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COMPLETELY PRIME SUBMODULES

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Abstract. We generalize completely prime ideals in rings to submodules in modules. The notion of multiplicative systems of rings is generalized to modules. Let N be a submodule of a left R-module M. Define co.$\sqrt{N} := \{m \in M$: every multiplicative system containing m meets $N\}$. It is shown that co.$\sqrt{N}$ is equal to the intersection of all completely prime submodules of M containing N, $\beta_{co}(N)$. We call $\beta_{co}(M) =$ co.$\sqrt{0}$ the completely prime radical of M. If R is a commutative ring, $\beta_{co}(M) = \beta(M)$ where $\beta(M)$ denotes the prime radical of M. β_{co} is a complete Hoehnke radical which is neither hereditary nor idempotent and hence not a Kurosh-Amistur radical. The torsion theory induced by β_{co} is discussed. The module radical $\beta_{co}({}_RR)$ and the ring radical $\beta_{co}(R)$ are compared. We show that the class of all completely prime modules, ${}_RM$ for which $RM \neq 0$ is special.

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1. Introduction

All rings in this paper are associative (not necessarily with identity) and all modules are left R-modules. A proper submodule P of an R-module M is a prime submodule of M [1], [8] if for all ideals $\mathcal{A}$ of R and submodules N of M such that $\mathcal{A}N \subseteq P$, we have $N \subseteq P$ or $\mathcal{A}M \subseteq P$. If R is a commutative ring, this definition is equivalent to: for all $a \in R$ and every $m \in M$, if $am \in P$ then $m \in P$ or $aM \subseteq P$. We call this the definition of a completely prime submodule P of a module ${}_RM$. Several authors have discussed prime submodules in modules over commutative rings, e.g., [2], [15] among others. In general (for example when R is not commutative), the two definitions above need not be equivalent - the later implies the former but not conversely. Simple modules (and maximal submodules) are always prime but need not be completely prime. This justifies our study of completely prime submodules in this paper, drawing motivation from how completely

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prime ideals in rings are defined. Although some authors, e.g., [1], [9], [18, p.1840] mentioned about completely prime modules, to the best of our knowledge none has ever studied them in detail.

An ideal I of a ring R is a prime ideal if for any ideals $\mathcal{A}, \mathcal{B}$ of R such that $\mathcal{A}\mathcal{B} \subseteq I$, we have $\mathcal{A} \subseteq I$ or $\mathcal{B} \subseteq I$. If R is commutative, this definition is equivalent to: for any elements $a, b \in R$ such that $ab \in I$, we have $a \in I$ or $b \in I$. If R is not a commutative ring, the latter implies the former but not conversely and we call the later the definition of a completely prime ideal of a ring R. If I is a completely prime ideal of a ring R, the complement $R \setminus I$ is a multiplicative system, i.e., closed under multiplication. We generalize this notion to modules and show that if P is a completely prime submodule of M, the complement $M \setminus P$ of M is a multiplicative system of M.

If N is a submodule of M and I is an ideal of R we respectively write $N \leq M$ and $I \lhd R$. If I is an essential ideal of R, we write $I \lhd \cdot R$. If $N, P \leq M$ such that $N \not\subseteq P$, we write $(P : N)$ to mean the ideal $\{r \in R \ : \ rN \subseteq P\}$, $(P : m) = \{r \in R \ : \ rm \in P, m \in M \setminus P\}$. $<m>$ is the submodule of ${}_RM$ generated by $m \in M$, i.e., $<m>= \mathbb{Z}m + Rm$. R-mod is used to mean the category of R-modules, where R is a ring.

In section 2, we define and give examples of completely prime submodules. A comparison of completely prime and other primes in literature is done. We give their definitions. A proper submodule P of an R-module M is 1) s-prime [12] if for every $\mathcal{A} \lhd R$ and for every $N \leq M$ if $x \in \mathcal{A}$ and $x^n N \subseteq P$ for some $n \in \mathbb{N}$, then $N \subseteq P$ or $\mathcal{A}M \subseteq P$; 2) classical completely prime [11] if for all $a, b \in R$ and every $m \in M$, such that $abm \in P$, then $a <m> \subseteq P$ or $b <m> \subseteq P$; 3) classical prime [4], [3] if for any $N \leq M$ and $\mathcal{A}, \mathcal{B} \lhd R$ such that $\mathcal{A}\mathcal{B}N \subseteq P$, then $\mathcal{A}N \subseteq P$ or $\mathcal{B}N \subseteq P$. In section 3, it is shown that the preradical β_{co} is a complete Hoehnke radical which is neither hereditary nor idempotent (hence not Kurosh-Amistur). In the same section, properties of the torsion class induced by the radical β_{co} are given. In section 4, a comparison of the completely prime radical of the module ${}_RR$ and that of the ring R is done. Lastly, in section 5 we show that the class of all completely prime modules ${}_RM$ for which $RM \neq 0$ is special.

2. Completely Prime Submodules and Multiplicative Systems of Modules

Definition 2.1. A proper submodule P of a left R-module M is a *completely prime submodule* if for each $a \in R$ and every $m \in M$ such that $am \in P$, we have $m \in P$ or $aM \subseteq P$.

An R-module M is *completely prime* if the zero submodule of M is a completely prime submodule of M. In general, an R-module M/P is a completely prime module if and only if P is a completely prime submodule of M.

Example 2.2. *Every torsion free module is a completely prime module. To prove this, let M be a torsion free module and suppose that $am = 0$ for $a \in R$ and $m \in M$. If $m = 0$, then we are through. Suppose $m \neq 0$, by definition of torsion free modules, $a = 0$ and $aM = 0$.*

An R-module M is *reduced* if for all $a \in R$ and $m \in M$, $am = 0$ implies $<m> \cap aM = 0$ (see [11], [14]).

Example 2.3. *A simple module which is reduced is completely prime. To prove this, let M be a simple module which is reduced and suppose that $am = 0$, where $a \in R$ and $m \in M$. If $m = 0$, then we are through. Suppose that $m \neq 0$. Then $0 = aM \cap <m> = aM \cap M = aM$.*

Proposition 2.4. *If $1 \in R$ and $P \triangleleft R$, then P is a completely prime ideal of R if and only if P is a completely prime submodule of ${}_RR$.*

Proof. Suppose $am \in P$ for $a \in R$ and $m \in M = R$. By the definition of a completely prime ideal, $a \in P$ or $m \in P$ such that $aM \subseteq P$ or $m \in P$. Conversely, if for $a, b \in R$, $ab \in P$, by the definition of a completely prime submodule, $a \in aR \subseteq P$ or $b \in P$. $\square$

The following proposition offers several other characterizations of completely prime modules.

Proposition 2.5. *Let M be an R-module. For a proper submodule P of M, the following statements are equivalent.*

1. *P is a completely prime submodule of M;*
2. *for all $a \in R$ and every $m \in M$, if $<am> \subseteq P$, then either $<m> \subseteq P$ or $<aM> \subseteq P$;*
3. *$(P : M) = (P : m)$ for all $m \in M \setminus P$;*
4. *$\mathcal{P} = (P : M)$ is a completely prime ideal of R, and $(P : m) = (\bar{0} : \bar{m}) = \mathcal{P}$ for each $m \in M \setminus P$;*
5. *the set $\{(P : m) \ : \ m \in M \setminus P\}$ is a singleton.*

Corollary 2.6. *If P is a completely prime submodule of ${}_RM$, then $(P : m)$ is a two sided ideal of R for all $m \in M \setminus P$.*

Theorem 2.7. *For any module ${}_RM$, we have the following implications.*

$$\begin{array}{ccccc}
\textit{completely prime} & \Rightarrow & \textit{classical completely prime} & \Rightarrow & \textit{classical prime} \\
\Downarrow & & & & \Uparrow \\
\textit{s-prime} & & \Rightarrow & & \textit{prime}
\end{array}$$

Proof. completely prime $\Rightarrow$ classical completely prime: Suppose $abm \in P$. If $m \in P$, then $a<m>\subseteq P$ and $b<m>\subseteq P$. Suppose $m \notin P$. By the definition of a completely prime submodule, $bm \in P$ or $aM \subseteq P$. If $aM \subseteq P$, then $a<m>\subseteq P$. Now let $bm \in P$. By the definition of a completely prime submodule, $b<m>\subseteq bM \subseteq P$.

completely prime $\Rightarrow$ s-prime: Suppose $a^n<m>\subseteq P$ for some $n \in \mathbb{N}$. Then $a^n m \in P$. Because P is completely prime, it is classical completely prime such that $am \in a<m>\subseteq P$. By the definition of a completely prime submodule, we have $aM \subseteq P$ or $m \in P$. For the implications classical completely prime $\Rightarrow$ classical prime, s-prime $\Rightarrow$ prime and prime $\Rightarrow$ classical prime, see [11], [12] and [4], respectively. $\square$

Examples 2.8 and 2.9 below illustrate that in general classical completely prime $\not\Rightarrow$ completely prime and s-prime $\not\Rightarrow$ completely prime, respectively.

Example 2.8. *Let R be a commutative domain, P a prime ideal of R, if $M = R \oplus R$ is an R-module, the submodules $0 \oplus P$ and $P \oplus 0$ are classical completely prime submodules of M which are not completely prime.*

Example 2.9. *A simple module is s-prime but it need not be completely prime.*

Lambek in [13, p.364] called a module *symmetric* if $abm = 0$ implies $bam = 0$ for $a, b \in R$ and $m \in M$. We call a submodule P of an R-module M *symmetric* if $abm \in P$ implies $bam \in P$ for $a, b \in R$ and $m \in M$. So, a module M is symmetric if its zero submodule is symmetric. From [6], a right (or left) ideal I of a ring R is said to have the *insertion-of-factor-property (IFP)* if whenever $ab \in I$ for $a, b \in R$, we have $aRb \subseteq I$. A submodule N of an R-module M is said to have an *IFP* if whenever $am \in N$ for $a \in R$ and $m \in M$, we have $aRm \subseteq N$. A module M has IFP if the zero submodule has IFP. A submodule P of an R-module M is *completely semiprime* if for every $a \in R$ and each $m \in M$ such that $a^2 m \in P$, we have $a<m>\subseteq P$ (see [11]). It is easy to show that completely semiprime $\Rightarrow$ symmetric $\Rightarrow$ IFP.

Theorem 2.10. *Let M be an R-module. A submodule P of M is a completely prime submodule if and only if it is a prime submodule and has IFP.*

Proof. If $P \leq M$ is completely prime it is easy to see that it is prime and has IFP. Suppose $P \leq M$ is prime and has IFP. Let $a \in R$ and $m \in M$ such that $am \in P$. Since P has IFP, $a<m>\subseteq P$. Furthermore, since P is prime we get $m \in P$ or $aM \subseteq P$. $\square$

Remark 2.11. *In the place of "has IFP" in Theorem 2.10, one can have "completely semiprime" or "symmetric". This leads us to the following example.*

Example 2.12. *Every maximal submodule which is completely semiprime (or symmetric) is completely prime.*

Corollary 2.13. *Since M reduced implies M is symmetric which implies M has IFP, we have the following.*

1. *M is completely prime if and only if M is prime and reduced,*
2. *M is completely prime if and only if M is prime and symmetric,*
3. *M is completely prime if and only if M is prime and has IFP.*

Theorem 2.14. *For any module M over a commutative ring R, we have*

$$\text{s-prime} \Leftrightarrow \text{prime} \Leftrightarrow \text{completely prime} \Rightarrow \text{classical prime}$$

$$\Leftrightarrow \text{classical completely prime}.$$

Remark 2.15. *In a multiplicative module over a commutative ring completely prime submodules coincide with classical completely prime submodules (see* [4, Section 2]*).*

Definition 2.16. Let ${}_RM$ be a module. A nonempty set $S \subseteq M \setminus \{0\}$ is called a *multiplicative system* of ${}_RM$ if for each $a \in R$, $m \in M$ and for all $K \leq M$ such that $(K+ <m>) \cap S \neq \emptyset$ and $(K+ <aM>) \cap S \neq \emptyset$, then $(K+ <am>) \cap S \neq \emptyset$.

Corollary 2.17. *Let M be an R-module. A submodule P of M is completely prime if and only if $M \setminus P$ is a multiplicative system of M.*

Proof. $(\Rightarrow)$. Suppose $S = M \setminus P$. For $a \in R$, $K \leq M$ and $m \in M$ suppose $(K+ <m>) \cap S \neq \emptyset$ and $(K+ <aM>) \cap S \neq \emptyset$. If $(K+ <am>) \cap S = \emptyset$, then $<am> \subseteq P$ and since P is completely prime $<m> \subseteq P$ or $<aM> \subseteq P$. Thus, $(K+ <m>) \cap S = \emptyset$ and $(K+ <aM>) \cap S = \emptyset$, a contradiction. $(\Leftarrow)$. Let $a \in R$ and $m \in M$ such that $<am> \subseteq P$ but $<m> \nsubseteq P$ and $<aM> \nsubseteq P$. Then, $<m> \cap S \neq \emptyset$ and $<aM> \cap S \neq \emptyset$. By definition of a multiplicative system, $<am> \cap S \neq \emptyset$ such that $<am> \nsubseteq P$, a contradiction. $\square$

Proposition 2.18. *For any proper submodule P of ${}_RM$, and $S := M \setminus P$, the following statements are equivalent.*

1. *P is a completely prime submodule of M;*
2. *S is a multiplicative system of M;*
3. *for all $a \in R$ and every $m \in M$, if $<m> \cap S \neq \emptyset$ and $<aM> \cap S \neq \emptyset$ then $<am> \cap S \neq \emptyset$;*
4. *for all $a \in R$ and every $m \in M$, such that $m \in S$ and $<aM> \cap S \neq \emptyset$ then $am \in S$.*

Lemma 2.19. *Let M be an R-module, $S \subseteq M$ a multiplicative system of M and P a submodule of M maximal with respect to the property that $P \cap S = \emptyset$. Then, P is a completely prime submodule of M.*

Proof. Suppose $a \in R$ and $m \in M$ such that $<am> \subseteq P$. If $<m> \not\subseteq P$ and $<aM> \not\subseteq P$ then $(<m> +P) \cap S \neq \emptyset$ and $(<aM> +P) \cap S \neq \emptyset$. Since S is a multiplicative system of M, $(<am> +P) \cap S \neq \emptyset$. Since $<am> \subseteq P$, we have $P \cap S \neq \emptyset$, a contradiction. Hence, P must be a completely prime submodule. $\square$

Definition 2.20. Let R be a ring and M an R-module. For $N \leq M$, if there is a completely prime submodule containing N, we define

$$\text{co.}\sqrt{N} := \{m \in M \ : \ \text{every multiplicative system containing } m \text{ meets } N\}.$$

We write $\text{co.}\sqrt{N} = M$ when there are no completely prime submodules of M containing N.

Theorem 2.21. *Let M be an R-module and $N \leq M$. Then, either co.$\sqrt{N} = M$ or co.$\sqrt{N}$ equals the intersection of all completely prime submodules of M containing N, which is denoted by $\beta_{co}(N)$.*

Proof. Suppose co.$\sqrt{N} \neq M$. Then, $\beta_{co}(N) \neq \emptyset$. Both co.$\sqrt{N}$ and N are contained in the same completely prime submodules. By definition of co.$\sqrt{N}$ it is clear that $N \subseteq$ co.$\sqrt{N}$. Hence, any completely prime submodule of M which contains co.$\sqrt{N}$ must necessarily contain N. Suppose P is a completely prime submodule of M such that $N \subseteq P$, and let $t \in$ co.$\sqrt{N}$. If $t \notin P$, then the complement of P, $C(P)$ in M is a multiplicative system containing t and therefore we would have $C(P) \cap N \neq \emptyset$. However, since $N \subseteq P$, $C(P) \cap P = \emptyset$ and this contradiction shows that $t \in P$. Hence co.$\sqrt{N} \subseteq P$ as we wished to show. Thus, co.$\sqrt{N} \subseteq \beta_{co}(N)$. Conversely, assume $s \notin$ co.$\sqrt{N}$, then there exists a multiplicative system S such that $s \in S$ and $S \cap N = \emptyset$. From Zorn's Lemma, there exists a submodule $P \supseteq N$ which is maximal with respect to $P \cap S = \emptyset$. From Lemma 2.19, P is a completely prime submodule of M and $s \notin P$. $\square$

Proposition 2.22. *Let R be a ring and $\mathcal{P} \lhd R$, $\mathcal{P} \neq R$. The following statements are equivalent.*

(1) *$\mathcal{P}$ is a completely prime ideal of R*
(2) *there exists a completely prime R-module M such that $\mathcal{P} = (0 : M)_R$.*

Proof. (1) $\Rightarrow$ (2). Let $\mathcal{P}$ be a completely prime ideal and $M = R/\mathcal{P}$. M is an R-module with the usual operation. If $p \in \mathcal{P}$ and $x \in R$ then, $p(x+\mathcal{P}) = px+\mathcal{P} = \mathcal{P}$. Hence, $\mathcal{P} \subseteq (0 : M)_R$. If $a \in (0 : M)_R$, $a(r + \mathcal{P}) = \mathcal{P}$ for all $r \in R$, hence $aR \subseteq \mathcal{P}$ and since $\mathcal{P}$ is a completely prime ideal we get $a \in \mathcal{P}$, hence $(0 : M)_R = \mathcal{P}$. M is completely prime, for if $a \in R$ and $m \in M = R/\mathcal{P}$ such that $am = \bar{0}$ then

$m = m_1 + \mathcal{P}$ and $am_1 \in P$. Since $\mathcal{P}$ is completely prime, we have $a \in \mathcal{P}$ or $m_1 \in \mathcal{P}$ and it follows that $aM = \bar{0}$ or $m = \bar{0}$.

(2) $\Rightarrow$ (1). Follows from Proposition 2.5. □

Corollary 2.23. *A ring R is a completely prime ring if and only if there exists a faithful completely prime R-module.*

Example 2.24. *If R is a domain, ${}_RR$ is a faithful completely prime module.*

Example 2.25. *If I is a completely prime ideal of R, R/I is a completely prime R-module.*

3. Preradicals and Radicals

The terminology of radicals is that of [17]. Throughout this section rings have unity and all modules are unital left modules. A functor γ: R-mod $\to$ R-mod is called a *preradical* if $\gamma(M)$ is a submodule of M and $f(\gamma(M)) \subseteq \gamma(N)$ for each homomorphism $f : M \to N$ in R-mod. A *radical* γ is a preradical for which $\gamma(M/\gamma(M)) = 0$ for all $M \in R$-mod. A preradical is *hereditary* or *left exact* if $\gamma(N) = N \cap \gamma(M)$ whenever $N \leq M \in R$-mod (equivalently, if γ is a left exact functor). N is a *characteristic submodule* of M if $f(N) \subseteq N$ for every $f \in \mathrm{Hom}_R(M, M)$. We have the following proposition.

Proposition 3.1. [16, Proposition 1] *Let $\mathcal{M}$ be any nonempty class of modules closed under isomorphisms, i.e., if $A \in \mathcal{M}$ and $A \cong B$, then $B \in \mathcal{M}$. For any $M \in \mathcal{M}$ define*

$$\gamma(M) = \cap\{K \ : \ K \leq M, M/K \in \mathcal{M}\}.$$

It is assumed that $\gamma(M) = M$ if $M/K \notin \mathcal{M}$ for all $K \leq M$. Then

1. *$\gamma(M/\gamma(M)) = 0$ for all modules M,*
2. *if $\mathcal{M}$ is closed under taking nonzero submodules, γ is a radical,*
3. *if $\mathcal{M}$ is closed under taking essential extensions, then $\gamma(M) \cap N \subseteq \gamma(N)$ for all $N \leq M$.*

In particular, γ is a left exact radical if $\mathcal{M}$ is closed under nonzero submodules and essential extensions.

For any module M, we define the completely prime radical $\beta_{co}(M)$ as co.$\sqrt{0}$.

From Theorem 2.21, we have

$$\beta_{co}(M) = \cap\{K \ : \ K \leq M, M/K \text{ is completely prime}\}$$

which is a radical by Proposition 3.1 since completely prime modules are closed under taking nonzero submodules.

Let $\beta(M)$ be the prime radical of M (the intersection of all prime submodules of M).

Theorem 3.2. *If R is a commutative ring, then $\beta_{co}(M) = \beta(M)$.*

Proof. If R is a commutative ring prime and completely prime submodules are indistinguishable. □

Proposition 3.3. *For any R-module M,*

1. *$\beta_{co}(M)$ is a characteristic submodule of M,*
2. *If M is projective then $\beta_{co}(R)M = \beta_{co}(M)$.*

Proof. Follows from [7, Proposition 1.1.3]. □

Proposition 3.4. *For any $M \in R$-mod,*

1. *if $M = \bigoplus\limits_{\Lambda} M_\lambda$ is a direct sum of submodules M_λ $(\lambda \in \Lambda)$, then $\beta_{co}(M) = \bigoplus\limits_{\Lambda} \beta_{co}(M_\lambda)$,*
2. *if $M = \prod\limits_{\Lambda} M_\lambda$ is a direct product of submodules M_λ $(\lambda \in \Lambda)$, then $\beta_{co}(M) \subseteq \prod\limits_{\Lambda} \beta_{co}(M_\lambda)$.*

Proof. Follows from [7, Proposition 1.1.2]. □

The following examples show that the preradical β_{co} is not hereditary.

Example 3.5. *Consider the $\mathbb{Z}$-module $\mathbb{Z}_4$, $\beta_{co}(\mathbb{Z}_4) = 2\mathbb{Z}_4$ and $\beta_{co}(2\mathbb{Z}_4) = (0)$, hence $\beta_{co}(\mathbb{Z}_4) \cap 2\mathbb{Z}_4 = 2\mathbb{Z}_4 \nsubseteq \beta_{co}(2\mathbb{Z}_4) = (0)$.*

Example 3.6. *Take a $\mathbb{Z}$-module $M = \mathbb{Z}_{p^\infty}$ where p is a prime number. By* [5, Remark 4.6], *$\beta_{co}(M) = \mathbb{Z}_{p^\infty}$ and if N is a proper submodule of M, N has a (maximal) completely prime submodule, say P. Thus, $\beta_{co}(N) \subset P \subset N = \beta_{co}(M) \cap N$ and $\beta_{co}(M) \cap N \nsubseteq \beta_{co}(N)$.*

However, for direct summands, we have the following proposition.

Proposition 3.7. *Let $M = X \oplus Y$. Then the following statements hold.*

1. *If P is a completely prime submodule of X, then $P \oplus Y$ is a completely prime submodule of M,*
2. *$\beta_{co}(M) \cap X \subseteq \beta_{co}(X)$.*

Proof. (1) Let $a \in R$ and $m = (m_1, m_2) \in M = X \oplus Y$. Suppose $am \in P \oplus Y$. Then $am_1 \in P$ and $am_2 \in Y$. Since P is a completely prime submodule of X, we have $aX \subseteq P$ or $m_1 \in P$. $aX \subseteq P$ implies $aM = aX \oplus aY \subseteq P \oplus Y$. If $m_1 \in P$, then $m = (m_1, m_2) \in P \oplus Y$.

(2) If Q is any completely prime submodule of X, then $\beta_{co}(M) \cap X \subseteq Q$. Hence, $\beta_{co}(M) \cap X \subseteq \beta_{co}(X)$. □

Corollary 3.8. *For any direct summand N of M, $\beta_{co}(M) \cap N = \beta_{co}(N)$.*

Remark 3.9. *It follows from Proposition 3.1 that completely prime modules are not closed under taking essential extensions.*

Definition 3.10. A functor γ: R-mod $\to$ R-mod is called a *Hoehnke radical* if $f(\gamma(M)) \subseteq \gamma(f(M))$ for every homomorphism $f : M \to f(M)$ (see [9, p. 454]) and moreover, $\gamma(M/\gamma(M)) = 0$ for all $M \in R$-mod. γ is *complete* if $\gamma(K) = K \leq M \in R$-mod implies $K \subseteq \gamma(M)$. γ is *idempotent* if $\gamma(\gamma(M)) = \gamma(M)$ for all $M \in R$-mod. A *Kurosh-Amitsur radical* is a complete idempotent Hoehnke radical.

Theorem 3.11. *The completely prime radical β_{co} is a complete Hoehnke radical which is not Kurosh Amistur.*

Proof. β_{co} is a Hoehnke radical by the nature of its definition, cf., [9, (4), p. 455]. All preradicals are complete, cf., [9, (3) p.455]. The radical β_{co} is not idempotent since $2\mathbb{Z}_4 = \beta_{co}(\mathbb{Z}_4) \neq \beta_{co}(\beta_{co}(\mathbb{Z}_4)) = (0)$, and hence not Kurosh-Amitsur. $\square$

4. Torsion Theory Induced by the Radical β_{co}

Definition 4.1. ([17, p.139]) A *torsion theory* in the category of modules R-mod is a pair $(\mathcal{T}, \mathcal{F})$ of classes of modules in R-mod such that

(1) $\mathrm{Hom}(T, F) = 0$ for all $T \in \mathcal{T}$, $F \in \mathcal{F}$.
(2) If $\mathrm{Hom}(C, F) = 0$ for all $F \in \mathcal{F}$, then $C \in \mathcal{T}$.
(3) If $\mathrm{Hom}(T, C) = 0$ for all $T \in \mathcal{T}$, then $C \in \mathcal{F}$.

Define $\mathcal{T}_{\beta_{co}} = \{M : \beta_{co}(M) = M\}$ and $\mathcal{F}_{\beta_{co}} = \{M : \beta_{co}(M) = 0\}$. $\mathcal{T}_{\beta_{co}}$ is a torsion class, $\mathcal{F}_{\beta_{co}}$ is a torsionfree class and the pair $(\mathcal{T}_{\beta_{co}}, \mathcal{F}_{\beta_{co}})$ is a torsion theory (see [17, p.140]). $\mathcal{T}_{\beta_{co}}$ coincides with the class of modules with no completely prime submodules.

Proposition 4.2. [17, Proposition 2.1] *$\mathcal{T}$ is a torsion class for some torsion theory if and only if it is closed under quotient objects, direct products and extensions.*

Corollary 4.3. *For any $M \in R$-mod and $N \leq M$, $\mathcal{T}_{\beta_{co}}$ is closed under quotients, direct products and extensions.*

By Example 3.6, $\mathcal{T}_{\beta_{co}}$ is not closed under taking submodules.

Corollary 4.4. *The following statements hold.*

(1) $\bar{\beta_{co}}(M) = \sum\{N : N \leq M$ *and* $\beta_{co}(N) = N\}$ *is an idempotent preradical,* $\bar{\beta_{co}} \subseteq \beta_{co}$, $\mathcal{T}_{\beta_{co}} = \mathcal{T}_{\bar{\beta_{co}}}$ *and* $\bar{\beta_{co}}$ *is the largest idempotent preradical contained in* β_{co}.
(2) $\hat{\beta_{co}}(M) = \cap\{N : N \leq M, M/N \in \mathcal{F}_{\beta_{co}}\}$ *is a radical.* $\beta_{co} \subseteq \hat{\beta_{co}}$, $\mathcal{F}_{\beta_{co}} = \mathcal{F}_{\hat{\beta_{co}}}$ *and* $\hat{\beta_{co}}$ *is the least radical containing* β_{co}.

Proof. Follows from [7, Proposition 1.1.5]. □

Proposition 4.5. *If $M \in \mathcal{T}_{\beta_{co}}$, then for each nonzero homomorphic image N of M there exists $K \leq N$ such that $0 \neq K \in \mathcal{T}_{\beta_{co}}$. This is the module analogue of (R1) in* [10, Theorem 2.1.5].

Proof. Since $\mathcal{T}_{\beta_{co}}$ is closed under quotients, the result follows from [17, Proposition 2.5]. □

Example 4.6. *Any completely prime module M is β_{co}-torsionfree.*

Remark 4.7. *All said about completely prime (sub)modules in section 3 also holds for prime (sub)modules. Thus, the prime radical $\beta(M)$ is a complete Hoenhke radical which is neither hereditary nor idempotent (hence not Kurosh-Amistur). Furthermore, prime modules are not closed under taking essential extensions. However, if we define a faithful prime radical, $\beta_0(M)$ as,*

$$\beta_0(M) = \{P \ : \ P \leq M, M/P \text{ is faithful and prime}\}$$

β_0 is a Kurosh-Amitsur radical (see [16, Section 3]*). Furthermore, the class of all faithful prime modules is closed under essential extensions. This leads us to the following.*

Question 4.1. Is the class of all faithful completely prime modules closed under essential extensions?

5. Comparison of the Radicals $\beta_{co}({}_RR)$ and $\beta_{co}(R)$

Lemma 5.1. *For any associative ring R, $\beta_{co}({}_RR) \subseteq \beta_{co}(R)$.*

Proof. Let $x \in \beta_{co}({}_RR)$ and I be a completely prime ideal of R. From Proposition 2.22, we have R/I is a completely prime R-module. Hence, $x \in I$ and we have $x \in \beta_{co}(R)$, i.e., $\beta_{co}({}_RR) \subseteq \beta_{co}(R)$. □

Remark 5.2. *In general the containment in Lemma 5.1 is strict.*

Example 5.3. *Let $R = \{\begin{pmatrix} x & y \\ 0 & 0 \end{pmatrix} \ : \ x, y \in \mathbb{Z}_2\}$ and $M = {}_RR$. (0) is a completely prime submodule of ${}_RR$. Hence, $\beta_{co}({}_RR) = 0$. $(0 : R)_R$ is a completely prime ideal of R but $(0 : R)_R \neq (0)$. For if $b \neq 0$, $b \in \mathbb{Z}_2$, then $\begin{pmatrix} 0 & b \\ 0 & 0 \end{pmatrix} R = 0$. Hence, $\beta_{co}(R) \subseteq (0 : R)_R$. But since $(0 : R)_R(0 : R)_R = 0$ we have $(0 : R)_R \subseteq \beta_{co}(R)$. Hence, $\beta_{co}(R) = (0 : R)_R \neq 0$.*

Lemma 5.4. *For any ring R and any R-module M we have*

$$\beta_{co}(R) \subseteq (\beta_{co}(M) : M)_R.$$

Proof. $(\beta_{co}(M) : M)_R = (\bigcap\limits_{P \leq M} P : M) = \bigcap\limits_{P \leq M} (P : M)$, where P is a completely prime submodule of M. Since $(P : M)_R$ is a completely prime ideal of R for each completely prime submodule P of M, we get $\beta_{co}(R) \subseteq (\beta_{co}(M) : M)_R$, i.e., $\beta_{co}(R)M \subseteq \beta_{co}(M)$. □

The containment in Lemma 5.4 is strict: Let $R = \mathbb{Z}$ and $M = \mathbb{Z}_{p^\infty} \oplus \mathbb{Z}$ for some prime number p. $\beta_{co}(M) = \mathbb{Z}_{p^\infty}$ and $\beta_{co}(R) = (0)$, i.e., $\beta_{co}(R)M = (0)$.

Proposition 5.5. *For any ring R, $\beta_{co}(R) = (\beta_{co}({}_RR) : R)_R$.*

Proof. From Lemma 5.4, $\beta_{co}(R) \subseteq (\beta_{co}({}_RR) : R)_R$. Since $\beta_{co}({}_RR) \subseteq \beta_{co}(R)$ we have $\beta_{co}(R) \subseteq (\beta_{co}({}_RR) : R) \subseteq (\beta_{co}(R) : R)$. Let $x \in (\beta_{co}(R) : R)$. Hence $xR \subseteq \beta_{co}(R) = \bigcap\limits_{\mathcal{P} \text{ completely prime in } R} \mathcal{P} \subseteq \mathcal{P}$ for all completely prime ideals $\mathcal{P}$ of R. Since $xR \subseteq \mathcal{P}$ for $\mathcal{P}$ completely prime, we have $x \in \mathcal{P}$ and $x \in \beta_{co}(R)$. Hence, $(\beta_{co}(R) : R) \subseteq \beta_{co}(R)$ and we are done. □

Lemma 5.6. *For all R-modules M,*

(1) $\beta_{co}(M) = \{x \in M \ : \ Rx \subseteq \beta_{co}(M)\}$,
(2) *If $\beta_{co}(R) = R$ then $\beta_{co}(M) = M$.*

Proof. (1) Since $\beta_{co}(M) \leq M$, we have $R\beta_{co}(M) \subseteq \beta_{co}(M)$. Conversely, let $x \in M$ with $Rx \subseteq \beta_{co}(M)$. Hence $Rx \subseteq P$ for all completely prime submodules P of M. Since P is also a prime submodule, we have $x \in P$ and hence $x \in \beta_{co}(M)$.

(2) $R = \beta_{co}(R)$ gives $R \subseteq (\beta_{co}(M) : M)$ from Lemma 5.4. Hence, $RM \subseteq \beta_{co}(M)$ and from (1), we have $M \subseteq \beta_{co}(M)$, i.e., $M = \beta_{co}(M)$. □

Proposition 5.7. *Let R be any ring. Then, any of the following conditions implies $\beta_{co}(R) = \beta_{co}({}_RR)$.*

(1) *R is commutative;*
(2) *$x \in xR$ for all $x \in R$, e.g., if R has an identity or R is Von Neumann regular.*

Proof. (1) Since R is commutative, it follows from Proposition 5.5 and Lemma 5.6 that $\beta_{co}(R) \subseteq \beta_{co}({}_RR) \subseteq \beta_{co}(R)$ and $\beta_{co}(R) = \beta_{co}({}_RR)$.

(2) Let $x \in \beta_{co}(R)$, then from Proposition 5.5, $xR \subseteq \beta_{co}({}_RR)$ and since $x \in xR$, we get $x \in \beta_{co}({}_RR)$ such that $\beta_{co}({}_RR) = \beta_{co}(R)$. □

6. A Special Class of Completely Prime Modules

A class ρ of associative rings is called a special class if ρ is hereditary, consists of prime rings and is closed under essential extensions (see [10, p.80]). Andrunakievich and Rjabuhin in [1] extended this notion to modules and showed that prime modules, irreducible modules, simple modules, modules without zero divisors, etc form special classes of modules. de la Rosa and Veldsman in [9] defined a weakly special class of modules. We follow the definition in [9] of a weakly special class of modules to define a special class of modules.

Definition 6.1. For a ring R, let $\mathcal{K}_R$ be a (possibly empty) class of R-modules. Let $\mathcal{K} = \cup\{\mathcal{K}_R \ : \ R \text{ a ring}\}$. $\mathcal{K}$ is a *special class* of modules if it satisfies:

- **S1** $M \in \mathcal{K}_R$ and $I \lhd R$ with $I \subseteq (0 : M)_R$ implies $M \in \mathcal{K}_{R/I}$.
- **S2** If $I \lhd R$ and $M \in \mathcal{K}_{R/I}$ then $M \in \mathcal{K}_R$.
- **S3** $M \in \mathcal{K}_R$ and $I \lhd R$ with $IM \neq 0$ implies $M \in \mathcal{K}_I$.
- **S4** $M \in \mathcal{K}_R$ implies $RM \neq 0$ and $R/(0 : M)_R$ is a prime ring.
- **S5** If $I \lhd R$ and $M \in \mathcal{K}_I$, then there exists $N \in \mathcal{K}_R$ such that $(0 : N)_I \subseteq (0 : M)_I$.

Following similar techniques of [19], we get the following theorem.

Theorem 6.2. *Let $\mathcal{M} = \cup\mathcal{M}_R$ be a special class of modules. Then,*

$$\mathcal{J} = \{R \ : \ \textit{there exists } M \in \mathcal{M}_R \textit{ with } (0 : M)_R = 0\} \cup \{0\}$$

is a special class of rings. If $\mathcal{R}$ is the corresponding special radical then,

$$\mathcal{R}(R) := \cap\{(0 : M)_R \ : \ M \in \mathcal{M}\}.$$

Theorem 6.3. *Let $\mathcal{J}$ be a special class of rings and for every ring R, let*

$$\mathcal{M}_R = \{M \ : \ M \textit{ is an } R\textit{-module}, RM \neq 0 \textit{ and } R/(0 : M)_R \in \mathcal{J}\}.$$

If $\mathcal{M} = \cup\mathcal{M}_R$, then $\mathcal{M}$ is a special class of modules. If r is the corresponding special radical and M is any R-module, then

$$r(M) := \cap\{P \leq M \ : \ M/P \in \mathcal{M}_R\}.$$

For the completely prime modules, we have the following theorem.

Theorem 6.4. *Let R be any ring and*

$$\mathcal{M}_R := \{M \ : \ M \textit{ is a completely prime } R\textit{-module with } RM \neq 0\}.$$

If $\mathcal{M} = \cup\mathcal{M}_R$, then $\mathcal{M}$ is a special class of R-modules.

Proof. S1 Let $M \in \mathcal{M}_R$ and $I \lhd R$ with $IM = 0$. M is an R/I-module via $(r + I)m = rm$. We show $M \in \mathcal{M}_{R/I}$. Let $a + I \in R/I$ and $m \in M$ such that $(a + I)m = 0$. Then $am = 0$ such that $aM = 0$ or $m = 0$ since $M \in \mathcal{M}_R$. Thus, $M \in \mathcal{M}_{R/I}$.

S2 Let $I \lhd R$ and $M \in \mathcal{M}_{R/I}$. M is an R-module w.r.t. $rm = (r+I)m$ for $r \in R$, $m \in M$. Let $a \in R$, $m \in M$ such that $am = 0 \Leftrightarrow (a + I)m = 0$. $(a + I)M = 0$ or $m = 0$ since $M \in \mathcal{M}_{R/I}$. Thus, $aM = 0$ or $m = 0$ and $M \in \mathcal{M}_R$.

S3 Suppose $M \in \mathcal{M}_R$, $I \lhd R$ and $IM \neq 0$. Let $a \in I$, $m \in M$ such that $am = 0$. Since $M \in \mathcal{M}_R$, $m = 0$ or $aM = 0$. Therefore, $M \in \mathcal{M}_I$.

S4 Let $M \in \mathcal{M}_R$. Hence $RM \neq 0$. Since $(0 : M)_R$ is a completely prime ideal of R, it is a prime ideal and $R/(0 : M)_R$ is a prime ring.

S5 Let $I \lhd R$ and $M \in \mathcal{M}_I$. $(0 : M)_I$ is a completely prime ideal of I. We have $(0 : M)_I \lhd I \lhd R$ and since $I/(0 : M)_I$ is a completely prime ring we have $(0 : M)_I \lhd R$. Choose $K/(0 : M)_I \lhd R/(0 : M)_I$ maximal w.r.t $I/(0 : M)_I \cap K/(0 : M)_I = 0$. Then $I/(0 : M)_I \cong (I + K)/K \lhd \cdot R/I$. Since the class of completely prime rings is essentially closed, R/I is completely prime. Let $N = R/K$. N is an R-module and $RN \neq 0$. From Proposition 2.22 we have $(0 : N)_R = K$. $(0 : N)_I \subseteq (0 : M)_I$, for let $x \in (0 : N)_I$, then $xR \subseteq I \cap K \subseteq (0 : M)_I$ and it follows that $x \in (0 : M)_I$ thus, $\mathcal{M}$ is a special class. □

Corollary 6.5. *If $\mathcal{M}_{co}$ is the special class of completely prime modules, then the special radical induced by $\mathcal{M}_{co}$ on a ring R is given by*

$$\beta_{co}(R) = \cap\{(0 : M)_R \ : \ M \text{ is a completely prime } R\text{-module}\}$$

$$= \cap\{I \lhd R \ : \ I \text{ is a completely prime ideal}\}$$

$$= \mathcal{N}_g(R) \text{ the generalized nil radical.}$$

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