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GENERALIZATIONS OF INJECTIVE MODULES

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Abstract. Let R be a ring with identity. Given a positive integer n, a unitary right R-module X is called *n–injective* provided, for every n-generated right ideal A of R, every R-homomorphism $\varphi : A \to X$ can be lifted to R. In this note we investigate this and related injectivity conditions and show that there are many rings R which have an n–injective module which is not $(n+1)$–injective.

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1. Introduction

In this paper all rings have an identity element and all modules are unitary right modules, unless stated otherwise. Let R be a ring. Recall that the Injective Test Lemma (see [1, 18.3]) states that an R-module X is injective if and only if for each right ideal E of R, every R-homomorphism $\varphi : E \to X$ can be lifted to R, equivalently, there exists $x \in X$ such that $\varphi(e) = xe\,(e \in E)$. Given a positive integer n, following [7, p. 103] (see also [10]), we call an R-module X *n–injective* provided, for each n-generated right ideal A of R, every homomorphism $\theta : A \to X$ lifts to R. Note that in [7], 1-injective modules are also called *principally injective* or simply *P-injective*. For information about n-injective modules see, for example, [8], [9], [10] and [11]. In addition, an R-module X is called *F–injective* if, for each finitely generated right ideal B of R, every homomorphism $\chi : B \to X$ lifts to R. Clearly a module is F–injective if and only if it is n-injective for every positive integer n. Next an R-module X will be called *C–injective* provided, for each countably generated right ideal C of R every homomorphism $\mu : C \to X$ can be lifted to R. It is clear that the following implications hold for a module X:

$$X \text{ is injective} \Rightarrow X \text{ is C–injective} \Rightarrow X \text{ is F–injective} \Rightarrow X \text{ is } n\text{–injective},$$

and

$$X \text{ is } (n+1)\text{–injective} \Rightarrow X \text{ is } n\text{–injective},$$

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for every positive integer n.

Note the following simple fact.

Lemma 1.1. *Let R be a ring, let X be an R-module, let G be a finitely generated submodule of a free R-module F and let $\varphi : G \to X$ be a homomorphism. Then φ lifts to F if and only if φ lifts to H for every finitely generated (free) submodule H of F containing G.*

Proof. The necessity is clear. Conversely, suppose that φ lifts to H for every finitely generated free submodule H of F containing G. Because G is finitely generated there exists a finite subset of any basis of F such that every generator can be written in terms of this finite subset. In other words, there exist free submodules F_1 and F_2 of F such that $F_1 \cap F_2 = 0$, $F = F_1 \oplus F_2$, F_1 is finitely generated and $G \subseteq F_1$. By hypothesis, φ lifts to F_1 and hence also to F. □

Following [7, p. 110], a module M over a ring R is called *finitely presented* provided there exists a finitely generated free R-module F and a finitely generated submodule K of F such that $M \cong F/K$. In addition, an R-module X is called *FP–injective* (or *absolutely pure*) if, for every finitely generated free R-module F and finitely generated submodule K of F, every homomorphism $\varphi : K \to X$ can be lifted to F. (Note that Lemma 1.1 gives that F need not be finitely generated in the definition of an FP–injective module.) It is proved in [7, Theorem 5.39] that an R-module X is FP–injective if and only if for every R-module M and submodule L of M such that the module M/L is finitely presented, every homomorphism $\alpha : L \to X$ can be lifted to M. Clearly the following implications hold for a module X:

$$X \text{ is injective} \Rightarrow X \text{ is FP–injective} \Rightarrow X \text{ is F–injective.}$$

Let n be a positive integer. We shall call a module X over a ring R *nP–injective* provided for every free R-module F and n-generated submodule G of F, every homomorphism $\varphi : G \to X$ can be lifted to F. Clearly a module is FP–injective if and only if it is nP –injective for every positive integer n. Moreover, for any module X we have the implications:

$$X \text{ is FP–injective} \Rightarrow X \text{ is } (n{+}1)\text{P-injective} \Rightarrow X \text{ is } n\text{P–injective,}$$

and

$$X \text{ is } n\text{P–injective} \Rightarrow X \text{ is } n\text{-injective,}$$

for every positive integer n.

The next result contains elementary facts that are proved by standard techniques.

Proposition 1.2. *Let R be any ring and n any positive integer. Then*

(i) *Every direct summand of a C–injective (respectively, FP–injective, nP–injective, F–injective, n–injective) R-module is C–injective (respectively, FP–injective, nP–injective, F–injective, n–injective).*

(ii) *Every direct product of C–injective (respectively, FP–injective, nP–injective, F–injective, n–injective) R-modules is C–injective (respectively, FP–injective, nP–injective, F–injective, n–injective).*

(iii) *Every direct sum of FP–injective (respectively, nP–injective, F–injective, n–injective) R-modules is FP–injective (respectively, nP–injective, F–injective, n-injective).*

Corollary 1.3. *The following statements are equivalent for a ring R and a positive integer n.*

(i) *R is right FP–injective (respectively, nP–injective, F–injective, n–injective).*

(ii) *Every projective right R-module is FP–injective (respectively, nP–injective, F–injective, n–injective).*

Proof. By Proposition 1.2. □

Lemma 1.4. *Let R be a ring and n any positive integer. Then*

(a) *An R-module X is n-injective if and only if for every n-generated R-module M such that there exists a monomorphism $\alpha : M \to R$ and every homomorphism $\varphi : M \to X$ there exists a homomorphism $\theta : R \to X$ such that $\varphi = \theta\alpha$.*

(b) *An R-module Y is nP-injective if and only if for every n-generated R-module N such that there exists a monomorphism $\lambda : N \to F$, for some free R-module F, and every homomorphism $\mu : N \to X$ there exists a homomorphism $\nu : F \to X$ such that $\mu = \nu\lambda$.*

Proof. Straightforward. □

Next note the following simple facts.

Lemma 1.5. *Let R be a ring and X an R-module. Then*

(a) *X is n-injective, for some positive integer n, if and only if for all $a_i \in R\,(1 \leq i \leq n)$ and every homomorphism $\varphi : \sum_{i=1}^{n} a_iR \to X$ there exists $x \in X$ such that $\varphi(a_i) = xa_i\,(1 \leq i \leq n)$.*

(b) *X is C-injective if and only if for all $a_i \in R\,(i \in \mathbb{N})$ and every homomorphism $\varphi : \sum_{i\in\mathbb{N}} a_iR \to X$ there exists $x \in X$ such that $\varphi(a_i) = xa_i\,(i \in \mathbb{N})$.*

Proof. Elementary. □

Given a non-empty subset T of a ring R, $\mathbf{r}(T)$ will denote the set of elements $r \in R$ such that $tr = 0$ for all $t \in T$. In case $T = \{t\}$, for some element $t \in R$, we write $\mathbf{r}(T)$ simply as $\mathbf{r}(t)$. Note that $\mathbf{r}(T)$ is a right ideal of R for every non-empty subset T of R. Let M be an R-module. Then $ann_M(T)$ will denote the set of elements $m \in M$ such that $mt = 0$ for all $t \in T$. Note that $ann_M(T)$ is a subgroup of the Abelian group $(M, +)$. If a is an element of R then we shall denote by Ma the set of elements of the form $ma \, (m \in M)$ of M. Note the following result (see [10, Corollary 2.3]).

Lemma 1.6. *A module X over a ring R is 1-injective if and only if $Xa = ann_X(\boldsymbol{r}(a))$ for all $a \in R$.*

Combining Lemma 1.6 with [6, Theorem 3.3] we have the following result.

Proposition 1.7. *Let R be a semiprime right Goldie ring. Then every torsion-free 1-injective R-module is injective.*

A ring R is called *right semihereditary* provided every finitely generated right ideal is projective. Following [12], given a positive integer n, a ring R will be called *right n-semihereditary* in case every n-generated right ideal is projective. Clearly a ring R is right semihereditary if and only if R is right n-semihereditary for every positive integer n. It is also clear that every right $(n+1)$-semihereditary ring is right n-semihereditary for every positive integer n. Camillo [3] proved that if a commutative ring R is 2-semihereditary then R is semihereditary. Later, for every positive integer n, we shall give examples of rings that are right n-semihereditary but not right $(n+1)$-semihereditary. Note the following fact. The proof is standard but we include it for completeness.

Lemma 1.8. *Let R be a right n-semihereditary ring and let F be a non-zero free R-module with basis $f_1, \dots, f_k$, for some positive integer k. Let M be any n-generated submodule of F. Then there exist n-generated right ideals $A_i \, (1 \leq i \leq k)$ of R such that $M \cong A_1 \oplus \cdots \oplus A_k \cong f_1 A_1 \oplus \cdots \oplus f_k A_k$. Moreover the R-module M is projective.*

Proof. If $k = 1$ then there is nothing to prove. Suppose that $k \geq 2$. Let $\pi : F \to f_k R$ denote the canonical projection. Then $\pi(M) = f_k A_k$ for some n-generated right ideal A_k of R and hence is projective by assumption. It follows that there exists a submodule K of M such that $K \cong f_k A_k$ and $M = (M \cap G) \oplus K$ where

G is the free R-module $f_1R \oplus \cdots \oplus f_{k-1}R$. By induction on k, the n-generated submodule $M \cap G$ of the free module G is isomorphic to $f_1A_1 \oplus \cdots \oplus f_{k-1}A_{k-1}$, for some n-generated right ideals $A_i\,(1 \leq i \leq k-1)$, and is projective. Thus $M \cong f_1A_1 \oplus \cdots \oplus f_kA_k$. Clearly $M \cong A_1 \oplus \cdots \oplus A_k$ and is projective. □

Corollary 1.9. *Let n be a positive integer. Then a ring R is right n-semihereditary if and only if every n-generated submodule of every free right R-module is isomorphic to a direct sum of n-generated right ideals of R and is projective.*

Proof. By Lemma 1.8. □

Corollary 1.10. *Let n be a positive integer and let R be a right n-semihereditary ring. Then a right R-module X is n–injective if and only if it is nP–injective.*

Proof. The sufficiency is clear. Conversely, suppose that X is n-injective. Let G be any n-generated submodule of a non-zero free R-module F. By Lemma 1.1 we can suppose without loss of generality that F is finitely generated. Let $f_1, \ldots, f_k$ be a basis of F, for some positive integer k. By Lemmas 1.4 and 1.8 we can suppose without loss of generality that $G = f_1G_1 \oplus \cdots \oplus f_kG_k$ for some n-generated right ideals $G_i\,(1 \leq i \leq k)$ of R. Let $\varphi : G \to X$ be any homomorphism. For each $1 \leq i \leq k$, φ induces a homomorphism $\varphi_i : f_iG_i \to X$ which lifts to a homomorphism $\theta_i : f_iR \to X$, because X is n-injective. Thus the mapping $\theta : F \to X$ defined by $\theta(f_1r_1 + \cdots + f_kr_k) = \theta_1(f_1r_1) + \cdots + \theta_k(f_kr_k)$ for all $r_i \in R\,(1 \leq i \leq k)$ lifts φ to F. It follows that X is nP-injective. □

Corollary 1.11. *Let R be a right semihereditary ring. Then a right R-module X is F–injective if and only if it is FP–injective.*

Proof. By Corollary 1.10. □

2. 1-injective Modules

In this section we shall consider some properties of 1–injective modules. The first result generalizes [7, Lemma 5.1].

Theorem 2.1. *Let R be any ring. Then the following statements are equivalent for an R-module X.*

(i) *X_R is 1–injective.*
(ii) *$x \in Xa$ for all $a \in R$, $x \in X$ with $\boldsymbol{r}(a) \subseteq ann_R(x)$.*
(iii) *$ann_X(bR \cap \boldsymbol{r}(a)) = ann_X(b) + Xa$ for all $a, b \in R$.*

Proof. (i) $\Rightarrow$ (ii) Suppose that $\mathbf{r}(a) \subseteq ann_R(x)$ for some $a \in R$, $x \in X$. Then $x\mathbf{r}(a) = 0$ and hence $x \in ann_X(\mathbf{r}(a)) = Xa$, by Lemma 1.6.

(ii) $\Rightarrow$ (iii) Let $a, b \in R$. Clearly $ann_X(b) + Xa \subseteq ann_X(bR \cap \mathbf{r}(a))$. Let $x \in ann_X(bR \cap \mathbf{r}(a))$. Note that $\mathbf{r}(ab) \subseteq ann_R(xb)$ and that (ii) gives that $xb = x'ab$ for some $x' \in X$. It follows that $x - x'a \in ann_X(b)$ and therefore $x \in ann_X(b) + Xa$.

(iii) $\Rightarrow$ (i) By (iii) with $b = 1$ and by Lemma 1.6, X is 1-injective. □

It is clear that if a and b are elements of a ring R and X is a faithful R-module such that $Xb \subseteq Xa$ then $\mathbf{r}(a) \subseteq \mathbf{r}(b)$. Now note the following immediate consequence of Lemma 1.6.

Corollary 2.2. *Let a and b be elements of a ring R such that $\mathbf{r}(a) \subseteq \mathbf{r}(b)$. Then $Xb \subseteq Xa$ for every 1-injective right R-module X.*

Compare the next result with [7, Proposition 5.9].

Corollary 2.3. *Let S and R be rings and let X be a left S-, right R-bimodule such that the right R-module X is 1–injective and let a and b be elements of R. Then for any homomorphism $\alpha : bR \to aR$ there exists an S-homomorphism $\varphi : Xa \to Xb$ such that*

(i) *α is a monomorphism implies that φ is an epimorphism,*
(ii) *α is an epimorphism implies that φ is a monomorphism, and*
(iii) *α is an isomorphism implies that φ is an isomorphism.*

Proof. Let $\alpha : bR \to aR$ be any homomorphism. There exists an element $c \in R$ such that $\alpha(b) = ac$. By Lemma 1.6 $Xac \subseteq Xb$. Then define a mapping $\varphi : Xa \to Xb$ by $\varphi(xa) = xac$ $(x \in X)$. It is easy to check that φ is an S-homomorphism from the left S-module Xa to the left S-module Xb.

(i) Suppose that α is a monomorphism. Then $\mathbf{r}(b) = \mathbf{r}(ac)$. By Corollary 2.2, $Xac = Xb$ and hence $\varphi : Xa \to Xb$ is an epimorphism.

(ii) Suppose that α is an epimorphism. Then $a = acd$ for some element $d \in R$. Clearly this implies that φ is a monomorphism.

(iii) By (i), (ii). □

Theorem 2.4. *Let R be a commutative ring. Then every 1-injective simple R-module is injective.*

Proof. Let U be any 1-injective simple R-module. Let A be an ideal of R and $\varphi : A \to U$ be a non-zero homomorphism. There exists $a \in A$ such that $\varphi(a) \neq 0$. Because U is 1-injective, the homomorphism $\varphi|_{aR} : aR \to U$ lifts to R and hence

$\varphi(a) = ua$ for some $u \in U$. Let $P = \text{ann}_R(u) = \text{ann}_R(U)$ which is a maximal ideal of R. Note that $a \notin P$ and hence $R = A + P$. Now

$$A \cap P = (A \cap P)A + (A \cap P)P = AP \subseteq \ker\varphi,$$

because $A/\ker\varphi \cong U$. Define a mapping $\alpha : R \to U$ by $\alpha(b + p) = \varphi(b)$ for all $b \in A, p \in P$. Note that α is well defined because $b + p = 0$ implies that $b = -p \in A \cap P \subseteq \ker\varphi$ which gives that $\varphi(b) = 0$. Thus α is a homomorphism which lifts φ to R. Therefore U_R is injective. □

We do not know if Theorem 2.4 is true without the hypothesis of R being a commutative ring.

3. Modules Over Certain Subrings

Let R be a ring and let e be any idempotent element of R. Note that eRe is a subring of R with identity element e. (Note that we do not insist that subrings of rings have the same identity element.) Given any right R-module M it is clear that Me is a unitary right module over the ring eRe. In [7, Proposition 5.35] it is proved that if a ring R is right P-injective then so too is any subring of the form eRe where e is an idempotent such that $R = ReR$. We shall generalize this result.

Theorem 3.1. *Let e be an idempotent in a ring R such that $R = ReR$, let S denote the subring eRe of R and let X be an n–injective (respectively, nP–injective) right R-module, for some positive integer n. Then the right S-module Xe is n–injective (respectively, nP–injective).*

Proof. Suppose first that X is nP–injective. There exist a positive integer k and elements p_i, $q_i \in R\,(1 \le i \le k)$ such that $1 = \sum_{i=1}^{k} p_i e q_i$. Let L be any n-generated submodule of the free S-module $S_S^{(m)}$, for some positive integer m, and let $\varphi : L \to Xe$ be any S-homomorphism. Note that $S_S^{(m)}$ is an S-submodule of the free R-module $R_R^{(m)}$. There exist elements $a_j \in L\,(1 \le j \le n)$ such that $L = a_1S + \cdots + a_nS$. Note that in this case the submodule LR of $R_R^{(m)}$ satisfies $LR = a_1R + \cdots + a_nR$. Now define a mapping $\overline{\varphi} : LR \to X$ by $\overline{\varphi}(\sum_{i=1}^{n} a_i r_i) = \sum_{i=1}^{n} \varphi(a_i) r_i$ for all $r_i \in R\,(1 \le i \le n)$. Suppose that $\sum_{i=1}^{n} a_i r_i = 0$, for some $r_i \in R\,(1 \le i \le n)$. Then

$$\sum_{i=1}^{n} \varphi(a_i) r_i = \sum_{i=1}^{n} \varphi(a_i e) r_i \sum_{j=1}^{k} p_j e q_j = \sum_{i=1}^{n} \sum_{j=1}^{k} \varphi(a_i) e r_i p_j e q_j =$$

$$\sum_{i=1}^{n} \sum_{j=1}^{k} \varphi(a_i e r_i p_j e) q_j = \sum_{j=1}^{k} \varphi(\sum_{i=1}^{n} a_i r_i p_j e) q_j = \sum_{j=1}^{k} \varphi(0 p_j e) q_j = 0,$$

so that $\overline{\varphi}$ is well-defined. It is easy to check that $\overline{\varphi}$ is an R-homomorphism.

Because X is nP–injective, $\overline{\varphi}$ can be lifted to an R-homomorphism $\theta : R_R^{(m)} \to X$. Note that, for each element $s \in S_S^{(m)}$, $\theta(s) = \theta(se) = \theta(s)e \in Xe$. Let $\chi : S_S^{(m)} \to Xe$ be the mapping defined by $\chi(s) = \theta(s)$ for all $s \in S_S^{(m)}$ and note that χ is an S-homomorphism. Moreover, for each $1 \leq i \leq n$, $\chi(a_i) = \theta(a_i) = \overline{\varphi}(a_i) = \varphi(a_i)$ and hence $\chi(b) = \varphi(b)$ for all $b \in L$. It follows that the S-module Xe is nP–injective.

Now suppose that X is an n-injective R-module. Then the above proof with $m = 1$ gives that the S-module Xe is n-injective. □

Corollary 3.2. *Let e be an idempotent in a ring R such that $R = ReR$ and let S denote the subring eRe of R. Let X be an F–injective (respectively, FP–injective) right R-module for some positive integer n. Then the right S-module Xe is F–injective (respectively, FP–injective).*

Proof. By Theorem 3.1. □

By adapting the proof of Theorem 3.1 we have the following result.

Proposition 3.3. *Let e be an idempotent in a ring R such that $R = ReR$ and let S denote the subring eRe of R. Let X be a C–injective right R-module for some positive integer n. Then the right S-module Xe is C–injective.*

Let R be a ring and n a positive integer. Again we consider a subring S of R of the form eRe for some idempotent e in R such that $R = ReR$. It might be tempting to think that if Y is an n–injective right S-module then the right R-module $Y \otimes_S R$ is also n–injective but this is not the case, as we shall show in the next section.

4. Examples

Note that for any ring R every direct sum $\oplus_{i\in I} X_i$ of injective R-modules is FP–injective and hence also F–injective. In fact, more is true, namely if N is any finitely generated submodule of an arbitrary R-module M then every homomorphism $\varphi : N \to X$, where X denotes the module $\oplus_{i\in I} X_i$, lifts to M. For, in this case, there exists a finite subset J of I such that $\varphi(N) \subseteq \oplus_{j\in J} X_j$ which is an injective module. It follows that φ lifts to M. For any module U let $\mathrm{E}(U)$ denote the injective envelope of U. The following result is essentially [1, Proposition 18.13] but we include a proof for completeness.

Lemma 4.1. *The following statements are equivalent for a ring R.*

(i) *R is right Noetherian.*
(ii) *Every direct sum of C–injective R-modules is C–injective.*
(iii) *Every direct sum of injective R-modules is C–injective.*

Proof. (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii) Clear.

(iii) $\Rightarrow$ (i) Suppose that R is not right Noetherian. Let $A_1 \subset A_2 \subset \dots$ be any properly ascending chain of right ideals of R. For every positive integer n let $a_n \in A_{n+1} \backslash A_n$. Let $A = a_1R + a_2R + \dots$. Define a mapping $\varphi : A \to \oplus_{n\geq 1} \mathrm{E}(R/A_n)$ by

$$\varphi(r) = (r + A_1, r + A_2, \dots)\ (r \in A).$$

Note that φ is well-defined because $A \subseteq \cup_{n\geq 1} A_n$. If φ lifts to R then the proof of [1, Proposition 18.13 (a) $\Rightarrow$ (c)] can be modified to show that $a_m \in A_m$ for some positive integer m, a contradiction. Thus φ does not lift to R. It follows that the module $\oplus_{n\geq 1} \mathrm{E}(R/A_n)$ is not C–injective. $\square$

It is easy to give examples of FP–injective (and hence also F–injective) modules which are not C–injective. Let R be a ring which is not right Noetherian. By Lemma 4.1 there exists a direct sum X of injective R-modules which is not C–injective and the above remarks show that X is FP–injective.

In [8], it is proved that if R is a domain such that every one-sided ideal is two-sided then the following statements are equivalent:

(i) R is semihereditary.
(ii) Every 1-injective right R-module is 2-injective.
(iii) Every 1-injective right R-module is FP-injective.

The above result was generalized by Tuganbaev [11, Theorem 1] to rings R which, instead of being a domain, are either right or left 1-semihereditary. Thus if R is a commutative domain which is not Prüfer (i.e. not semihereditary) then there exists a 1-injective R-module which is not 2-injective.

Recall that a ring R is right self-injective in case the module R_R is injective. Now we shall call a ring R a *right P-injective ring* if R_R is 1–injective. In addition, for any positive integer $n \geq 2$ we shall call a ring R *right n–injective* provided R_R is n-injective. We shall use "right P–injective" instead of "right 1-injective" to be consistent with the usual terminology in the literature. The next example is essentially due to Björk [2].

Example 4.2. *Let F be a field such that there exists an isomorphism $a \to \overline{a}$ from F to a proper subfield $\overline{F}$ of F. Let n be any integer with $n \geq 2$. Let R denote the left vector space over F with basis $\{1, t, \dots, t^{n-1}\}$ and make R into an F-algebra by defining $t^n = 0$ and $ta = \overline{a}t\,(a \in F)$. Then*

(i) *The Jacobson radical J of R is given by $J = Rt$.*
(ii) *$R/J \cong F$.*

(iii) *The only left ideals of R are $R \supset J \supset J^2 \supset \cdots \supset J^{n-1} \supset 0$.*
(iv) *R is right P–injective but not right 2-injective.*
(v) *R is not left P–injective.*

Proof. See [7, Example 2.5]. □

Let R be the ring in Example 4.2 and let A be the ring of 2×2 matrices with entries in R. If e is the matrix

$$\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix},$$

then e is an idempotent in A such that $A = AeA$. Moreover $B = eAe$ is the subring of A consisting of all matrices of the form

$$\begin{bmatrix} r & 0 \\ 0 & 0 \end{bmatrix},$$

for all $r \in R$, and B is isomorphic to R. By Example 4.2, the right B-module B is 1–injective. Moreover, $B \otimes_B A \cong A$ as right A-modules. However the right A-module A is not 1–injective as Nicholson and Yousif point out (see [7, Proposition 5.36 and Example 5.37]).

In view of Björk's example we ask the following question:
If n is any positive integer does there exist a ring R such that R is right n–injective but not right $(n{+}1)$–injective?

When we pass to non-commutative rings it turns out that, for any positive integer n, there exist rings R and n–injective R-modules which are not $(n{+}1)$–injective and hence not F-injective. To see why this is the case we first note the following fact which is due to Tuganbaev [11, Lemma 1].

Lemma 4.3. *Let n be a positive integer. Then a ring R is right n-semihereditary if and only if every homomorphic image of every n–injective right R-module is n–injective.*

Corollary 4.4. *Let n be a positive integer and let R be a right n-semihereditary ring such that every n-injective R-module is $(n{+}1)$-injective. Then R is right $(n{+}1)$-semihereditary.*

Proof. Let X be any $(n{+}1)$–injective R-module and let Y be any submodule of X. Clearly X is n–injective and hence so too is X/Y by Lemma 4.3. By hypothesis, X/Y is $(n{+}1)$–injective. Thus every homomorphic image of an $(n{+}1)$–injective R-module is $(n{+}1)$-injective. Again applying Lemma 4.3 we conclude that R is right $(n{+}1)$-semihereditary. □

Proposition 4.5. *Let R be any ring and let n be a positive integer.*

(a) *Let A be an n-generated right ideal of R such that for some free R-module F and submodule K of F with $A \cong F/K$ the module $E(F)/K$ is n–injective. Then the right R-module A is projective*

(b) *Let M be an n-generated submodule of a projective R-module P such that for some free R-module G and submodule L of G with $M \cong G/L$ the module $E(G)/L$ is nP–injective. Then the R-module M is projective.*

Proof. We shall prove statement (b); the proof of (a) is similar. Let $\alpha : M \to G/L$ be an isomorphism. Let $\iota_1 : M \to P$ and $\iota_2 : G/L \to \mathrm{E}(G)/L$ denote the inclusion mappings and let $\pi_1 : G \to G/L$ and $\pi_2 : \mathrm{E}(G) \to \mathrm{E}(G)/L$ denote the canonical projections. Because $\mathrm{E}(G)/L$ is nP–injective, there exists a homomorphism $\beta : P \to \mathrm{E}(G)/L$ such that $\beta\iota_1 = \iota_2\alpha$. Next P projective implies that there exists a homomorphism $\gamma : P \to \mathrm{E}(G)$ such that $\beta = \pi_2\gamma$. Note that

$$\pi_2\gamma\iota_1 = \beta\iota_1 = \iota_2\alpha,$$

and hence $\gamma\iota_1(M) \subseteq G$. Let $\delta : M \to G$ be the homomorphism defined by $\delta(m) = \gamma\iota_1(m)$ for all $m \in M$. For each $g \in G$ there exists $m \in M$ such that $g + L = \alpha(m) = \delta(m) + L$. It follows that $G = L + \delta(M)$. Moreover, if $m_1 \in L \cap \delta(M)$ then $m_1 = \delta(m_2) \in L$ and hence $\alpha(m_2) = \pi_1\delta(m_2) = 0$. This implies that $m_2 = 0$ and hence $m_1 = 0$. Thus $L \cap \delta(M) = 0$ and $G = L \oplus \delta(M)$. It follows that M is projective. □

Combining these facts together we have the following result.

Theorem 4.6. *Let R be a ring such that R is right n-semihereditary but not right $(n+1)$-semihereditary, for some positive integer n. Let A be any $(n+1)$-generated right ideal of R such that A is not a projective R-module and $A \cong F/K$ for some free R-module F and submodule K of F. Then the R-module $E(F)/K$ is nP-injective but not $(n+1)$-injective.*

Proof. By Proposition 4.5, the module $Y = \mathrm{E}(F)/K$ is not $(n+1)$–injective. However, Y is an n–injective module by Lemma 4.3. Moreover, by Corollary 1.10, Y is an nP–injective module. □

In view of Theorem 4.6 to find examples of n-injective (even, nP-injective) modules which are not $(n+1)$-injective it is sufficient to find rings R which are right n-semihereditary but not right $(n+1)$-semihereditary and this we do next. First we shall show that for every field F and positive integer n there exists an algebra

R over F which is right n-semihereditary but not right $(n+1)$-semihereditary and then we shall show how to use such a ring to produce others of the same type.

Lemma 4.7. *For every field F and positive integer n there exists an F-algebra A which is a right n-semihereditary domain but is not right $(n+1)$-semihereditary.*

Proof. Let F be any field and let n be any positive integer. Let A denote the F-algebra on the $2(n+1)$ generators x_i, y_i $(1 \leq i \leq n+1)$ subject to the relation

$$\sum_{i=1}^{n+1} x_i y_i = 0.$$

It is proved in [5, Theorem 2.3] that A is a right n-semihereditary domain (in fact, every n-generated right or left ideal is free) but A is not a right $(n+1)$-semihereditary ring. □

Before we proceed we prove an elementary result whose proof is given for completeness.

Lemma 4.8. *Let e be an idempotent of a ring R such that $eR(1-e) = 0$ and let T be the subring eRe of R. Let X be a right R-module such that $X(1-e) = 0$ and the right T-module Xe is projective. Then the right R-module X is projective.*

Proof. Note that $T = eR$ and hence T is a projective right R-module. Note also that $X = Xe$. Because X_T is projective, there exist an index set I and a T-epimorphism $\pi : T^{(I)} \to X$ such that $\pi = \pi^2$. Note that for all $u \in T^{(I)}, r \in R$, we have:

$$\pi(ur) = \pi((ue)r) = \pi(u(er)) = \pi(u)(er) = (\pi(u)e)r = \pi(u)r.$$

Thus π is an idempotent R-homomorphism. It follows that X is a direct summand of the projective R-module $T^{(I)}$ and hence X is a projective R-module. □

Let S and T be rings and let M be a left S-, right T-bimodule. Then $[s, m : 0, t]$ will denote the "matrix"

$$\begin{bmatrix} s & m \\ 0 & t \end{bmatrix},$$

with $s \in S$, $t \in T$ and $m \in M$. The collection of all such matrices will be denoted by $[S, M : 0, T]$ and forms a ring with respect to matrix addition and multiplication in the usual way. Using Lemma 4.7 the next result can be used to produce many examples of the required type.

Theorem 4.9. *Let F be a field and let n be any positive integer. Let T be an algebra over F such that T is right n-semihereditary but not right $(n+1)$-semihereditary and let P be any submodule of a free right T-module. Then the F-algebra $R = [F, P : 0, T]$ is right n-semihereditary but not right $(n+1)$-semihereditary.*

Proof. Let A be any n-generated right ideal of R. Let e be the idempotent element [1,0:0,0] of R. Note that $1 - e = [0, 0 : 0, 1]$ is an idempotent in R, $(1 - e)Re = 0$, $(1-e)R(1-e)$ is the subring of R consisting of all matrices of the form $[0, 0 : 0, t]$ $(t \in T)$ and that $(1-e)R(1-e) \cong T$. Next, $eR = [F, P : 0, 0]$ and $(1-e)R = [0, 0 : 0, T]$ are both projective right ideals of R. Suppose that there exists an element $[f, p : 0, t]$ in A with $f \neq 0$. Then $A = eR \oplus B$ where $B = [0, 0 : 0, C]$ for some (clearly) n-generated right ideal C of T. By hypothesis, C is a projective right T-module. By Lemma 4.5, B_R is a projective R-module, and hence so too is A_R. Otherwise $A = [0, p_1 : 0, t_1]R + \cdots + [0, p_n : 0, t_n]R$ for some $p_i \in P, t_i \in T\,(1 \leq i \leq n)$. Let N denote the T-submodule of the projective T-module $P \oplus T$ generated by the n elements $(p_i, t_i)\,(1 \leq i \leq n)$. Since $P \oplus T$, and hence also N, is a submodule of a free T-module it follows that N is a projective T-module by Lemma 1.8. As T-modules, $N \cong A$ and hence A_T is projective. Now $Ae = 0$ so that Lemma 4.5 gives that A_R is projective. Thus the ring R is right n-semihereditary.

On the other hand, there exists an $(n+1)$-generated right ideal D of T such that D_T is not projective. Let E denote the right ideal $[0, 0 : 0, D]$ of R. It is easy to check that E is an $(n+1)$-generated right ideal of R. Suppose that E_R is projective. Note that eR is an idempotent two-sided ideal of R such that $R/eR \cong T$. Moreover $Ee = 0$ so that E is a right R/eR-module. By [4, Theorem 1], E_R being projective implies that $E_{R/eR}$ is projective. But this implies that E_T is projective and hence D_T is projective. Thus E is not a projective R-module. We have proved that the ring R is not right $(n+1)$-semihereditary. □

Rings R such that there exists an n-injective R-module which is not $(n+1)$-injective need not be of the type found in Lemma 4.7 or Theorem 4.9, as we show next.

Proposition 4.10. *Let S be any ring and let n be any positive integer. Then there exists a ring R such that S is a ring direct summand of R and an n-injective right R-module X which is not $(n+1)$-injective.*

Proof. Let T be any ring which has the property that there exists an n-injective T-module X which is not $(n+1)$-injective. Let R denote the ring direct sum $S \oplus T$, where we shall think of S and T as ideals of R. We can make X into an R-module by defining $x(s+t) = xt$ for all $x \in X$, $s \in S$ and $t \in T$. Let A be any n-generated right ideal of R and let $\varphi : A \to X$ be an R-homomorphism. Then $A = B \oplus C$ for some n-generated right ideal B of S and n-generated right ideal C of T. Clearly the restriction of φ to C is a T-homomorphism from C to X and can be lifted to a T-homomorphism $\theta : T \to X$. Now define a mapping $\chi : R \to X$ by $\chi(s+t) = \theta(t)$ for all $s \in S$ and $t \in T$. Clearly χ is an R-homomorphism. Let $a \in A$. Then $a = b + c$ for some $b \in B$, $c \in C$. Note that

$$\varphi(b) \in \varphi(bS) \subseteq \varphi(b)S \subseteq XS = (XT)S = X(TS) = X0 = 0,$$

and hence $\varphi(b) = 0$. It follows that $\varphi(a) = \varphi(c) = \theta(c) = \chi(c)$. Thus χ lifts φ to R. It follows that the R-module X is n-injective.

Now suppose that the R-module X is $(n+1)$-injective. Let D be an $(n+1)$-generated right ideal of T and $\alpha : D \to X$ be a T-homomorphism. There exist elements $d_i \in D\,(1 \leq i \leq n+1)$ such that $D = d_1T + \cdots + d_{n+1}T = d_1R + \cdots + d_{n+1}R$. Since $XS = 0$ (see above) it follows that α is an R-homomorphism from the $(n+1)$-generated right ideal D of R to the R-module X. By hypothesis, α lifts to an R-homomorphism $\beta : R \to X$. But this implies that the restriction of β to T is a T-homomorphism which extends α. It follows that X_T is $(n+1)$-injective, a contradiction. Thus X_R is not $(n+1)$-injective. □

We do not know an example of a ring R and an F–injective R-module X such that X is not FP–injective (compare Corollary 1.11).

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