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0-1 EMBEDDINGS OF M_ℓ IN ABELIAN SUBGROUP LATTICES

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Abstract. In this note we consider the question of how large ℓ can be so that M_ℓ fits inside $\mathbf{Sub}(G)$ as a 0-1 sublattice, for a finite abelian group G. We answer the question for $G = \mathbb{Z}_n \times \mathbb{Z}_n$: in this case $\ell = p + 1$, where p is the smallest prime dividing n.

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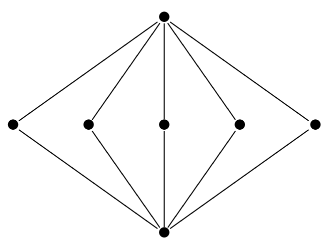
1. Introduction

Let G be a group, and let $\mathbf{Sub}(G)$ denote the lattice of subgroups of G. These subgroup lattices have been studied extensively. (See, for instance, the tome [7].) Every lattice is isomorphic to a sublattice of $\mathbf{Sub}(G)$ for some group G [9], and every finite lattice is isomorphic to a sublattice of $\mathbf{Sub}(G)$ for some finite group G [6]. These two results speak to the complexity of subgroup lattices.

One of the most important problems in lattice theory is as follows: is every finite lattice isomorphic to the congruence lattice of some finite algebra? Palfy and Pudlak showed in [4] that this is equivalent to the problem, Is every finite lattice isomorphic to an interval in the subgroup lattice of some finite group? The answer is Yes for all finite distributive lattices; for modular lattices, it is not even known for which ℓ it is the case that M_ℓ—the lattice with top, bottom, and ℓ incomparable atoms—is isomorphic to an interval (for a summary of what is known, see [3]).

The lattice M_5



The subgroup lattices of finite abelian groups are better-behaved. They are modular and self-dual, for instance. In [8], Vogt shows how to decompose such lattices via tolerance relations (which are "non-transitive congruence relations") and computes the cardinality of $\mathbf{Sub}(G)$ for any p-group G. Counting the total number of subgroups of a finite abelian group is a difficult problem; for a recent advance, see [2]. A very nice result from [1] is that every abelian group is determined by a subgroup lattice—that is, if A is abelian and $\mathbf{Sub}(A \times \mathbb{Z}) \cong \mathbf{Sub}(B \times \mathbb{Z})$, then $A \cong B$. Interestingly, for finite abelian groups, the intervals isomorphic to M_ℓ have been determined: ℓ must be one more than a prime power [5].

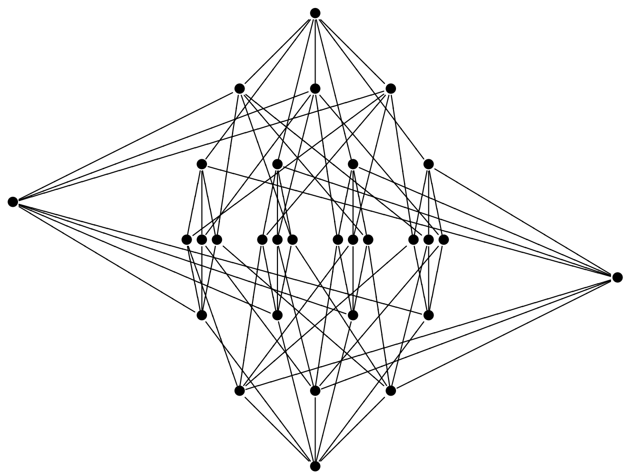
In this paper, we consider M_ℓ not as an interval sublattice but as a 0-1 sublattice of a finite abelian group. In particular, we ask for which ℓ does M_ℓ appear as a maximal 0-1 sublattice:

> For which $\ell \in \mathbb{Z}^+$ does there exist a finite abelian group G such that M_ℓ embeds as a 0-1 sublattice of $\mathbf{Sub}(G)$ but $M_{\ell+1}$ does not?

We answer this question for a certain class of groups in Theorem 1.1:

Theorem 1.1. *Let $n \geq 2$ be a positive integer, and let ℓ be the largest integer such that M_ℓ embeds as a 0-1 sublattice of $\mathbf{Sub}(\mathbb{Z}_n \times \mathbb{Z}_n)$. Then $\ell = p + 1$, where p is the smallest prime dividing n. Furthermore, if M_ℓ so embeds but $M_{\ell+1}$ does not, then the images of the atoms of M_ℓ are all cyclic subgroups of order n.*

The lattice $\mathbf{Sub}(\mathbb{Z}_6 \times \mathbb{Z}_6)$



The authors were led to consider $\mathbf{Sub}(\mathbb{Z}_n \times \mathbb{Z}_n)$ through relation-algebraic considerations. The fact that, for these groups, ℓ must be one more than a prime (power) is certainly intriguing.

2. Proof of the Main Theorem

Before we prove Theorem 1.1, it will be useful to establish the following lemma, which is interesting in its own right.

Lemma 2.1. *Let $\Xi(d)$ be the number of cyclic subgroups of order d in $\mathbb{Z}_n \times \mathbb{Z}_n$, where $d|n$. Then*

$$\Xi(d) = \sum_{\substack{d_1,d_2|\, d \\ \mathrm{lcm}(d_1,d_2)=d}} \frac{\varphi(d_1)\cdot\varphi(d_2)}{\varphi(d)}.$$

Proof. Let $d_1, d_2|d$, with $\mathrm{lcm}(d_1, d_2) = d$. Consider the collection $\{\langle(m,k)\rangle : m, k \in \mathbb{Z}_n, |m| = d_1, |k| = d_2\}$. The number of pairs (m,k) with $|m| = d_1$ and $|k| = d_2$ will be $\varphi(d_1)\cdot\varphi(d_2)$, where φ is Euler's function. Each subgroup $\langle(m,k)\rangle$ has $\varphi(d)$ generators, and so the number of distinct cyclic subgroups in the set above is $\frac{\varphi(d_1)\cdot\varphi(d_2)}{\varphi(d)}$. Now summing over all such pairs (d_1, d_2) gives the desired result. □

Proof of Theorem 1.1. Let ℓ be the largest integer such that M_ℓ embeds in $\mathbf{Sub}(\mathbb{Z}_n \times \mathbb{Z}_n)$ as a 0-1 sublattice. First we show that $\ell \le \Xi(p)$, where p is the smallest prime dividing n.

Suppose there exist subgroups $H_1, \ldots, H_\ell$ that meet pairwise to the bottom and join pairwise to the top. By cardinality considerations, each H_i must have at least n elements. Since $H_i \cap H_j = \{0\}$ for $i \ne j$, there exist ℓ independent paths from $\{0\}$ to each of the subgroups H_i. These paths must each pass through the rank of cyclic subgroups of order q for every prime q dividing n. Thus $\ell \le \Xi(q)$. Since $\Xi(q) = q + 1$ for all primes q, the smallest is $\Xi(p)$, where p is the smallest prime dividing n. So $\ell \le \Xi(p) = p + 1$.

Now we must show that the $p+1$ subgroups can always be found. Cyclic subgroups of order n can be pictured as lines in $[n]^2$. In $\mathbb{Z}_p \times \mathbb{Z}_p$, the $p+1$ cyclic subgroups are given by the "vertical line" $\langle(0,1)\rangle$, as well as $\{(x,y) \in \mathbb{Z}_p\times\mathbb{Z}_p : y \equiv mx\}$ for each $m \in \mathbb{Z}_p$; these are the lines of "slope" m. Since $\mathbb{Z}_n \times \mathbb{Z}_n$ contains an isomorphic copy of $\mathbb{Z}_p \times \mathbb{Z}_p$, it would seem natural that the $p+1$ lines on $\mathbb{Z}_p \times \mathbb{Z}_p$ could be extended to lines in $\mathbb{Z}_n \times \mathbb{Z}_n$.

Consider the subgroups $\langle(0,1)\rangle, \langle(1,0)\rangle, \langle(1,1)\rangle, \langle(1,2)\rangle, \ldots, \langle(1,p-1)\rangle$. The last p are described by the equivalences

$$\begin{aligned} y &\equiv 0\cdot x \\ y &\equiv 1\cdot x \\ &\vdots \\ y &\equiv (p-1)\cdot x \end{aligned}$$

To show that these subgroups intersect trivially, suppose $(a,b) \in \langle(1,m_1)\rangle \cap \langle(1,m_2)\rangle$, with $m_1 \not\equiv m_2$. Then it must be that $b \equiv m_1 a$ and $b \equiv m_2 a$. Hence $m_1 a \equiv m_2 a$, and so $0 \equiv (m_1 - m_2)a$. The additive order of a in $\mathbb{Z}_n$ must be a divisor of n; since $m_1, m_2 < p$, $(m_1 - m_2) < p$ also. Thus $|a| < p$. Since $|a|\big|n$ and p is the smallest prime dividing n, $|a| = 1$. It follows that $|b| = 1$ also. Hence $(a,b) = (0,0)$, and $\langle(1,m_1)\rangle$ and $\langle(1,m_2)\rangle$ intersect trivially. Then by cardinality considerations, it follows that the subgroups $\langle(1,m_1)\rangle$ and $\langle(1,m_2)\rangle$ must join to $\mathbb{Z}_n \times \mathbb{Z}_n$.

Finally, let us prove that if $p+1$ subgroups of $\mathbb{Z}_n \times \mathbb{Z}_n$ are pairwise trivially intersecting and pairwise join to $\mathbb{Z}_n \times \mathbb{Z}_n$, all $p+1$ subgroups must be cyclic subgroups of order n. Since we know that $|H_i \vee H_j| = |H_i||H_j|$ for H_i, H_j with trivial intersection, it must be that $|H_i| = |H_j| = n$. So we must exclude the possibility that some of the H_i's are "rectangles", i.e., isomorphic to $\mathbb{Z}_k \times \mathbb{Z}_m$, with $km = n$ and $(k,m) > 1$. Write $n = p_1^{\alpha_1} \cdots p_k^{\alpha_k}$ for distinct primes p_i. Then

$$\mathbb{Z}_n \times \mathbb{Z}_n \cong \prod \mathbb{Z}_{p_i^{\alpha_i}} \times \prod \mathbb{Z}_{p_i^{\alpha_i}},$$

and this decomposition is unique up to reordering of factors. If H_i and H_j meet to the bottom and join to the top, then $\mathbb{Z}_n \times \mathbb{Z}_n \cong H_i \times H_j$. Since $|H_i| = |H_j| = n$, both H_i and H_j are isomorphic to products of factors $\mathbb{Z}_{p_i^{\alpha_i}}$ with total cardinality n. The only way for this to happen is for

$$H_i \cong H_j \cong \prod \mathbb{Z}_{p_i^{\alpha_i}} \cong \mathbb{Z}_n.$$

This concludes the proof. □

3. Final Remarks

The authors leave as an open problem the determination of the maximal ℓ such that M_ℓ embeds as a 0-1 sublattice of $\mathbf{Sub}(G)$ for arbitrary finite abelian G. Of course, for cyclic G, $\ell = 1$ or $\ell = 2$, since $\mathbf{Sub}(G)$ is distributive. If ℓ were restricted in every other case to being one more than a prime power, this would be a very nice result.

If M_ℓ embeds in $\underline{L}$ as a 0-1 sublattice but $M_{\ell+1}$ does not, we can think of ℓ as a sort of width of the lattice $\underline{L}$. The usual definition of width for a lattice is the size of the largest antichain, so we might call ℓ the *girth* of $\underline{L}$. If we are concerned with the poset structure only, we might consider the following alternate definition of the width of $\underline{L}$ (as a poset): the smallest m such that $\underline{L} \hookrightarrow Q_m$, the boolean poset with m atoms. (This was suggested to the first author by Ryan Martin.) It might be interesting to explore the connection between the lattice girth ℓ and the poset width m.

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References

[1] S. Breaz and G. Călugăreanu, *Every abelian group is determined by a subgroup lattice*, Studia Sci. Math. Hungar., 45 (2008), 135–137.

[2] G. Călugăreanu, *The total number of subgroups of a finite abelian group*, Sci. Math. Jpn., 60 (2004), 157–167.

[3] R. Freese, *A Review of Roland Schmidt's book Subgroup lattices of groups*, Bull. Amer. Math. Soc., 33(4) (1996), 487–492.

[4] P. P. Pálfy and P. Pudlák, *Congruence lattices of finite algebras and intervals in subgroup lattices of finite groups*, Algebra Universalis, 11 (1980), 22–27.

[5] P. P. Pálfy and J. Saxl, *Congruence lattices of finite algebras and factorizations of groups*, Comm. Algebra, 18 (1990), 2783–2790.

[6] P. Pudlák and J. Tůma, *Every finite lattice can be embedded in a finite partition lattice*, Algebra Universalis, 10 (1980), 74–95.

[7] R. Schmidt, Subgroup lattices of groups, de Gruyter Expositions in Mathematics, 14, Walter de Gruyter & Co., Berlin, 1994.

[8] F. Vogt, *Subgroup lattices of finite abelian groups: structure and cardinality*, Lattice theory and its applications (Darmstadt, 1991), vol. 23, Res. Exp. Math., Heldermann, Lemgo, 1995, 241–259.

[9] P. M. Whitman, *Lattices, equivalence relations, and subgroups*, Bull. Amer. Math. Soc., 52 (1946), 507–522.

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