BLOCK TRANSITIVE $2-(v, 17, 1)$ DESIGNS AND REE GROUPS

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Received: 27 April 2011; Revised: 15 March 2012
Communicated by Abdullah Harmanci

Abstract. This article is a contribution to the study of the automorphism groups of $2-(v, k, 1)$ designs. Let $\mathcal{D}$ be $2-(v, 17, 1)$ design, $G \leq Aut(\mathcal{D})$ be block transitive and point primitive. If $G$ is unsolvable, then $\text{Soc}(G)$, the socle of $G$, is not $\text{S}_G(q)$.

Mathematics Subject Classification (2010): 05B05, 20B25
Keywords: block transitive, design, Ree groups

1. Introduction

A $2-(v, k, 1)$ design $\mathcal{D} = (\mathcal{P}, \mathcal{B})$ is a pair consisting of a finite set $\mathcal{P}$ of $v$ points and a collection $\mathcal{B}$ of $k$-subsets of $\mathcal{P}$, called blocks, such that any 2-subsets of $\mathcal{P}$ is contained in exactly one block. We will always assume that $2 < k < v$.

Let $G \leq Aut(\mathcal{D})$ be a group of automorphisms of a $2-(v, k, 1)$ design $\mathcal{D}$. Then $G$ is said to be block transitive on $\mathcal{D}$ if $G$ is transitive on $\mathcal{B}$ and is said to be point transitive(point primitive on $\mathcal{D}$ if $G$ is transitive (primitive) on $\mathcal{P}$. A flag of $\mathcal{D}$ is a pair consisting of a point and a block through that point. Then $G$ is flag transitive on $\mathcal{D}$ if $G$ is transitive on the set of flags.

The classification of block transitive $2-(v, 3, 1)$ designs was completed about thirty years ago (see [2]). In [3], Camina and Siemons classified $2-(v, 4, 1)$ designs with a block transitive, solvable group of automorphisms. Li classified $2-(v, 4, 1)$ designs admitting a block transitive, unsolvable group of automorphisms (see [7]). Tong and Li [11] classified $2-(v, 5, 1)$ designs with a block transitive, solvable group of automorphisms. Han and Li [4] classified $2-(v, 5, 1)$ designs with a block transitive, unsolvable group of automorphisms. Liu [9] classified $2-(v, k, 1)(\text{where } k = 6, 7, 8, 9, 10)$ designs with a block transitive, solvable group of automorphisms. In [5], Han and Ma classified $2-(v, 11, 1)$ designs with a block transitive classical Simple groups of automorphisms.

Supported by the NNSFC (Grant No. 10871205) and the Research Fund of Tianjin Polytechnic University.
This article is a contribution to the study of the automorphism groups of \(2-(v,k,1)\) designs. Let \(\mathcal{D}\) be \(2-(v,17,1)\) design, \(G \leq \text{Aut}(\mathcal{D})\) be block transitive and point primitive. We prove that following theorem.

**Main Theorem.** Let \(\mathcal{D}\) be \(2-(v,17,1)\) design, \(G \leq \text{Aut}(\mathcal{D})\) be block transitive and point primitive. If \(G\) is unsolvable, then \(\text{Soc}(G) \not\cong 2G_2(q)\).

2. Preliminary Results

Let \(\mathcal{D}\) be a \(2-(v,k,1)\) design defined on the point set \(\mathcal{P}\) and suppose that \(G\) is an automorphism group of \(\mathcal{D}\) that acts transitively on blocks. For a \(2-(v,k,1)\) design, as usual, \(b\) denotes the number of blocks and \(r\) denotes the number of blocks through a given point. If \(B\) is a block, \(G_B\) denotes the setwise stabilizer of \(B\) in \(G\) and \(G_B^{\text{w}}\) is the pointwise stabilizer of \(B\) in \(G\). Also, \(G_B\) denotes the permutation group induced by the action of \(G_B\) on the points of \(B\), and so \(G_B \cong G_B^{\text{w}}/G_B^{\text{w}}(B)\).

The Ree groups \(2G_2(q)\) form an infinite family of simple groups of Lie type, and were defined in [10] as subgroups of \(\text{GL}(7,q)\). Let \(\text{GF}(q)\) be finite field of \(q\) elements, where \(q = 3^{2n+1}\) for some positive integer \(n \geq 1\). Set \(t = 3^{n+1}\) so that \(t^2 = 3q\). We give the following information about subgroups of \(2G_2(q)\). For each \(l\) dividing \(2n + 1\), \(2G_2(3^l)\) denotes the subgroup of \(2G_2(q)\) consisting of all matrices in \(2G_2(q)\) with entries in subfield of \(3^l\). We use the symbols \(Q\) and \(K\) to note a Sylow 3-subgroup and a cyclic subgroup of order \(q - 1\) of \(2G_2(q)\), respectively.

**Lemma 2.1.** ([6]) Let \(T \leq 2G_2(q)\) and \(T\) be maximal in \(2G_2(q)\). Then either \(T\) is conjugate to \(P_6(l) = 2G_2(3^l)\) for some divisor \(l\) of \(2n + 1\), or \(T\) is conjugate to one of the subgroups \(P_i\) in Table 1.

**Table 1: Group conjugate to \(T\)**

<table>
<thead>
<tr>
<th>Group</th>
<th>Structure</th>
<th>Remarks</th>
</tr>
</thead>
<tbody>
<tr>
<td>(P_1)</td>
<td>(Q : K)</td>
<td>The normaliser of (Q) in (2G_1(q))</td>
</tr>
<tr>
<td>(P_2)</td>
<td>((Z_2 \times D_{(q+1)/2}) : Z_3)</td>
<td>The normaliser of a fours-group</td>
</tr>
<tr>
<td>(P_3)</td>
<td>(Z_2 \times \text{PSL}(2,q))</td>
<td>An involution centraliser</td>
</tr>
<tr>
<td>(P_4)</td>
<td>(Z_{q+t+1} : Z_6)</td>
<td>The normaliser of (Z_{q+t+1})</td>
</tr>
<tr>
<td>(P_5)</td>
<td>(Z_{q-t+1} : Z_6)</td>
<td>The normaliser of (Z_{q-t+1})</td>
</tr>
</tbody>
</table>

**Lemma 2.2.** ([8]) Let \(T = 2G_2(q)\) be an exceptional simple group of Lie type over \(\text{GF}(q)\), and let \(G\) be a group with \(T \trianglelefteq G \leq \text{Aut}(T)\). Suppose that \(M\) is a maximal subgroup of \(G\) not containing \(T\), then one of the following holds:

1. \(|M| < q^3|G : T|\);
2. \(T \cap M\) is a parabolic subgroup of \(T\).
Lemma 2.3. ([5]) Let $G$ and $D = (\mathcal{P}, \mathcal{B})$ be a group and a design, and $G \leq \text{Aut}(D)$ be block transitive, point-primitive but not flag-transitive. Let $\text{Soc}(G) = T$. Then

$$|T| \leq \frac{v}{\lambda} \cdot |T_\alpha|^2 \cdot |G : T|,$$

where $\alpha \in \mathcal{P}$, $\lambda$ is the length of the longest suborbit of $G$ on $\mathcal{P}$.

3. Proof of the Main Theorem

Proposition 3.1. Let $D$ be a $(v, 17, 1)$ design, $G$ be block transitive, point-primitive but not flag transitive, then $v = 272b_2 + 1$.

Proof. Let $b_1 = (b, v)$, $b_2 = (b, v - 1)$, $k_1 = (k, v)$, $k_2 = (k, v - 1)$. Obviously, $k = k_1k_2$. If $k = 17$, we get $k_1 = 1$. Otherwise, $k \mid v$, by [8], $G$ is flag transitive, a contradiction. Thus $v = k(k - 1)b_2 + 1 = 272b_2 + 1$. $\square$

Proposition 3.2. Let $D$ be a $(v, 17, 1)$ design, $G$ be block transitive, point-primitive but not flag transitive and $|T|$ be even. If $G$ is unsolvable, then $|T| \leq 137|T_\alpha|^2|G : T|$.

Proof. Let $B = \{1, 2, \cdots, 17\} \in \mathcal{B}$. Since $G$ is unsolvable, then the structure of $G^B$, the rank and subdegree of $G$ do not occur:

<table>
<thead>
<tr>
<th>Type of $G^B$</th>
<th>Rank of $G$</th>
<th>Subdegree of $G$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\langle 1 \rangle$</td>
<td>273</td>
<td>$1, b_2, \cdots, b_2$</td>
</tr>
</tbody>
</table>

Otherwise, $G^B$ is odd and $G$ is also odd, a contradiction with $|T|$ be even. Thus $\lambda \geq 2b_2$. By Lemma 2.3 and Proposition 3.1,

$$\frac{|T|}{|T_\alpha|^2} \leq \frac{v}{\lambda} \cdot |G : T| \leq \frac{272b_2 + 1}{2b_2} \cdot |G : T| \leq 137|G : T|.$$

$\square$

Now we may prove our main theorem.

Suppose that $\text{Soc}(G) = ^2G_2(q) = T$, then $^2G_2(q) \leq G \leq \text{Aut}(^2G_2(q))$. We have $G = T : \langle x \rangle$, where $x \in \text{Out}(T)$, the outer automorphisms group of $T$ which may be generated by an automorphism of field. We may assume that $x$ is an automorphism of field. Set $\phi(x) = m$, then $m \mid (2n + 1)$. Obviously, $|^2G_2(q)| = q^3(q^3 + 1)(q - 1)$. By [1] and $k = 17$, $G$ is not flag transitive. Since $G$ is point primitive, $G_\alpha$ ($\alpha \in \mathcal{P}$) is the maximal subgroup of $G$, $T$ is block transitive in $\mathcal{D}$. Hence $M = G_\alpha$ satisfies one of the two cases in Lemma 2.2. We will rule out these cases one by one.
**Case (1) \(|M| < q^3|G : T|\).**

By Proposition 3.2, we have an upper bound of \(|T|\),

\[
|T| < 137|T_\alpha|^2|G : T| < 137q^6|G : T| = 137q^6m.
\]

We get

\[
q - 1 < 137(2n + 1).
\]

Let \(2n + 1 = s \geq 3\), then \(3^s < 138s\). Thus \(s = 3, 5\).

If \(s = 3\), then \(|^2G_2(3^3)| = 3^9 \cdot 2^3 \cdot 7 \cdot 13 \cdot 19 \cdot 37\). Since \(v = 272b_2 + 1\) is odd, then \(2^3 \mid |T_\alpha|\). Clearly \(T_\alpha\) is contained in some maximal subgroups of \(T\). By Lemma 2.1, \(T_\alpha \cong ^2G_2(3), (Z_2^2 \times D_{(q+1)/2}) : Z_3\) or \(Z_2 \times PSL(2, q)\), where \(q = 3^3\).

(i) \(T_\alpha \cong ^2G_2(3)\). We have

\[
v - 1 = \frac{|T|}{|T_\alpha|} - 1 = \frac{3^9 \cdot 2^3 \cdot 7 \cdot 13 \cdot 19 \cdot 37}{3^3 \cdot 2^3 \cdot 7} - 1 = 6662330.
\]

By Proposition 3.1, \(156b_2 = 6662330\), a contradiction.

(ii) \(T_\alpha \cong (Z_2^2 \times D_{(q+1)/2}) : Z_3\). We have

\[
v - 1 = \frac{|T|}{|T_\alpha|} - 1 = \frac{3^9 \cdot 2^3 \cdot 7 \cdot 13 \cdot 19 \cdot 37}{3 \cdot 2^3 \cdot 7} - 1 = 59960978.
\]

By Proposition 3.1, \(156b_2 = 59960978\), a contradiction.

(iii) \(T_\alpha \cong Z_2 \times PSL(2, q)\). We have

\[
v - 1 = \frac{|T|}{|T_\alpha|} - 1 = \frac{3^9 \cdot 2^3 \cdot 7 \cdot 13 \cdot 19 \cdot 37}{3^3 \cdot 2^3 \cdot 7 \cdot 13} - 1 = 512486.
\]

By Proposition 3.1, \(156b_2 = 512486\), a contradiction.

If \(s = 5\), then \(|^2G_2(3^5)| = 3^{15} \cdot (3^{15} + 1)) \cdot (3^5 - 1)\). Since \(v = 272b_2 + 1\) is odd, then \(2^3 \mid |T_\alpha|\). Clearly \(T_\alpha\) is contained in some maximal subgroups of \(T\). By Lemma 2.1,

\[
T_\alpha \cong ^2G_2(3), (Z_2^2 \times D_{(q+1)/2}) : Z_3\) or \(Z_2 \times PSL(2, q)\),
\]

where \(q = 3^5\). It is not difficult to exclude them by Proposition 3.1.

**Case (2) \(T \cap M\) is a parabolic subgroup of \(T\).**

By Lemma 2.1, the parabolic subgroup of \(^2G_2(q)\) is conjugate to \(QK\). Then the order of parabolic subgroup is \(q^3(q - 1)\) and \(v = q^3 + 1\). By Proposition 3.1, we have \(q^3 = v - 1 = 272b_2\) and so \(272 \mid q^3\), a contradiction.

This completes the proof the Main Theorem.
References


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