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ON n-COPRESENTED MODULES AND n-CO-COHERENT RINGS

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Abstract. In this paper, we introduce and study dual notions of both n-presented modules and n-coherent rings, which we call respectively n-copresented modules and n-co-coherent rings.

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1. Introduction

Throughout this paper all rings are commutative with identity element and all modules are unital.

In 1968 [15], Vamos introduced the notion of finitely embedded modules as a dual of finitely generated modules such that, for a ring R, an R-module M is called finitely embedded if there is a finite set $\{S_i, i = 1, ..., n\}$ of simple R-modules, such that $E(M) = E(S_1) \oplus E(S_2) \oplus \cdots \oplus E(S_n)$ (where $E(X)$ denotes the injective hull of the R-module X). Finitely embedded modules were called, by Jans [12], finitely cogenerated modules when he introduced co-Noetherian rings as a dual notion of Noetherian rings. Such that a ring R is called co-Noetherian if factors of finitely cogenerated R-modules are finitely cogenerated R-modules. In that paper, Jans mentioned that Vamos' property coincides with the following Pareigis' one on a module M: "for every family $\{M_i\}_{i\in I}$ of submodules of M with $\cap_{i\in I} M_i = 0$, there is a finite subset $J \subset I$ such that $\cap_{i\in J} N_i = 0$". Since then, several authors have been interested in this notion such that various characterizations of finitely cogenerated modules were given (see [1] and [16] for more details). The following result gives some of them including the two conditions above.

Theorem 1.1. *For a ring R, an R-module M is called finitely cogenerated if it satisfies one of the following equivalent conditions:*

(1) *There is a finite set $\{S_i, i = 1, ..., n\}$ of simple R-modules, such that $E(M) = E(S_1) \oplus E(S_2) \oplus \cdots \oplus E(S_n)$.*
(2) *There is a finite set $\{S_i, i = 1, ..., n\}$ of simple R-modules, such that M is isomorphic to a submodule of $E(S_1) \oplus E(S_2) \oplus \cdots \oplus E(S_n)$. In others words, M is isomorphic to a submodule of a finitely cogenerated cofree R-module, where a cofree R-module means a direct product of the injective hull of simple R-modules (see* [10]*).*
(3) *M is isomorphic to a submodule of direct product of finitely many cocyclic modules (where a cocyclic module means an essential extension of a simple module* [16, pages 115-116]*).*
(4) *For every injective homomorphism $M \to \prod_{i \in I} N_i$, where $\{N_i\}_{i \in I}$ is a family of R-modules, there is a finite subset $J \subset I$ and an injective homomorphism $M \to \prod_{i \in J} N_i$.*
(5) *For every family $\{M_i\}_{i \in I}$ of submodules of M with $\cap_{i \in I} M_i = 0$ there is a finite subset $J \subset I$ such that $\cap_{i \in J} N_i = 0$.*
(6) *The socle of M, $Soc(M)$, is finitely generated and essential in M (where the socle of M is by definition the sum of all simple (minimal) submodules of M* [16, pages 174-175]*).*

In the same way, Hiremath [10] introduced the notion of finitely copresented module as a dual of finitely presented modules such that a module M is said to be finitely copresented, if it is finitely cogenerated and for every short exact sequence $0 \to M \to L \to K \to 0$, if L is finitely cogenerated then also K is finitely cogenerated (see [16, pages 248-249]). As the classical case for coherent rings (see [11]), the notion of finitely copresented modules was served to define co-coherent rings as a dual notion of coherent rings (see [16, page 249]).

In this paper, we introduce and study dual notions of both n-presented modules and n-coherent rings which were first introduced by Costa [7] and developed by Dobbs, Mahdou and Kabbaj [8] as extensions of respectively the classical finitely generated (presented) modules and Noetherian (coherent) rings. Recall that a module is said to be n-presented, for some positive integer n, if there is an exact sequence of modules of the form $F_n \to F_{n-1} \to \cdots \to F_0 \to M \to 0$ where F_i are free and finitely generated. A ring R is said to be n-coherent, if every n-presented R-module is $(n+1)$-presented. Clearly, 0-presented and 1-presented modules are respectively the same as finitely generated and finitely presented modules. Then, 0-coherent and 1-coherent rings are respectively the same as Noetherian and coherent rings. Notice that the terminology of "n-coherence" in this paper is Costa's n-coherence but it is not the same as that of [8]. These notions have been the subject of several papers (see for example [2,5,6,9,17,18]).

In Section 2, we define and study n-copresented modules as a dual notion of n-presented modules (see Definition 2.1). The main result (Theorem 2.4) studies the behavior of this notion in short exact sequences. It is a generalization of [16, Theorem 30.2] and a dual result of a well known one on n-presented modules (see [4, Exercice 6, p. 60]). We close Section 2 with some change of rings results (Propositions 2.6 and 2.8). In Section 3, we are interested in n-co-coherent rings, a dual notion of n-coherent rings (see Definition 3.1). We show that for semi-local rings, n-co-coherence implies the classical coherence (Theorem 3.6). We close the paper with some change of rings results (Proposition 3.7 and Theorem 3.9).

2. n-Copresented modules

In this section, we investigate the notion of n-copresented modules which is defined as follows.

Definition 2.1. For a ring R and a positive integer n, an R-module M is called *n-copresented* if there is an exact sequence of R-modules of the form

$$0 \to M \to I_0 \to I_1 \to \cdots \to I_n$$

where, for $i = 0, ..., n$, I_i is injective and finitely cogenerated. If M is n-copresented for every positive integer n, we say that M is *infinitely copresented*. If we ignore that M is n-copresented for some positive integer n, we say that M is (-1)-*copresented*.

Obviously, every n-copresented module is m-copresented for every positive integer $m \leq n$. Also, one can see easily that every injective and finitely cogenerated R-module I is infinitely copresented associated to the exact sequence $0 \to I = I \to 0$.

The following propositions shows that 0-copresented and 1-copresented modules are just, respectively, the finitely cogenerated and finitely copresented modules.

Proposition 2.2. *For a ring R, an R-module is* 0*-copresented if and only if it is finitely cogenerated.*

Proof. Since every submodule of finitely cogenerated module is finitely cogenerated, 0-copresented R-modules are finitely cogenerated. Conversely, from Theorem 1.1 (2), we can see easily that every finitely cogenerated module is 0-copresented. □

Proposition 2.3. *For a ring R, an R-module is* 1*-copresented if and only if it is finitely copresented.*

Proof. $(\Rightarrow)$ Suppose that M is 1-copresented. Then there exists an exact sequence $0 \to M \to E_0 \to E_1$ where, E_0 and E_1 are injective and finitely cogenerated. Then, M is finitely cogenerated as a submodule of E_0. Now, Let $0 \to M \to L \to N \to 0$ with L is finitely cogenerated. We claim that N is also finitely cogenerated. Consider the short exact sequence $0 \to M \to E_0 \to K \to 0$ with $K = \mathrm{Im}(E_0 \to E_1)$. Then we get the following pushout diagram:

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & & & \\
 & & \downarrow & & \downarrow & & & & \\
0 & \to & M & \to & E_0 & \to & K & \to & 0 \\
 & & \downarrow & & \downarrow & & \| & & \\
0 & \to & L & \to & D & \to & K & \to & 0 \\
 & & \downarrow & & \downarrow & & & & \\
 & & N & = & N & & & & \\
 & & \downarrow & & \downarrow & & & & \\
 & & 0 & & 0 & & & &
\end{array}$$

By the middle horizontal exact sequence and since L and K are finitely cogenerated (since $K \subset E_1$), D is finitely cogenerated. Since E_0 is injective, the sequence $0 \to E_0 \to D \to N \to 0$ splits and so $D \cong N \oplus E_0$. And, since D is finitely cogenerated, N is finitely cogenerated. Therefore, M is finitely copresented.
$(\Leftarrow)$ Suppose that M is finitely copresented. Then, M is finitely cogenerated and so there is an exact sequence $0 \to M \to E_0$ such that E_0 is injective and finitely cogenerated (by Proposition 2.2). Consider the short exact sequence $0 \to M \to E_0 \to K \to 0$, where $K = E_0/M$. By definition, K is finitely cogenerated, which implies that there is an exact sequence $0 \to K \to E_1$ such that E_1 is an injective and finitely cogenerated. Then, we get the following commutative diagram

$$\begin{array}{ccccccccc}
0 & \to & M & \to & E_0 & & \longrightarrow & & E_1 \\
 & & & & & \searrow & & \nearrow & \\
 & & & & & & K & & \\
 & & & & & \nearrow & & \searrow & \\
 & & & & 0 & & & & 0
\end{array}$$

with the sequence $0 \to M \to E_0 \to E_1$ is exact. This implies that M is 1-copresented. □

Now we give the main result in this section which is dual to a well known result on n-presented modules (see [4, Exercice 6, p. 60]) and a generalization of [16, Theorem 30.2].

Theorem 2.4. *Let R be a ring and let $0 \to A \to B \to C \to 0$ be a short exact sequence of R-modules. Then, for a positive integer n, we have:*

(1) *If A and C are n-copresented, then B is n-copresented.*
(2) *If C is $(n-1)$-copresented and B is n-copresented, then A is n-copresented.*
(3) *If A is $(n+1)$-copresented and B is n-copresented, then C is n-copresented.*
(4) *If $B = A \oplus C$, then B is n-copresented if and only if A and C are n-copresented.*

Proof. 1. Since A and C are n-copresented, there are exact sequences of R-modules

$$0 \to A \to A_0 \to A_1 \to \cdots \to A_n \quad \text{and} \quad 0 \to C \to C_0 \to C_1 \to \cdots \to C_n,$$

where, for $i = 0, ..., n$, A_i and C_i are injective and finitely cogenerated. By the dual result of Horseshoe Lemma ([14, Remark after Lemma 6.20]), we get the following commutative diagram of R-modules with exact sequences:

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & A & \to & B & \to & C & \to & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & A_0 & \to & A_0 \oplus C_0 & \to & C_0 & \to & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & A_1 & \to & A_1 \oplus C_1 & \to & C_1 & \to & 0 \\
 & & \vdots & & \vdots & & \vdots & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & A_n & \to & A_n \oplus C_n & \to & C_n & \to & 0
\end{array}$$

By the middle vertical sequence and since $A_i \oplus C_i$ are injective and finitely cogenerated, we deduce that B is n-copresented.

2. Now suppose that C is $(n-1)$-copresented and B is n-copresented, then there is an exact sequence of R-modules $0 \to B \to B_0 \to B_1 \to \cdots \to B_n$, where, for $i = 0, ..., n$, B_i is injective and finitely cogenerated. Then, we get the following exact sequences

$$0 \to B \to B_0 \to K \to 0 \quad \text{and} \quad 0 \to K \to B_1 \to B_2 \to \cdots \to B_n,$$

where $K = B_0/B$. Then K is $(n-1)$-copresented. Consider the pushout diagram

$$\begin{array}{ccccccccc} & & & & 0 & & 0 & & \\ & & & & \downarrow & & \downarrow & & \\ 0 & \to & A & \to & B & \to & C & \to & 0 \\ & & \| & & \downarrow & & \downarrow & & \\ 0 & \to & A & \to & B_0 & \to & D & \to & 0 \\ & & & & \downarrow & & \downarrow & & \\ & & & & K & = & K & & \\ & & & & \downarrow & & \downarrow & & \\ & & & & 0 & & 0 & & \end{array}$$

By (1), D is $(n-1)$-copresented (since C and K are $(n-1)$-copresented). Then, there is an exact sequence of R-modules $0 \to D \to D_0 \to D_1 \to \cdots \to D_{n-1}$, where each D_i is injective and finitely cogenerated. We combine this sequence with the sequence $0 \to A \to B_0 \to D \to 0$, we get the following commutative diagram

$$\begin{array}{ccccccccccccc} 0 & \longrightarrow & A & \longrightarrow & B_0 & & \longrightarrow & & D_0 & \longrightarrow & D_1 & \longrightarrow \cdots \longrightarrow & D_{n-1} \\ & & & & & \searrow & & \nearrow & & & & & \\ & & & & & & D & & & & & & \\ & & & & & \nearrow & & \searrow & & & & & \\ & & & & 0 & & & & 0 & & & & \end{array}$$

with the top sequence is exact. Hence, A is n-copresented.

3. We have that A is $(n+1)$-copresented, then there is an exact sequence of R-modules $0 \to A \to A_0 \to A_1 \to \cdots \to A_n \to A_{n+1}$ where each A_i is injective and finitely cogenerated. Thus we get the two exact sequences

$$0 \to A \to A_0 \to K \to 0 \quad \text{and} \quad 0 \to K \to A_1 \to A_2 \to \cdots \to A_n \to A_{n+1}$$

where $K = A_0/A$. Then, K is n-copresented. Consider the pushout diagram

$$\begin{array}{ccccccccc} & & 0 & & 0 & & & & \\ & & \downarrow & & \downarrow & & & & \\ 0 & \to & A & \to & B & \to & C & \to & 0 \\ & & \downarrow & & \downarrow & & \| & & \\ 0 & \to & A_0 & \to & D & \to & C & \to & 0 \\ & & \downarrow & & \downarrow & & & & \\ & & K & = & K & & & & \\ & & \downarrow & & \downarrow & & & & \\ & & 0 & & 0 & & & & \end{array}$$

Since B and K are n-copresented, D is n-copresented by (1). And since A_0 is injective, the middle horizontal sequence splits and so $D = A_0 \oplus C$. Thus we get the following short exact sequence $0 \to C \to D = A_0 \oplus C \to A_0 \to 0$. Since D is n-copresented, there is an exact sequence of R-modules $0 \to D \to D_0 \to D_1 \to$

$\cdots \to D_n$, where each D_i is injective and finitely cogenerated. This gives a short exact sequence $0 \to D \to D_0 \to K \to 0$ such that $K = D_0/D$ is $(n-1)$-copresented. Then we have the following pushout diagram

$$\begin{array}{ccccccccc}
 & & & & 0 & & 0 & & \\
 & & & & \downarrow & & \downarrow & & \\
0 & \to & C & \to & D & \to & A_0 & \to & 0 \\
 & & \| & & \downarrow & & \downarrow & & \\
0 & \to & C & \to & D_0 & \to & E & \to & 0 \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & K & = & K & & \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & 0 & & 0 & &
\end{array}$$

Being a finitely cogenerated and injective R-module, A_0 is infinitely copresented. Then, by the right vertical exact sequence and (1), E is $(n-1)$-copresented. Then there is an exact sequence of R-modules $0 \to E \to E_0 \to E_1 \to \cdots \to E_{n-1}$ where each E_i is injective and finitely cogenerated. Combining this sequence with $0 \to C \to D_0 \to E \to 0$ we get the following exact sequence:

$$0 \to C \to D_0 \to E_0 \to E_1 \to \cdots \to E_{n-1}$$

Therefore, C is n-copresented.

4. Assume that A and C are n-copresented. Applying (1) to the following short exact sequence $0 \to A \to B = A \oplus C \to C \to 0$, we get that B is n-copresented. Conversely, suppose that $B = A \oplus C$ is n-copresented. Then B is finitely cogenerated and so are A and C. Let $0 \to A \to A_0 \to Z_0 \to 0$ and $0 \to C \to C_0 \to X_0 \to 0$ be exact sequences. We add these sequences such that we get a short exact sequence

$$0 \to B = A \oplus C \to A_0 \oplus C_0 \to Z_0 \oplus X_0 \to 0$$

By (3), $Z_0 \oplus X_0$ is $(n-1)$-copresented. Therefore, applying (2) to the above two short exact sequences, we get that A and C are n-copresented. □

Corollary 2.5. *Let R be a ring and let $0 \to M \to I_0 \to I_1 \to \cdots \to I_n \to K \to 0$ be an exact sequence, where n is a positive integer and, for $i = 0, ..., n$, I_i is $(m-(i+1))$-copresented for a positive integer $m \geq n$. Then, M is m-copresented if and only if K is $(m-n-1)$-copresented.*

Proof. We decompose the sequence $0 \to M \to I_0 \to I_1 \to \cdots \to I_n \to K \to 0$ into short exact sequences as follows:

$$0 \to K_i \to I_i \to K_{i+1} \to 0, \quad \text{for} \quad i = 0, ..., n$$

such that $K_0 = M$ and $K_{n+1} = K$, and by applying recursively Theorem 2.4 to each of these sequences we obtain the desired result. □

Note that Corollary 2.5 holds true, in particular, if I_i are injective and finitely cogenerated.

We close this Section with the following change of rings results.

Proposition 2.6. *Let $R \longrightarrow S$ be a ring homomorphism and consider a positive integer n. If the injective hull of every simple S-module is n-copresented as an R-module, then every n-copresented S-module is an n-copresented R-module.*

Proof. Let M be an n-copresented S-module. Then there is an exact sequence of S-modules

$$0 \to M \to E_0 \to E_1 \to \cdots \to E_n$$

where, for $i = 0, ..., n$, E_i is injective and finitely cogenerated. We decompose this sequence into two exact sequences:

$$0 \to M \to E_0 \to E_1 \to \cdots \to E_{n-1} \to K \to 0 \quad \text{and} \quad 0 \to K \to E_n \to E_n/K \to 0$$

Being a finite direct sum of the injective hull of simple S-modules, each E_i is an n-copresented R-module (by hypothesis). On the other hand, K is finitely cogenerated as an R-module because it is embedded in E_n which is finitely cogenerated as an R-module. Therefore, by Corollary 2.5, M is an n-copresented R-module. □

Corollary 2.7. *Let $R \longrightarrow R/I$ be the canonical ring homomorphism. Then every finitely cogenerated R/I-module is a finitely cogenerated R-module.*

Proof. Let T be a simple R/I-module, then there is a maximal ideal M of R such that $I \subset M$ and $T = ((R/I)/(M/I)) \cong R/M \subset E(R/M)$ then T is a finitely cogenerated R-module. Then, using Proposition 2.6, every finitely cogenerated R/I-module is a finitely cogenerated R-module. □

Proposition 2.8. *Let $R \longrightarrow S$ be a ring homomorphism such that, the injective hull of every simple S-module is $(n-1)$-copresented as an R-module, where n is a positive integer. Then, for every S-module M, if M is n-copresented as an R-module then it is n-copresented as an S-module.*

Proof. We prove by induction on n. The case $n = 0$ means that M is a finitely cogenerated R-module. We prove that M is a finitely cogenerated S-module. Then consider $(M_i)_{i \in I}$ to be a family of submodules of the S-module M with $\cap_{i \in I} M_i = 0$. Since $M_i \subset M$ as S-modules, $M_i \subset M$ as R-modules. Since M is a finitely cogenerated R-module, there exists a finite subset J of I such that $\cap_{j \in J} M_j =$

0, Then M is a finitely cogenerated S-module. Now assume that M is an n-copresented R-module for $n \geq 1$. In particular, M is a finitely cogenerated R-module. Then, from the first case, M is also finitely cogenerated as S-module. Hence, there exists a short exact sequence of S-modules $0 \to M \to E_0 \to K \to 0$ where E_0 is injective and finitely cogenerated. By hypothesis E_0 is an $(n-1)$-copresented R-module, then, by Theorem 2.4 (3), K is an $(n-1)$-copresented R-module. Then, by induction, K is an $(n-1)$-copresented S-module. Therefore, by Theorem 2.4 (2), M is an n-copresented S-module. □

3. n-Co-coherent rings

In this section we give some properties of a dual notion of n-coherent rings.

Definition 3.1. For a positive integer n, a ring R is called *n-co-coherent*, if every n-copresented R-module is $(n+1)$-copresented.

Proposition 3.2. *For a positive integer n, if R is an n-co-coherent ring, then every n-copresented R-module M is infinitely copresented.*

Proof. Since M is n-copresented, there is an exact sequence $0 \to M \to I_0 \to M_1 \to 0$ such that I_0 is injective and finitely cogenerated. Since R is n-co-coherent, M is $(n+1)$-copresented, then, by Theorem 2.4, M_1 is n-copresented and so it is $(n+1)$-copresented (since R is n-co-coherent). This implies, also by Theorem 2.4 (3), that M is $(n+2)$-copresented. We continue, using the same argument, such that we obtain that M is m-copresented for every positive integer $m \geq n$, which means that M is infinitely copresented. □

Proposition 3.3. *For a positive integer n, if a ring R is n-co-coherent, then, for every positive integer $m \geq n$, R is m-co-coherent.*

Proof. Consider an m-copresented R-module M. Then, M is n-copresented (since $m \geq n$). And, since R is n-co-coherent, M is infinitely copresented (by Proposition 3.2), and so it is $m+1$-copresented. This means that R is m-co-coherent. □

The following results show that n-co-coherent rings are extensions of the known co-Noetherian and co-coherent rings.

Recall that a ring R is called co-Noetherian if factors of finitely cogenerated R-modules are finitely cogenerated [12].

Proposition 3.4. *A ring R is 0-co-coherent if and only if it is co-Noetherian.*

Proof. (⇒) Suppose that R is 0-co-coherent and consider a 0-copresented R-module M. Then every submodule K of M is 0-copresented and so it is 1-copresented (since R is 0-co-coherent). Then Theorem 2.4 (3) applied to the short

exact sequence $0 \to K \to M \to M/K \to 0$ implies that M/K is finitely cogenerated. Hence R is co-Noetherian.
($\Leftarrow$) Now suppose that R is co-Noetherian. Let M be a 0-copresented R-module M. Then, there is an exact sequence $0 \to M \to E_0 \to E_0/M \to 0$ with E_0 is injective and finitely cogenerated, and E_0/M is finitely cogenerated (since R is co-Noetherian). Then, form Theorem 2.4 (2), M is 1-copresented. Therefore, R is 0-co-coherent. □

The co-coherence is defined as a dual of the classical coherence, such that a ring R is called co-coherent if every finitely cogenerated factor module of a finitely cogenerated injective R-module is finitely copresented [13]. Then clearly we get the following result.

Proposition 3.5. *A ring R is 1-co-coherent if and only if it is co-coherent.*

The following result gives a condition that relies n-coherent rings with n-co-coherent rings.

Recall that an R-module C is called cogenerator if every R-module M can be embedded in a product of copies of C [1, page 210]. From [1, Proposition 18.14], every cogenerator R-module C is faithful, that is every sequence of R-modules $A \to B \to D$ is exact if the sequence $\text{Hom}(D, C) \to \text{Hom}(B, C) \to \text{Hom}(A, C)$ is exact. From [1, Corollary 18.16], the direct sum of injective hull of all simple modules is a cogenerator.

Theorem 3.6. *Let R be a semi-local ring (i.e. R has a finite set of maximal ideals). If R is an n-co-coherent ring for a positive integer n, then it is n-coherent.*

Proof. Let $\{I_1, I_2, ..., I_m\}$ be the set of all maximal ideals of R. Then $\{R/I_1, R/I_2, ..., R/I_m\}$ is the set of all simple modules, then the cogenerator R-module $C = \oplus_{0 \leq i \leq m} E(R/I_i)$ is injective, faithful and finitely cogenerated. Let M be an n-presented R-module. Then there exists an exact sequence of R-modules

$$0 \to K \to F_n \to \cdots \to F_1 \to F_0 \to M \to 0$$

where each F_i is finitely generated and free and K is finitely generated. Then

$$0 \to \text{Hom}(M, C) \to \text{Hom}(F_0, C) \to \cdots \to \text{Hom}(F_n, C) \to \text{Hom}(K, C) \to 0$$

is exact (since C is injective). Since each F_i is free and finitely generated, there exists a positive integer n_i such that $F_i \cong R^{n_i}$, then

$$\text{Hom}(F_i, C) \cong \text{Hom}(R^{n_i}, C) \cong C^{n_i}$$

And since C is injective and finitely cogenerated, $\mathrm{Hom}(F, C) \cong C^{n_i}$ is injective and finitely cogenerated. On the other hand, $\mathrm{Hom}(K, C)$ is finitely cogenerated, since K is finitely generated (by [16, Proposition 30.6 (1)]). Then, $\mathrm{Hom}(M, C)$ is n-copresented by Corollary 2.5. Then, since R is n-co-coherent, $\mathrm{Hom}(M, C)$ is $(n+1)$-copresented which implies, by corollary 2.5, that $\mathrm{Hom}(K, C)$ is 1-copresented. Then, from [16, Proposition 30.6 (1)], K is 1-presented and therefore M is $(n+1)$-presented. This means that R is n-coherent. □

We close this paper with some change of rings results.

In [12, first part of Section 2], Jans noted that if a ring R is co-Noetherian then so is R/I for every ideal I of R. The following result generalize this fact to n-co-coherent rings (see Corollary 2.5 and its proof).

Proposition 3.7. *Let $R \longrightarrow S$ be a ring homomorphism such that, for a positive integer n, the injective hull of every simple S-module is n-copresented as an R-module. Then, S is n-co-coherent if R is n-co-coherent.*

Proof. Suppose that R is an n-co-coherent ring and consider an n-copresented S-module M. From Proposition 2.6, M is n-copresented as an R-module. Thus M is an $(n{+}1)$-copresented R-module (since R is n-co-coherent). Then, from Proposition 2.8, M is an $(n{+}1)$-copresented S-module. Therefore, S is n-co-coherent. □

Now we study n-co-coherence of a direct product of rings. For that we recall the following results on the structure of modules over a direct product of rings. Let $R = \prod_{i=1}^{n} R_i$ be a direct product of rings, where $n > 0$ is a positive integer. If M_i is an R_i-module for $i = 1, ..., n$, then $M = M_1 \oplus \cdots \oplus M_n$ is an R-module. Conversely, if M is an R-module, then it is of the form $M = M_1 \oplus \cdots \oplus M_n$, where M_i is an R_i-module for $i = 1, ..., n$ [3, Subsection 2.6.6]. Also, the homomorphisms of R-modules are determined by their actions on the R_i-module components. Using [3, Theorem 2.6.8], we get that an R-module $M = M_1 \oplus \cdots \oplus M_n$ is injective if and only if M_i is an injective R_i-module for every $1 \leq i \leq n$.

For the structure of n-copresented modules over direct product of rings we give the following result.

Lemma 3.8. *Let $R = \prod_{i=1}^{n} R_i$ be a direct product of rings, where $n > 0$ is a positive integer, and let $M = M_1 \oplus \cdots \oplus M_n$ be a decomposition of an R-module M into R_i-modules M_i. Then, for a positive integer m, M is an m-copresented R-module if and only if each M_i is an m-copresented R_i-module.*

Proof. Using the definition of a finitely cogenerated module (the assertion 5 in Theorem 1.1) and the structure of modules over a direct product of rings, we get immediately the result for the case $m = 0$. Now suppose that $m \geq 1$. First note that for the R-module M, there exists a short exact sequence of R-modules $0 \to M \to I \to K \to 0$ such that I is injective. By [3, Theorem 2.6.8], this sequence can be decomposed into short exact sequences of R_i-modules $0 \to M_i \to I_i \to K_i \to 0$ such that $I = I_1 \oplus \cdots \oplus I_n$ and $K = K_1 \oplus \cdots \oplus K_n$. Now, if M is m-copresented, we can suppose that I is finitely cogenerated and K is $(m-1)$-copresented. Then, by induction each K_i is an $(m-1)$-copresented R_i-module and I_i is a finitely cogenerated R_i-module. Then, using the short exact sequences, we get that each M_i is an m-copresented R_i-module. Conversely, if each M_i is an m-copresented R_i-module, then we can suppose that each I_i is a finitely cogenerated R_i-module and each K_i is an $(m-1)$-copresented R_i-module. Then, by induction, I is a finitely cogenerated R-module and each K is an $(m-1)$-copresented R-module. This implies that M is an m-copresented R-module. □

The following result is a dual one of [8, Theorem 2.13].

Theorem 3.9. *Let $n > 0$ be a positive integer. A direct product of rings $R = \prod_{i=1}^{n} R_i$ is an n-co-coherent ring if and only if each R_i is n-co-coherent.*

Proof. The result follows immediately from Lemma 3.8. □

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