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ON THE MAXIMAL CARDINALITY OF AN INFINITE CHAIN OF VECTOR SUBSPACES

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Abstract. For each infinite cardinal number κ, let $\Omega(\kappa)$ be the supremum of the cardinalities of chains of subsets of a set of cardinality κ. ($\Omega(\kappa)$ is equal to what has been called $\mathrm{ded}(\kappa)$ in the literature.) Let K be a field and V a vector space over K. Let $\Lambda(V)$ be the supremum of the cardinalities of chains of vector subspaces of V. Let the dimension of V as a vector space over K be the infinite cardinal number κ. Then $\Omega(\kappa) \leq \Lambda(V) \leq \Omega(|V|)$, and so $\Lambda(V) > \kappa$, contrary to a result of Menth. If, in addition, K is either finite or infinite with $|K| \leq \kappa$, then $\Omega(\kappa) = \Omega(|V|)$ $(= \Lambda(V))$.

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1. Introduction

Since the appearance of [5], it has been fruitful to intuitively measure the size of a field extension L/K by means of a certain invariant that has been denoted by $\lambda(L/K)$. For an infinitely, but not countably, generated field extension L/K, the calculation of $\lambda(L/K)$ was recently reduced to a question of set theory in [4, Theorem 4.3]. Our main interest here is in defining an analogous way to measure the size of an infinite-dimensional vector space V and to see if its calculation reduces analogously to set-theory. Our work will be pursued while assuming only ZFC (that is, the usual Zermelo-Fraenkel foundations for set theory, together with the Axiom of Choice).

Consider a vector space V over a field K. We will be interested in calculating, or at least finding lower and upper bounds for, an invariant $\Lambda(V)$, which is defined as the supremum of the cardinalities of chains of K-subspaces of V. (As usual, $|S|$ will denote the cardinal number of a set S.) It is easy to see that if V is finite-dimensional, with $n := \dim_K(V) < \infty$, then every maximal chain $\mathcal{C}$ of K-subspaces of V has length n (that is, $|\mathcal{C}| = n + 1$), and so $\Lambda(V) = n + 1$. Accordingly, our interest here is focused on the case where V is infinite-dimensional over K, with $\dim_K(V)$ being an infinite cardinal number κ. It was asserted in [9] (without the Λ notation) that $\Lambda(V) = \kappa$.

We will show in Remark 2.5 (a) that the above assertion from [9] is incorrect. While our work will assume only ZFC, one could detect the error in [9] rather quickly if one also assumes the Generalized Continuum Hypothesis (GCH). Indeed, suppose that κ is an infinite cardinal number and let B be any set such that $|B| = \kappa$. It follows from ZFC+GCH that there is a chain $\mathcal{C} = \{B_i \mid i \in I\}$ of subsets B_i of B (with $B_i \neq B_j$ if $i \neq j$ in I) such that $|\mathcal{C}| = 2^\kappa$. (For a correct explanation of which set-theoretic principles imply the preceding assertion, see [2, Remark 2.2].) Now suppose that V is a vector space over a field K with $\dim_K(V) = \kappa$ and that the above B arose as a K-basis of V (so that $|B| = \kappa$). For each $i \in I$, let W_i denote the K-subspace of V that is spanned by B_i. It is clear that if $i, j \in I$ with $B_i \subset B_j$, then $W_i \subset W_j$. (As usual, $\subset$ denotes proper containment.) Consequently, $\mathcal{D} := \{W_i \mid i \in I\}$ is a chain of K-subspaces of V such that $|\mathcal{D}| = |\mathcal{C}| = 2^\kappa$, and so $\Lambda(V) \geq |\mathcal{D}| > \kappa$, contrary to the assertion in [9]. The treatment in Remark 2.5 will have much of the same tempo as in the preceding argument, but instead of appealing to GCH (which will play no further role in this paper), our work will depend on the behavior of a quantity $\Omega(\kappa)$ whose definition is recalled next from [4].

Let κ be an infinite cardinal number and U a set of cardinality κ. Let T denote the set of all chains of subsets of U. Then, as in [4], we define $\Omega(\kappa) := \sup\{|\mathcal{C}| \mid \mathcal{C} \in T\}$. For any (infinite cardinal number) κ, the $\Omega(\kappa)$ concept has appeared in the literature as follows (cf. [1, page 87]): $\mathrm{ded}(\kappa) := \sup\{\mu \mid$ there is a linear order of cardinal κ with μ Dedekind cuts$\}$. The interested reader can easily verify that $\Omega(\kappa)$ and $\mathrm{ded}(\kappa)$ define the same quantity. Our form of the definition of $\Omega(\kappa)$ seems more intuitively suited for the task at hand, and so we will not mention Dedekind cuts (or $\mathrm{ded}(\kappa)$) again.

It is convenient next to recall the definition of $\lambda(L/K)$ from [5] and its interpretation using $\Omega(-)$ from [4]. Let L/K be a field extension. Then by definition, $\lambda(L/K)$ is the supremum of the set of cardinal numbers that arise as lengths of chains $\{F_i\}$ involving fields F_i such that $K \subseteq F_i \subseteq L$. (As usual, the "length" of a finite chain is defined as the number of "jumps" in it; to avoid possible ambiguity, we take the "length" of any infinite chain to be its cardinality. Note that one could consider a vectorial analogue of $\lambda(L/K)$ by positing an inclusion of vector spaces $V_1 \subseteq V_2$ and taking the supremum of the cardinalities of chains $\{W_i\}$ of vector spaces W_i such that $V_1 \subseteq W_i \subseteq V_2$. By a standard isomorphism theorem, there is an order-isomorphism between the set of such chains and the set of chains of vector subspaces of the vector space V_2/V_1, and so we can equate this supremum with the above definition of $\Lambda(V)$, for the case $V = V_2/V_1$.) Now, suppose that the field extension L/K is infinitely generated (in the sense that there is no finite set S such that $K(S) = L$) but not countably generated. By [4, Theorem 4.3], if κ is the (infinite cardinal number that is the) infimum of the cardinalities of sets S such that $K(S) = L$, then $\lambda(L/K) = \Omega(\kappa)$. The present work was motivated by the question whether an infinite-dimensional vector space V (over a field K) must satisfy the analogous assertion that $\Lambda(V) = \Omega(\dim_K(V))$.

In fact, Corollary 2.4 shows that $\Lambda(V) = \Omega(\dim_K(V))$ does hold whenever K is a finite field (and V is infinite-dimensional). This follows from our main result, Theorem 2.3, which gives the following lower and upper bounds: regardless of whether the field K is finite, any infinite-dimensional vector space V over K, with $\kappa := \dim_K(V)$, satisfies $\Omega(\kappa) \leq \Lambda(V) \leq \Omega(|V|)$. Using this fact, we obtain the above-mentioned counterexample to [9] in Remark 2.5 (a) with the aid of a fact from [4] about the behavior of $\Omega(-)$. For convenience, the latter fact is stated in Lemma 2.2.

Note that our assumption of the ZFC foundations ensures that we can use the usual laws of arithmetic for infinite cardinal numbers. For the appropriate background on related transfinite matters, we recommend [8].

2. Results

For the sake of completeness, we begin by recording the value of $\Lambda(V)$ in case V is finite-dimensional. The proof of Proposition 2.1 may safely be left to the reader.

Proposition 2.1. *Let V be a vector space over a field K, with $n := \dim_K(V) < \infty$. Then every maximal chain of K-subspaces of V has length n, and so $\Lambda(V) = n+1$.*

Henceforth, fix the following notation and assumptions: K is a field, V is an infinite-dimensional vector space over K of dimension $\kappa := \dim_K(V)$, and $\mathcal{B}$ is a basis of V over K (so that $|\mathcal{B}| = \kappa$). To analyze $\Lambda(V)$ in the spirit of an argument that was given in the Introduction (but, this time, without using GCH), we will need to use the following result. (Note that the cited result, which is stated in [4] as a result about an arbitrary infinite cardinal number, applies here because every cardinal number is the dimension of some vector space over any field.)

Lemma 2.2. ([4, Lemma 4.4]) $\kappa < \Omega(\kappa) \leq 2^\kappa$.

We can now give our main result.

Theorem 2.3. $\Omega(\kappa) \leq \Lambda(V) \leq \Omega(|V|)$.

Proof. Recall that $\Lambda(V)$ (resp., $\Omega(|V|)$) is the supremum of the cardinalities of chains of K-subspaces (resp., of subsets) of V. Since every K-subspace of V is a subset of V, it is now clear that $\Lambda(V) \leq \Omega(|V|)$.

It will take slightly more work to establish the lower bound. Insofar as possible, we modify an argument that was given in the Introduction. Since $|\mathcal{B}| = \kappa$, one sees that $\Omega(\kappa)$ is the supremum of the cardinalities of chains of subsets of $\mathcal{B}$. Consider any such chain $\mathcal{C} = \{B_i \mid i \in I\}$. Without loss of generality, $B_i \neq B_j$ if $i \neq j$ in I. For each $i \in I$, let W_i denote the K-subspace of V that is spanned by B_i. It is clear that if $i, j \in I$ with $B_i \subset B_j$, then $W_i \subset W_j$. Consequently, $\mathcal{D} := \{W_i \mid i \in I\}$ is a chain of K-subspaces of V such that $|\mathcal{C}| = |\mathcal{D}| \leq \Lambda(V)$. Taking the supremum as $\mathcal{C}$ varies leads to $\Omega(\kappa) \leq \Lambda(V)$, as required. □

We next give some important cases where $\Lambda(V) = \Omega(\kappa)$.

Corollary 2.4. *Suppose that the field K either is finite or is infinite but satisfies $|K| \leq \kappa$. Then $\Omega(\kappa) = \Omega(|V|) = \Lambda(V)$.*

Proof. The usual laws of arithmetic with infinite cardinal numbers give that $\kappa \cdot \kappa$ and $\aleph_0 \cdot \kappa$ each equal κ. Since the K-basis $\mathcal{B}$ has cardinality κ, it follows easily that the (cardinal) number of finite subsets of $\mathcal{B}$ is κ. Given the assumptions on $|K|$, this, in turn, leads to $|V| = \kappa$. In view of Theorem 2.3 (and the Schroeder-Bernstein Theorem), the assertion is now immediate. □

Part (a) of Remark 2.5 shows that the result from [9] that was mentioned in the Introduction is false for every infinite cardinal number κ. In Remark 2.5 (c), we close by using the field-theoretic material on $\lambda(L/K)$ from [4] to give an amusing second proof (which is much more elaborate than the above proof) of the easier part of Theorem 2.3 in case $\kappa > \aleph_0$ and K has characteristic 0.

Remark 2.5. *(a) Let K, V and κ be as in the riding assumptions. Then by combining Lemma 2.2 with Theorem 2.3, we get that $\kappa < \Omega(\kappa) \leq \Lambda(V)$, whence $\kappa < \Lambda(V)$. In particular, $\Lambda(V) \neq \kappa$, contrary to what was asserted in* [9].

(b) It is natural to ask if the conclusion of Corollary 2.4 holds if $|K| > \kappa$. This question remains open. Perhaps the case $|K| > \kappa$ will prove to be analogous to the requirement that $\kappa > \aleph_0$ in [4, Theorem 4.3]. *That requirement was necessary because of the behavior of J-extensions, in the sense of* [7]. *(Indeed, if L/K is a*

J-extension, then each maximal chain of fields going from K to L is denumerable and $\lambda(L/K) = \aleph_0$ [3, Proposition 2.7]*, although* $\Omega(\aleph_0) = 2^{\aleph_0}$ [4, Proposition 4.2]*.) When one studies vector spaces, there may be no anomaly of the kind presented by J-extensions in the field-theoretic setting. In any case, despite the above Theorem and Corollary, we have not fully realized the goal of reducing the calculation of $\Lambda(V)$ to set theory (in the spirit of* [4, Theorem 4.3]*).*

(c) We next sketch how to use field theory to give an alternate proof that $\Lambda(V) \leq \Omega(|V|)$ in case $\kappa > \aleph_0$ and K has characteristic 0*. It is enough to show that if $\mathcal{C} := \{W_i \mid i \in I\}$ is any chain of K-subspaces of V, then $|\mathcal{C}| \leq \Omega(|V|)$. Without loss of generality, $W_i \neq W_j$ if $i \neq j$ in I. For each $i \in I$, let D_i denote the group ring $K[W_i]$. Since K has characteristic* 0*, a standard fact about monoid rings* [6, Theorem 8.1] *shows that D_i is an integral domain. Let F_i (resp., L) denote the quotient field of D_i (resp., of $K[V]$). If $W_i \subset W_j$, with $v \in W_j \setminus W_i$, one can check that X^v is in both $D_j \setminus D_i$ and $F_j \setminus F_i$. Thus, $\mathcal{D} := \{F_i \mid i \in I\}$ is a chain of fields that are contained between K and L such that $|\mathcal{D}| = |\mathcal{C}|$. As $|\mathcal{D}| \leq \lambda(L/K)$, it now suffices to prove that $\lambda(L/K) \leq \Omega(|V|)$. In fact, $\lambda(L/K) = \Omega(|V|)$ (which is perhaps the most interesting part of this argument). Indeed, by* [4, Theorem 4.3] *(which applies since $\kappa > \aleph_0$), it suffices to show that $|V|$ is the infimum of the cardinalities of generating sets of the field extension L/K. This, in turn, follows from* [4, Corollary 3.3] *since L is purely transcendental over K with transcendence degree $|V|$; in fact, $\{X^v \mid v \in V\}$ is a transcendence basis of L/K.*

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