathfrak{a}}^i(M)$ for all $i > 0$. Thus we may assume that $\Gamma_{\mathfrak{a}}(M) = 0$. Let E be an injective hull of M and put $N = E/M$. Then it is easy to see that $\Gamma_{\mathfrak{a}}(E) = 0$. Consequently $H^i_{\mathfrak{a}}(N) \cong H^{i+1}_{\mathfrak{a}}(M)$ for all $i \geq 0$. Since $\mathrm{Ext}^i_R(R/\mathfrak{a}, N) = \mathrm{Ext}^{i+1}_R(R/\mathfrak{a}, M)$ for all $i \geq 0$, the induction hypothesis yields that $\mathrm{Hom}(R/\mathfrak{a}, H^{n-1}_{\mathfrak{a}}(N))$ is finitely generated and hence $\mathrm{Hom}(R/\mathfrak{a}, H^n_{\mathfrak{a}}(M))$, which is isomorphic to it, is finitely generated. □

Zöschinger [28] introduced the interesting class of minimax modules. The R-module M is said to be a *minimax module*, if there is a finitely generated submodule N of M, such that M/N is Artinian. The class of minimax modules includes all finitely generated and all Artinian modules. Moreover it is closed under taking submodules, quotients and extensions, i.e., it is a Serre subcategory of the category of R-modules; see, Rudlof [24] and Zöschinger [28,29]. Obviously this class is strictly larger than the class of all finitely generated modules and also than the class of all Artinian modules; see Belshoff et al [3].

The next result has been shown using a spectral sequence argument by Mafi in [19, Theorem 2.1] under the assumption that M is finitely generated. We should mention that by definition every finitely generated R-module M is $\mathfrak{a}$-cofinite, where $\mathfrak{a} \subseteq \mathrm{Ann}(M)$ is an ideal of R.

Corollary 2.2. *Let M be $\mathfrak{a}$-cofinite. If $H^n_{\mathfrak{b}}(M)$ is minimax and $H^i_{\mathfrak{b}}(M)$ is $\mathfrak{b}$-cofinite for all $i < n$, then $H^n_{\mathfrak{b}}(M)$ is $\mathfrak{b}$-cofinite.*

Proof. Since M is $\mathfrak{a}$-cofinite, it follows from [8, Corollary 1] that $\mathrm{Ext}^n_R(R/\mathfrak{b}, M)$ is finitely generated. The above theorem implies that $\mathrm{Hom}_R(R/\mathfrak{b}, H^n_{\mathfrak{b}}(M))$ is finitely generated. Now the assertion follows from [21, Proposition 4.3] and the fact that $\mathrm{Supp} H^n_{\mathfrak{b}}(M) \subseteq V(\mathfrak{b})$. □

3. Cofiniteness and Grade

The proof of the following result is standard and we give its proof for completeness.

Lemma 3.1. *Let M be $\mathfrak{a}$-cofinite and $\mathfrak{c}$ be an ideal of R such that $M \neq \mathfrak{c}M$. Then $\mathfrak{c}$ contains an M-regular element if and only if $\mathrm{Hom}_R(R/\mathfrak{c}, M) = 0$.*

Proof. Let $f \in \mathrm{Hom}_R(R/\mathfrak{c}, M)$ and $a \in \mathfrak{c}$ be an M-regular element. Since $ax = 0$ for all $x \in R/\mathfrak{c}$, $f(ax) = af(x) = 0$ for all $x \in R/\mathfrak{c}$. Since a is M-regular, $f(x) = 0$. Therefore $\mathrm{Hom}_R(R/\mathfrak{c}, M) = 0$. Conversely, let $\mathfrak{c}$ have no M-regular elements. Then by [20, Corollary 1.4] there exists an associated prime $\mathfrak{p} \in \mathrm{Ass} M$ such that $\mathfrak{c} \subseteq \mathfrak{p}$. There is a monomorphism $\phi : R/\mathfrak{p} \longrightarrow M$; the composition of the natural epimorphism $R/\mathfrak{c} \longrightarrow R/\mathfrak{p}$ and ϕ yields a non-zero homomorphism $R/\mathfrak{c} \longrightarrow M$. □

The following is a generalization of Rees' characterization of grade (see [5, Theorem 1.2.5]).

Theorem 3.2. *Let M be $\mathfrak{a}$-cofinite and let $\mathfrak{c}$ be an ideal of R such that $M \neq \mathfrak{c}M$. Then all maximal M-sequences in $\mathfrak{c}$ have same length n and it is given by*

$$n = \inf\{i : \operatorname{Ext}^i_R(R/\mathfrak{c}, M) \neq 0\}.$$

Proof. Let $\mathbf{x} = x_1, x_2, \ldots, x_n$ be a maximal M-regular sequence in $\mathfrak{c}$. If $M_i = M/(x_1, x_2, \ldots, x_i)M$, then $\mathfrak{c}$ has an M_i-regular element for $i = 0, 1, \ldots, n-1$. Hence by [5, Lemma 1.2.4] we have $\operatorname{Ext}^i_R(R/\mathfrak{c}, M) \cong \operatorname{Hom}_R(R/\mathfrak{c}, M_i) = 0$ for all $i = 0, 1, \ldots, n-1$. On the other hand, since $M \neq \mathfrak{c}M$ and $\mathfrak{c}$ has no $M/\mathbf{x}M$-regular elements,

$$\operatorname{Ext}^n_R(R/\mathfrak{c}, M) \cong \operatorname{Hom}_R(R/\mathfrak{c}, M/\mathbf{x}M) \neq 0.$$

This concludes the proof. □

We are now in a position to define the concept of grade and depth for $\mathfrak{a}$-cofinite modules.

Definition 3.3. Let M be $\mathfrak{a}$-cofinite and $\mathfrak{c}$ an ideal of R such that $\mathfrak{c}M \neq M$. Then the common length of all maximal M-sequences in $\mathfrak{c}$ is called the *grade* of $\mathfrak{c}$ on M and it is denoted by $\operatorname{grade}(\mathfrak{c}, M)$. If $(R, \mathfrak{m})$ is local, then we put $\operatorname{depth} M = \operatorname{grade}(\mathfrak{m}, M)$.

Now we are ready to present the main result of this section.

Theorem 3.4. *Let M be $\mathfrak{a}$-cofinite and let $\mathfrak{c}$ be an ideal of R such that $M \neq \mathfrak{c}M$. Then* $\operatorname{grade}(\mathfrak{c}, M)$ *is the least integer i such that $H^i_{\mathfrak{c}}(M) \neq 0$*

Proof. Let $n = \operatorname{grade}(\mathfrak{c}, M)$. Apply induction on n. If $n = 0$ then $\mathfrak{c}$ contains only zero divisors of M. Thus $H^0_{\mathfrak{c}}(M) = \Gamma_{\mathfrak{c}}(M) \neq 0$. Suppose $n > 0$. Then there exists $x \in \mathfrak{c}$, a non-zerodivisor on M. Set $\overline{M} = M/xM$. We have still $\mathfrak{c}\overline{M} \neq \overline{M}$ and note that $\operatorname{grade}(\mathfrak{c}, \overline{M}) = \operatorname{grade}(\mathfrak{c}, M) - 1 = n - 1$. By [20, Remark(a)], $\overline{M}$ is $\mathfrak{a}$-cofinite. Therefore, by the induction hypothesis, $H^i_{\mathfrak{c}}(\overline{M}) = 0$ for $i < n-1$ and $H^{n-1}_{\mathfrak{c}}(\overline{M}) \neq 0$. Consider the exact sequence

$$0 \longrightarrow M \xrightarrow{x} M \longrightarrow \overline{M} \longrightarrow 0.$$

Applying the long exact cohomology sequence we get

$$H^{i-1}_{\mathfrak{c}}(M) \longrightarrow H^{i-1}_{\mathfrak{c}}(\overline{M}) \longrightarrow H^i_{\mathfrak{c}}(M) \xrightarrow{x} H^i_{\mathfrak{c}}(M).$$

If $i < n$ then $i - 1 < n - 1$ and so $H^{i-1}_{\mathfrak{c}}(\overline{M}) = 0$. Thus x is a non-zero divisor of $H^i_{\mathfrak{c}}(M)$. As $H^i_{\mathfrak{c}}(M)$ is $\mathfrak{c}$-torsion module it follows that $H^i_{\mathfrak{c}}(M) = 0$. On the other hand, from $H_{\mathfrak{c}}^{n-1}(M) = 0$ we get the injective map $H_{\mathfrak{c}}^{n-1}(\overline{M}) \longrightarrow H_{\mathfrak{c}}^{n}(M)$. As $H_{\mathfrak{c}}^{n-1}(\overline{M}) \neq 0$ it follows that $H_{\mathfrak{c}}^{n}(M) \neq 0$. □

The following result is a generalization of Grothendieck's Theorem (see, for example, [5, Theorem 3.5.7]).

Corollary 3.5. *Let $(R, \mathfrak{m})$ be a local ring and let M be a non-zero $\mathfrak{a}$-cofinite module of depth t and dimension d. Then*

(1) $H_{\mathfrak{m}}^{i}(M) = 0$ *for all* $i < t$ *and* $i > d$,

(2) $H_{\mathfrak{m}}^{t}(M) \neq 0$ *and* $H_{\mathfrak{m}}^{d}(M) \neq 0$.

Proof. (1) Use Theorem 3.4 and Grothendieck's Vanishing Theorem [4, Theorem 6.1.2].

(2) Use Theorem 3.4 and [19, Theorem 2.9]. □

We end this section by establishing upper bounds for depth$M$. First we need two lemmas.

Lemma 3.6. *Let $(R, \mathfrak{m})$ be a local ring and let M be $\mathfrak{a}$-cofinite. If $\mathfrak{m}M = M$, then $\mathfrak{a}M = M$.*

Proof. Since $\mathfrak{m}(M/\mathfrak{a}M) = (\mathfrak{m}M + \mathfrak{a}M)/\mathfrak{a}M = M/\mathfrak{a}M$, the assertion follows from [20, Corollary 1.2] and Nakayama's Lemma. □

Lemma 3.7. *Let $(R, \mathfrak{m})$ be a local ring, $\mathfrak{p} \in \operatorname{Supp}M$ and M be an $\mathfrak{a}$-cofinite module such that $\mathfrak{a}M \neq M$. If $\dim R/\mathfrak{p} \leq \operatorname{depth}M$, then $\operatorname{Ext}_R^i(R/\mathfrak{p}, M) = 0$ for $i < \operatorname{depth}M - \dim R/\mathfrak{p}$.*

Proof. We proceed by induction on $n = \dim R/\mathfrak{p}$. If $n = 0$, then $\mathfrak{p} = \mathfrak{m}$. By Lemma 3.6, we have $\mathfrak{m}M \neq M$. Hence the assertion follows from Theorem 3.2. Now suppose $n > 0$. Then $\mathfrak{p} \neq \mathfrak{m}$. Choose an element $x \in \mathfrak{m} \setminus \mathfrak{p}$. The element x is a non-zero divisor on $R/\mathfrak{p}$, and therefore we get the exact sequence

$$0 \longrightarrow R/\mathfrak{p} \xrightarrow{x} R/\mathfrak{p} \longrightarrow R/(\mathfrak{p}, x) \longrightarrow 0.$$

There is a chain $0 = M_0 \subsetneq M_1 \subsetneq \cdots \subsetneq M_{n-1} \subsetneq M_n = R/(\mathfrak{p}, x)$ of submodules of $R/(\mathfrak{p}, x)$ such that $M_j/M_{j-1} \cong R/\mathfrak{p}_j$ for some $\mathfrak{p}_j \in \operatorname{Spec}R$. Moreover, $\mathfrak{p} \subsetneq \mathfrak{p}_j$ and so $\mathfrak{p}_j \in \operatorname{Supp}M$ and $\dim R/\mathfrak{p}_j < \dim R/\mathfrak{p}$ for each j. So by induction, for each j, $\operatorname{Ext}_R^i(R/\mathfrak{p}_j, M) = 0$ for $i \leq \operatorname{depth}M - \dim R/\mathfrak{p}$. Hence $\operatorname{Ext}_R^i(R/(\mathfrak{p}, x), M) = 0$ for $i \leq \operatorname{depth}M - \dim R/\mathfrak{p}$. If $i < \operatorname{depth}M - \dim R/\mathfrak{p}$, the exact sequence

$$0 \longrightarrow \operatorname{Ext}_R^i(R/\mathfrak{p}, M) \xrightarrow{x} \operatorname{Ext}_R^i(R/\mathfrak{p}, M) \longrightarrow \operatorname{Ext}_R^{i+1}(R/(\mathfrak{p}, x), M) = 0$$

gives $x\mathrm{Ext}^i_R(R/\mathfrak{p}, M) = \mathrm{Ext}^i_R(R/\mathfrak{p}, M)$ and thus the conclusion follows from [8, Corollary 1] and Nakayama's Lemma. □

We are now in a position to prove the following result.

Theorem 3.8. *Let* $(R, \mathfrak{m})$ *be a local ring and* M *be an* $\mathfrak{a}$*-cofinite module such that* $\mathfrak{a}M \neq M$*. Then*

(1) $\mathrm{depth}M \leq \mathrm{dim}R/\mathfrak{p}$ *for all* $\mathfrak{p} \in \mathrm{Ass}M$*,*

(2) $\mathrm{depth}M \leq \mathrm{dim}M$*.*

Proof. (1) If $\mathfrak{p} \in \mathrm{Ass}M$, then $\mathrm{Hom}(R/\mathfrak{p}, M) \neq 0$, and so the result follows from Lemma 3.7.

(2) This follows from Part (1). □

4. Associated Primes of Local Cohomology Modules

The following theorem is one of the main results of this paper. We shall use the following theorem to deduce that, if M is $\mathfrak{a}$-cofinite, then the first non-vanishing local cohomology module $H^t_{\mathfrak{c}}(M)$, where $\mathfrak{c}$ is an ideal of R such that $\mathfrak{c}M \neq M$ and $t = \mathrm{grade}(\mathfrak{c}, M)$, has only finitely many associated primes.

Theorem 4.1. (See [11, Theorem 1]) *Let* M *be an* $\mathfrak{a}$*-cofinite module and let* $\mathfrak{c}$ *be an ideal of* R *such that* $M \neq \mathfrak{c}M$*; let* $t = \mathrm{grade}(\mathfrak{c}, M)$*. Then*

$$\mathrm{Hom}_R(R/\mathfrak{c}, H^t_{\mathfrak{c}}(M)) \cong \mathrm{Ext}^t_R(R/\mathfrak{c}, M).$$

Proof. We use induction on t. If $t = 0$, then $H^0_{\mathfrak{c}}(M) = \Gamma_{\mathfrak{c}}(M)$ and the assertion follows from the fact that $(0 :_{\Gamma_{\mathfrak{c}}(M)} \mathfrak{c}) = (0 :_M \mathfrak{c})$.
Now suppose that $t > 0$. Let $x_1, x_2, ..., x_t$ be an M-regular sequence in $\mathfrak{c}$ and let $\overline{M} = M/x_1M$. Since $H^{t-1}_{\mathfrak{c}}(M) = 0$ by Theorem 3.2, the exact sequence

$$0 \longrightarrow M \xrightarrow{x_1} M \longrightarrow \overline{M} \longrightarrow 0$$

induces an exact sequence

$$0 \longrightarrow H^{t-1}_{\mathfrak{c}}(\overline{M}) \longrightarrow H^t_{\mathfrak{c}}(M) \xrightarrow{x_1} H^t_{\mathfrak{c}}(M).$$

Therefore $H^{t-1}_{\mathfrak{c}}(\overline{M}) \cong (0 :_{H^t_{\mathfrak{c}}(M)} x_1)$ and so $\mathrm{Hom}_R(R/\mathfrak{c}, H^{t-1}_{\mathfrak{c}}(\overline{M})) \cong \mathrm{Hom}_R(R/\mathfrak{c}, H^t_{\mathfrak{c}}(M))$. By the inductive hypothesis and [5, Lemma 1.2.4], we have

$$\begin{aligned} \mathrm{Hom}_R(R/\mathfrak{c}, H^t_{\mathfrak{c}}(M)) &\cong \mathrm{Ext}^{t-1}_R(R/\mathfrak{c}, \overline{M}) \\ &\cong \mathrm{Hom}_R(R/\mathfrak{c}, M/(x_1, x_2, ..., x_t)M) \\ &\cong \mathrm{Ext}^t_R(R/\mathfrak{c}, M). \end{aligned}$$

This concludes the proof. □

Corollary 4.2. *Let M be $\mathfrak{a}$-cofinite, $\mathfrak{b}M \neq M$ and $t = \mathrm{grade}(\mathfrak{b}, M)$. Then $\mathrm{Ass}_R H^t_{\mathfrak{b}}(M)$ is finite.*

Proof. The assertion follows from Theorem 4.1, [8, Corollary 1] and the fact that

$$\mathrm{Ass}_R(H^t_{\mathfrak{b}}(M)) = \mathrm{Ass}_R(\mathrm{Hom}_R(R/\mathfrak{b}, H^t_{\mathfrak{b}}(M))). \qquad \square$$

Corollary 4.3. *Let $(R, \mathfrak{m})$ be a local ring and let M be an $\mathfrak{a}$-cofinite module such that $M \neq \mathfrak{a}M$. Then $\mathrm{Ass}_R H^t_{\mathfrak{b}}(M)$ is finite, where $t = \mathrm{grade}(\mathfrak{b}, M)$.*

Proof. By Lemma 3.6, we have $M \neq \mathfrak{b}M$. Now the assertion follows from Corollary 4.2. □

5. Auslander-Buchsbaum Formula

We start with the following lemma which is the dual of [5, Lemma 3.1.11].

Lemma 5.1. *Let M be an R-module, N a finite R-module and $n \geq 0$ an integer. If $\mathrm{Tor}^R_n(R/\mathfrak{p}, M) = 0$ for all $\mathfrak{p} \in \mathrm{Supp}N$, then $\mathrm{Tor}^R_n(N, M) = 0$.*

Proof. This is dual to the proof of [5, Lemma 3.1.11]. □

Corollary 5.2. *Let M be an R-module and $n \geq 0$ be an integer. Then the following are equivalent.*

(1) $\mathrm{fl\,dim}_R M < n$,
(2) $\mathrm{Tor}^R_n(R/\mathfrak{p}, M) = 0$ *for all* $\mathfrak{p} \in \mathrm{Spec}R$.

Proof. This follows easily from the above lemma and the Flat Dimension Theorem (see, for example, [22, Proposition 4.5]). □

Motivated by the above corollary, we make the following definition which provides a generalization of the concept of flat dimension.

Definition 5.3. An R-module M is said to be of $\mathfrak{a}$-*relative flat dimension* $\leq n$ if $\mathrm{Tor}^R_k(R/\mathfrak{p}, M) = 0$ for all $\mathfrak{p} \in V(\mathfrak{a})$ and all $k \geq n+1$. We write $\mathrm{flat\,dim}_{\mathfrak{a}} M$ for $\mathfrak{a}$-relative flat dimension of M.

Proposition 5.4. *Let $(R, \mathfrak{m}, k)$ be a local ring, $\mathfrak{p} \in V(\mathfrak{a})$ different from $\mathfrak{m}$, M be an $\mathfrak{a}$-cofinite R-module, and let n be an integer. If $\mathrm{Tor}^R_n(R/\mathfrak{q}, M) = 0$ for all prime ideals $\mathfrak{q} \in V(\mathfrak{p})$, $\mathfrak{q} \neq \mathfrak{p}$, then $\mathrm{Tor}^R_n(R/\mathfrak{p}, M) = 0$.*

Proof. Choose an element $x \in \mathfrak{m} \setminus \mathfrak{p}$. The element x is a nonzero divisor on $R/\mathfrak{p}$, and therefore we get the exact sequence

$$0 \longrightarrow R/\mathfrak{p} \xrightarrow{x.} R/\mathfrak{p} \longrightarrow R/(\mathfrak{p}, x) \longrightarrow 0$$

which induces the exact sequence

$$\operatorname{Tor}_n^R(R/\mathfrak{p}, M) \xrightarrow{x.} \operatorname{Tor}_n^R(R/\mathfrak{p}, M) \longrightarrow \operatorname{Tor}_n^R(R/(\mathfrak{p}, x), M).$$

Since $V(x, \mathfrak{p}) \subseteq \{\mathfrak{q} \in V(\mathfrak{p}) | \mathfrak{q} \neq \mathfrak{p}\}$, the above lemma and our assumption imply

$$\operatorname{Tor}_n^R(R/(\mathfrak{p}, x), M) = 0.$$

Therefore $x\operatorname{Tor}_n^R(R/\mathfrak{p}, M) = \operatorname{Tor}_n^R(R/\mathfrak{p}, M)$ and hence the conclusion follows from [21, Theorem 2.1], [8, Corollary 1] and Nakayama's Lemma. □

It is now easy to obtain the following useful formula for an $\mathfrak{a}$-relative flat dimension of an $\mathfrak{a}$-cofinite module.

Theorem 5.5. *Let $(R, \mathfrak{m}, k)$ be a local ring and M be an $\mathfrak{a}$-cofinite R-module. Then*

$$\operatorname{flat\,dim}_{\mathfrak{a}} M = \operatorname{fd}_R(k, M).$$

Proof. Clearly $\operatorname{flat\,dim}_{\mathfrak{a}} M \geq \operatorname{fd}_R(k, M)$. For the opposite inequality, let $s = \operatorname{fd}_R(k, M)$. Repeated applications of Proposition 5.4 show $\operatorname{Tor}_i^R(R/\mathfrak{p}, M) = 0$ for all $\mathfrak{p} \in V(\mathfrak{a})$ and all $i > s$. This gives the desired inequality. □

The following theorem is our first main result of this section which generalizes the classical Auslander-Buchsbaum formula.

Theorem 5.6. *Let $(R, \mathfrak{m}, k)$ be a local ring and M be a nonzero $\mathfrak{a}$-cofinite R-module of finite flat dimension. Then* $\operatorname{flat\,dim}_{\mathfrak{a}} M = \operatorname{depth} R - \operatorname{depth} M$.

Proof. We have $\operatorname{fd}_R(k, M) = \operatorname{depth} R - \operatorname{depth} M$, by [7, Proposition 1]. Now the desired result follows from Theorem 5.5. □

The following result is due to S. Choi and S. Iyengar [6, Theorem 3]:
"Let $(R, \mathfrak{m}, k)$ be a local ring, let M and N be finitely generated R-modules and let M have finite complete intersection dimension. Then

$$p = \operatorname{fd}_R(M, N) \geq \operatorname{depth} R - \operatorname{depth} M - \operatorname{depth} N$$

with equality if and only if $\operatorname{depth}\operatorname{Tor}_p^R(M, N) = 0$".

A similar result holds for an $\mathfrak{a}$-cofinite R-module M and any R-module N.

Theorem 5.7. *Let $(R, \mathfrak{m}, k)$ be a local ring and M be a nonzero $\mathfrak{a}$-cofinite R-module of finite flat dimension. Then for any nonzero R-module N,*

$$p = \operatorname{fd}_R(M, N) \geq \operatorname{depth} R - \operatorname{depth} M - \operatorname{depth} N$$

with equality if and only if $\operatorname{depth}\operatorname{Tor}_p^R(M, N) = 0$.

Proof. If we combine [26, Lemma 2.2] with [26, Lemmas 2.5 and 2.6(a)], we see that $\mathrm{fd}_R(M,N) \geq \mathrm{fd}_R(k,M) - \mathrm{depth}N$ with equality if and only if $\mathrm{depth}\mathrm{Tor}_p^R(M,N) = 0$. Now the desired result follows from Theorem 5.5 and Theorem 5.6. □

6. Bass Formulas

We will consider a generalization of the concept of injective dimension.

Definition 6.1. An R-module M is said to be of $\mathfrak{a}$-*relative injective dimension* $\leq n$ if $\mathrm{Ext}_R^k(R/\mathfrak{p}, M) = 0$ for all $\mathfrak{p} \in V(\mathfrak{a})$ and all $k \geq n+1$. We write $\mathrm{inj\,dim}_{\mathfrak{a}} M$ for the $\mathfrak{a}$-relative injective dimension of M.

Proposition 6.2. *Let $(R, \mathfrak{m}, k)$ be a local ring, $\mathfrak{p} \in V(\mathfrak{a})$ different from $\mathfrak{m}$, M be an $\mathfrak{a}$-cofinite R-module, and let n be an integer. If $\mathrm{Ext}_R^{n+1}(R/\mathfrak{q}, M) = 0$ for all prime ideals $\mathfrak{q} \in V(\mathfrak{p})$, $\mathfrak{q} \neq \mathfrak{p}$, then $\mathrm{Ext}_R^n(R/\mathfrak{p}, M) = 0$.*

Proof. Choose an element $x \in \mathfrak{m} \setminus \mathfrak{p}$. Then the exact sequence

$$0 \longrightarrow R/\mathfrak{p} \xrightarrow{x.} R/\mathfrak{p} \longrightarrow R/(\mathfrak{p}, x) \longrightarrow 0$$

induces the exact sequence

$$\mathrm{Ext}_R^n(R/\mathfrak{p}, M) \xrightarrow{x.} \mathrm{Ext}_R^n(R/\mathfrak{p}, M) \longrightarrow \mathrm{Ext}_R^{n+1}(R/(\mathfrak{p}, x), M).$$

Since $V(x, \mathfrak{p}) \subseteq \{\mathfrak{q} \in V(\mathfrak{p}) | \mathfrak{q} \neq \mathfrak{p}\}$, our assumption and [5, Lemma 3.1.11] imply $\mathrm{Ext}_R^{n+1}(R/(\mathfrak{p}, x), M) = 0$. Therefore $x\mathrm{Ext}_R^n(R/\mathfrak{p}, M) = \mathrm{Ext}_R^n(R/\mathfrak{p}, M)$, and hence, the conclusion follows from [8, Corollary 1] and Nakayama's Lemma. □

Theorem 6.3. *Let $(R, \mathfrak{m}, k)$ be a local ring and M be an $\mathfrak{a}$-cofinite R-module. Then*

$$\mathrm{inj\,dim}_{\mathfrak{a}} M = \mathrm{id}_R(k, M).$$

Proof. Clearly $\mathrm{inj\,dim} M \geq \mathrm{id}_R(k, M)$. For the opposite inequality, let $t = \mathrm{id}_R(k, M)$. Repeated applications of Proposition 6.2 show $\mathrm{Ext}_R^i(R/\mathfrak{p}, M) = 0$ for all $\mathfrak{p} \in V(\mathfrak{a})$ and all $i > t$. This gives the desired inequality. □

The proof of the following corollary is standard and we include it here for completeness.

Corollary 6.4. *Let $(R, \mathfrak{m}, k)$ be a local ring and M be an $\mathfrak{a}$-cofinite R-module.*

(1) *If $x \in \mathfrak{m}$ is R-regular and M-regular, then*

$$\mathrm{inj\,dim}_{(\mathfrak{a}+(x))/(x)} M/xM = \mathrm{inj\,dim}_{\mathfrak{a}} M - 1,$$

(2) $\dim M \leq \mathrm{inj\,dim}_{\mathfrak{a}} M$.

Proof. (1) Consider the exact sequence

$$0 \longrightarrow M \xrightarrow{x\cdot} M \longrightarrow M/xM \longrightarrow 0.$$

By [20, Remark(a)], M/xM is an $\mathfrak{a}$-cofinite R-module and hence M/xM is an $(\mathfrak{a}+(x))/(x)$-cofinite $R/(x)$-module, by [8, Proposition 2]. Therefore the assertion follows from [5, Lemma 3.1.16] and Theorem 6.3.

(2) Let $d = \dim M$ and consider a chain $\mathfrak{p}_0 \subsetneq \mathfrak{p}_1 \subsetneq \cdots \subsetneq \mathfrak{p}_d = \mathfrak{m}$ of prime ideals in $\mathrm{Supp} M$ where for all i there is no prime ideal strictly between $\mathfrak{p}_i$ and $\mathfrak{p}_{i+1}$. By induction on i, we show that $\mathrm{Ext}^i_{R_{\mathfrak{p}_i}}(R_{\mathfrak{p}_i}/\mathfrak{p}_i R_{\mathfrak{p}_i}, M_{\mathfrak{p}_i}) \neq 0$. If $i = 0$, then $\mathfrak{p}_0 R_{\mathfrak{p}_0} \in \mathrm{Ass}_{R_{\mathfrak{p}_0}} M_{\mathfrak{p}_0}$, and therefore $\mathrm{Hom}_{R_{\mathfrak{p}_0}}(R_{\mathfrak{p}_0}/\mathfrak{p}_0 R_{\mathfrak{p}_0}, M_{\mathfrak{p}_0}) \neq 0$. Now suppose $i \geq 1$. Then by [23, Theorem 9.50] and the induction hypothesis,

$$\begin{aligned}
(\mathrm{Ext}^{i-1}_{R_{\mathfrak{p}_i}}(R_{\mathfrak{p}_i}/\mathfrak{p}_{i-1}R_{\mathfrak{p}_i}, M_{\mathfrak{p}_i}))_{\mathfrak{p}_{i-1}R_{\mathfrak{p}_i}} &\cong ((\mathrm{Ext}^{i-1}_R(R/\mathfrak{p}_{i-1}, M))_{\mathfrak{p}_i})_{\mathfrak{p}_{i-1}R_{\mathfrak{p}_i}} \\
&\cong \mathrm{Ext}^{i-1}_R(R/\mathfrak{p}_{i-1}, M)_{\mathfrak{p}_{i-1}} \\
&\cong \mathrm{Ext}^{i-1}_{R_{\mathfrak{p}_{i-1}}}(R_{\mathfrak{p}_{i-1}}/\mathfrak{p}_{i-1}R_{\mathfrak{p}_{i-1}}, M_{\mathfrak{p}_{i-1}}) \\
&\neq 0,
\end{aligned}$$

and so $\mathrm{Ext}^{i-1}_{R_{\mathfrak{p}_i}}(R_{\mathfrak{p}_i}/\mathfrak{p}_{i-1}R_{\mathfrak{p}_i}, M_{\mathfrak{p}_i}) \neq 0$. It follows from [20, Proposition 1.5] that $M_{\mathfrak{p}_i}$ is an $\mathfrak{a}R_{\mathfrak{p}_i}$-cofinite $R_{\mathfrak{p}_i}$-module. Therefore, by Proposition 6.2, we have that

$$\mathrm{Ext}^i_{R_{\mathfrak{p}_i}}(R_{\mathfrak{p}_i}/\mathfrak{p}_i R_{\mathfrak{p}_i}, M_{\mathfrak{p}_i}) \neq 0.$$

In particular, it follows that $\mathrm{Ext}^d_{R_{\mathfrak{p}_d}}(R_{\mathfrak{p}_d}/\mathfrak{p}_d R_{\mathfrak{p}_d}, M_{\mathfrak{p}_d}) \neq 0$ and so $\mathrm{Ext}^d_R(R/\mathfrak{p}_d, M) \neq 0$, by [23, Theorem 9.50]. Thus $d \leq \mathrm{inj}\dim_{\mathfrak{a}} M$, and we are done. □

The following theorem is our second main result of this section which generalizes the classical Bass formula.

Theorem 6.5. *Let $(R, \mathfrak{m}, k)$ be a local ring and M be a nonzero $\mathfrak{a}$-cofinite R-module of finite injective dimension. Then* $\mathrm{inj}\dim_{\mathfrak{a}} M = \mathrm{depth} R - \mathrm{width} M$. *In particular, if M is finite then* $\mathrm{inj}\dim_{\mathfrak{a}} M = \mathrm{depth} R$.

Proof. We have $\mathrm{id}_R(k, M) = \mathrm{depth} R - \mathrm{width} M$, by [7, Proposition 2]. Now the desired result follows from Theorem 6.3. □

Ischebeck [15, p. 517] proved the following formula from which the classical Bass formula can be recovered by setting N equal to the residue field of the base ring.

Theorem. Let $(R, \mathfrak{m}, k)$ be a local ring and let M and N be nonzero finitely generated R-modules with $\mathrm{inj}\dim M < \infty$. Then

$$\mathrm{id}_R(N, M) = \mathrm{depth} R - \mathrm{depth} N.$$

The next theorem is a generalization of Ischebeck's result.

Theorem 6.6. *Let* $(R, \mathfrak{m}, k)$ *be a local ring and* M *be a nonzero* $\mathfrak{a}$*-cofinite* R*-module of finite injective dimension and* N *be an arbitrary* R*-module. Then*

$$q = \mathrm{id}_R(N, M) \geq \mathrm{depth} R - \mathrm{width} M - \mathrm{depth} N$$

with equality if and only if $\mathrm{width}\mathrm{Ext}^q_R(N, M) = 0$*. In particular, the equality holds for any finite* R*-module* N*.*

Proof. By [26, Lemma 2.5 and 2.6(a)], we have $\mathrm{id}_R(N, M) \geq \mathrm{id}_R(k, M) - \mathrm{depth} N$ with equality if and only if $\mathrm{width}\mathrm{Ext}^q_R(N, M) = 0$. Now the desired result follows from Theorem 6.3 and Theorem 6.5. □

Acknowledgment. The author is deeply grateful to the referees for their valuable comments and suggestions.

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