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INVERSION OF RATIONAL SURFACES PARAMETERIZABLE BY QUADRATICS

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Abstract. Given a rational surface parameterized by quadratics, the main aim of the paper is to describe a method for finding an inversion map for such parameterized surfaces.

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1. Introduction

Let $\mathbb{k}$ be an algebraically closed field with $\mathrm{char}(\mathbb{k}) = 0$, and let $\mathbb{P}^3$ be the three dimensional projective space over $\mathbb{k}$. Let $S \subset \mathbb{P}^3$ be a rational surface which is given as the image of the generic 1-to-1 parametrization

$$\begin{array}{cccc} \phi : & \mathbb{P}^2 & \longrightarrow & S \\ & [s;t;u] & \mapsto & [f_0;f_1;f_2;f_3] \end{array} \tag{1}$$

where $f_0,\ f_1,\ f_2,\ f_3$ are linearly independent homogeneous quadratic polynomials in the standard $\mathbb{Z}$-graded algebra $R := \mathbb{k}[s,t,u]$, and $\gcd(f_0,\ f_1,\ f_2,\ f_3) = 1$. We define the ideal $I = \langle f_0,\ f_1,\ f_2,\ f_3\rangle \subset R$.

An inversion map of the rational surfaces S is a dominant rational map

$$\begin{array}{cccc} \psi : & S & \longrightarrow & \mathbb{P}^2 \\ & [x;y;z;w] & \mapsto & [F_0(x,y,z,w);F_1(x,y,z,w);F_2(x,y,z,w)] \end{array} \tag{2}$$

such that $\psi\circ\phi = id_{\mathbb{P}^2}$ and $\psi\circ\phi = id_S$, where $F_0(x,y,z,w), F_1(x,y,z,w), F_2(x,y,z,w)$ are homogeneous polynomials (forms) of $\mathbb{k}[x,y,z,w]$ with $\gcd(F_0,\ F_1,\ F_2) = 1$.

The birationality of ψ means:

$$\phi(\psi[x;y;z;w]) = [x;y;z;w], \text{ for almost all } [x;y;z;w] \in S \tag{3}$$

and

$$\psi(\phi[s;t;u]) = [s;t;u], \text{ for almost all } [s;t;u] \in \mathbb{P}^2. \tag{4}$$

In this paper, our focus is to describe a method of finding the inversion map ψ of rational surfaces given by the quadratically parametrization ϕ. We proceed in the following fashion. In Section 2, we review some important information concerning quadratically parametrized surfaces. In particular, we recall some known results concerning the syzygies of quadratically parametrized surfaces. In Section 3, we use the syzygies to find the inversion map. Finally, we will provide some illustrative examples.

2. Syzygies of Parametrized Surfaces

A point $[s_0;t_0;u_0] \in \mathbb{P}^2$ is called a *base point* of the parametrization given by Equation (1), if $[s_0;t_0;u_0] \in Z = \mathbb{V}(I)$, i.e., $[s_0;t_0;u_0]$ is a common root of the polynomials f_i for $i = 0,1,2,3$. Base points play an important role in studying the image of the parametrization. Bézout's Theorem stated below (see [2] for detailed proof) provides a relationship between the multiplicity of base points and the degree of a variety.

Theorem 2.1. (Bézout's Theorem) *Let S be a surface in projective 3-space given by the image of a generic 1-to-1 rational parametrization as in Equation* (1) *with* $\deg(f_i) = d$*, then*

$$\deg(S) = d^2 - \sum_{p \in Z} e_p,$$

where $Z = \mathbb{V}(f_0, f_1, f_2, f_3)$ is the set of base points, and e_p is the algebraic multiplicity of the base point p.

It is shown in [3] that if S be a surface in projective 3-space given by the image of a generic 1-to-1 rational parametrization as in Equation (1) with $\deg(f_i) = 2$, then

$$\deg Z = \sum_{p \in Z} e_p \leq 2, \text{ where } Z = \mathbb{V}(f_0, f_1, f_2, f_3),$$

and hence

$$\deg(S) = 2, \text{ or } 3, \text{ or } 4.$$

Moreover, in [3, Theorem 3.10], the detailed structures of the free resolutions of quadratically parametrized surfaces with base points and their multiplicities were given. Let I be the ideal generated by $\{f_0,\ f_1,\ f_2,\ f_3\}$, then the free resolution of R/I has the form

1. If $Z = \emptyset$, then

$$0 \to R^2(-5) \to R^2(-3) \bigoplus R^3(-4) \xrightarrow{\mathbf{P}_1,\mathbf{P}_2,\mathbf{P}_3,\mathbf{P}_4,\mathbf{P}_5} R^4(-2) \to R \to R/I \to 0,$$

where $\deg(\mathbf{P}_1) = \deg(\mathbf{P}_2) = 1$, and $\deg(\mathbf{P}_3) = \deg(\mathbf{P}_4) = \deg(\mathbf{P}_5) = 2$ in s,t,u.

2. If $\deg Z = 1$, then

$$0 \to R(-5) \to R^3(-3) \bigoplus R(-4) \xrightarrow{\mathbf{P}_1,\mathbf{P}_2,\mathbf{P}_3,\mathbf{P}_4} R^4(-2) \to R \to R/I \to 0,$$

where $\deg(\mathbf{P}_1) = \deg(\mathbf{P}_2) = \deg(\mathbf{P}_3) = 1$, and $\deg(\mathbf{P}_4) = 2$ in s, t, u.

3. If $\deg Z = 2$, then

$$0 \to R(-4) \to R^4(-3) \xrightarrow{\mathbf{P}_1,\mathbf{P}_2,\mathbf{P}_3,\mathbf{P}_4} R^4(-2) \to R \to R/I \to 0,$$

where $\deg(\mathbf{P}_1) = \deg(\mathbf{P}_2) = \deg(\mathbf{P}_3) = \deg(\mathbf{P}_4) = 1$ in s, t, u.

Note, the $\mathbf{P}_i$ are the generators of the first syzygy module $\mathrm{Syz}(I)$.

Now, recall a **moving surface** of degree r is given by a polynomial

$$\sum_{i+j+\ell+k=r} A_{ij\ell k}(s,t,u)x^i y^j z^\ell w^k, \quad A_{ij\ell k} \in R.$$

This **follows** the parametrization (1) if

$$\sum_{i+j+\ell+k=r} A_{ij\ell k}(s,t,u) f_0(s,t,u)^i f_1(s,t,u)^j f_2(s,t,u)^\ell f_3(s,t,u)^k \equiv 0.$$

Hence, a moving surface of degree r follows the parametrization if and only if

$$(A_{ij\ell k})_{i+j+\ell+k=r} \in \mathrm{Syz}(I^r), \text{ where } I \text{ is the ideal } \langle f_0, f_1, f_2, f_3 \rangle \subset R.$$

The set of all moving surfaces that follow the parametrization is an ideal in $\mathbb{k}[s,t,u,x,y,z,w]$. This is called the **moving surface ideal**. When $r = 1$, the moving surface is a moving plane.

From the structure of the free resolution of quadratically parametrized surfaces S, we obtain the following moving planes that follow the parametrization (1) of degree one in s, t, u:

1. If $Z = \emptyset$, then

$$\begin{aligned} P_1 &= \mathbf{P}_1 \cdot (x,y,z,w) = p_{1s}(x,y,z,w)s + p_{1t}(x,y,z,w)t + p_{1u}(x,y,z,w)u, \\ P_2 &= \mathbf{P}_2 \cdot (x,y,z,w) = p_{2s}(x,y,z,w)s + p_{2t}(x,y,z,w)t + p_{2u}(x,y,z,w)u. \end{aligned}$$

2. If $\deg Z = 1$, then

$$\begin{aligned} P_1 &= \mathbf{P}_1 \cdot (x,y,z,w) = p_{1s}(x,y,z,w)s + p_{1t}(x,y,z,w)t + p_{1u}(x,y,z,w)u, \\ P_2 &= \mathbf{P}_2 \cdot (x,y,z,w) = p_{2s}(x,y,z,w)s + p_{2t}(x,y,z,w)t + p_{2u}(x,y,z,w)u, \\ P_3 &= \mathbf{P}_3 \cdot (x,y,z,w) = p_{3s}(x,y,z,w)s + p_{3t}(x,y,z,w)t + p_{3u}(x,y,z,w)u. \end{aligned}$$

3. If $\deg Z = 2$, then

$$\begin{aligned}
P_1 &= \mathbf{P}_1 \cdot (x,y,z,w) = p_{1s}(x,y,z,w)s + p_{1t}(x,y,z,w)t + p_{1u}(x,y,z,w)u,\\
P_2 &= \mathbf{P}_2 \cdot (x,y,z,w) = p_{2s}(x,y,z,w)s + p_{2t}(x,y,z,w)t + p_{2u}(x,y,z,w)u,\\
P_3 &= \mathbf{P}_3 \cdot (x,y,z,w) = p_{3s}(x,y,z,w)s + p_{3t}(x,y,z,w)t + p_{3u}(x,y,z,w)u,\\
P_4 &= \mathbf{P}_4 \cdot (x,y,z,w) = p_{4s}(x,y,z,w)t + p_{4t}(x,y,z,w)t + p_{4u}(x,y,z,w)u,
\end{aligned}$$

where p_{is}, p_{it}, p_{iu} are linear forms in x, y, z, w.

It is shown in [3] that for any quadratically parametrized surface S, there are at least two linearly independent moving planes P_1 and P_2, and the implicit equation F for the parametrized surface are given by

$$\begin{cases} F = \mathrm{Sylv}_{s,t,u}(P_1, P_2, Q), & \text{if} \quad Z = \emptyset,\\ F = \mathrm{Sylv}_{s,t,u}(P_1, P_2, P_3), & \text{if} \quad \deg Z = 1,\\ F = \gcd(\mathrm{Sylv}_{s,t,u}(P_1, P_2, P_3), \mathrm{Sylv}_{s,t,u}(P_1, P_2, P_4)), & \text{if} \quad \deg Z = 2, \end{cases} \tag{5}$$

where $\mathrm{Sylv}_{s,t,u}(f, g, h)$ stands for the Sylvester determinant of the polynomials f, g, h with respect to s, t, u.

It is proven by [1] that in the non-homogenous setting, there are three moving planes $\mathbf{p}, \mathbf{q}, \mathbf{r}$ in s, t forming a μ-basis for the rational surface such that

$$[\mathbf{p}, \mathbf{q}, \mathbf{r}] = \kappa\ [f_0(s,t);\ f_1(s,t);\ f_2(s,t);\ f_3(s,t)] \tag{6}$$

for some nonzero constant κ, where $\mathbf{p} = (p_1, p_2, p_3, p_4), \mathbf{q} = (q_1, q_2, q_3, q_4),\ \mathbf{r} = (r_1, r_2, r_3, r_4)$, and

$$[\mathbf{p}, \mathbf{q}, \mathbf{r}] = \left[\det\begin{pmatrix} p_2 & q_2 & r_2\\ p_3 & q_3 & r_3\\ p_4 & q_4 & r_4 \end{pmatrix}, -\det\begin{pmatrix} p_1 & q_1 & r_1\\ p_2 & q_2 & r_2\\ p_3 & q_3 & r_3 \end{pmatrix}, \det\begin{pmatrix} p_1 & q_1 & r_1\\ p_2 & q_2 & r_2\\ p_4 & q_4 & r_4 \end{pmatrix}, -\det\begin{pmatrix} p_1 & q_1 & r_1\\ p_2 & q_2 & r_2\\ p_3 & q_3 & r_3 \end{pmatrix}\right]. \tag{7}$$

Moreover, [1] provided an algorithm to obtain the μ-basis. Thus, in the homogenous setting, we may choose P_1, P_2 as the homogenous forms of $\mathbf{p}, \mathbf{q}$.

3. Inversion Map Via Syzygies

We conclude with our main result and illustrative examples.

Theorem 3.1. *Let $\phi : \mathbb{P}^2 \longrightarrow \mathbb{P}^3 : [s;t;u] \mapsto [f_0; f_1; f_2; f_3] \in \mathbb{P}^3$ be a generic one-to-one parametrization of a surface S, where $f_0, f_1, f_2, f_3 \in R = \Bbbk[s,t,u]$ are homogenous polynomials of degree $d = 2$, and $\gcd(f_0, f_1, f_2, f_3) = 1$. If $I =$*

$\langle f_0, f_1, f_2, f_3\rangle \subset R$, *the inversion map* ψ *is given as the determinant of the* 2×2 *minor of the matrix* $\begin{bmatrix} p_{1s} & p_{1t} & p_{1u} \\ p_{2s} & p_{2t} & p_{2u}\end{bmatrix}$, *that is*

$$[s;t;u] = \psi(x,y,z,w) = \left[\det\begin{pmatrix} p_{1t} & p_{1u} \\ p_{2t} & p_{2u}\end{pmatrix};\ -\det\begin{pmatrix} p_{1s} & p_{1u} \\ p_{2s} & p_{2u}\end{pmatrix};\ \det\begin{pmatrix} p_{1s} & p_{1t} \\ p_{2s} & p_{2t}\end{pmatrix}\right]$$

where p_{is}, p_{it}, p_{iu} *are the linear forms in* x, y, z, w *of the two linearly independent moving planes* P_1, P_2.

Proof. First of all, we note that the parametrization is parametrization is a generic one-to-one map, hence the surface is of degree two, three or four according to the total multiplicity of the base points. In any case, based on the free resolution results obtained in [3], there are at least two linearly independent moving planes that follow the parametrization. Without loss of generality, let P_1 and P_2 be two linearly independent moving planes that follow the parametrization (1) of the following forms

$$\begin{aligned} P_1 &= p_{1s}(x,y,z,w)s + p_{1t}(x,y,z,w)t + p_{1u}(x,y,z,w)u, \\ P_2 &= p_{2s}(x,y,z,w)s + p_{2t}(x,y,z,w)t + p_{2u}(x,y,z,w)u. \end{aligned}$$

Now, we claim at the general point of the surface S

$$\operatorname{rank}\begin{bmatrix} p_{1s}(x,y,z,w) & p_{1t}(x,y,z,w) & p_{1u}(x,y,z,w) \\ p_{2s}(x,y,z,w) & p_{2t}(x,y,z,w) & p_{2u}(x,y,z,w)\end{bmatrix} = 2.$$

Otherwise, if the rank was one, then

$$\det\begin{pmatrix} p_{1t} & p_{1u} \\ p_{2t} & p_{2u}\end{pmatrix} = \det\begin{pmatrix} p_{1s} & p_{1u} \\ p_{2s} & p_{2u}\end{pmatrix} = \det\begin{pmatrix} p_{1s} & p_{1t} \\ p_{2s} & p_{2t}\end{pmatrix} = 0.$$

But by the formula provided by Hoffman-Wang [3] stated in Equation (5), this would imply that the implicit equation of the surfaces were identically zero, which is absurd.

Hence, the matrix equation

$$\begin{bmatrix} p_{1s}(x,y,z,w) & p_{1t}(x,y,z,w) & p_{1u}(x,y,z,w) \\ p_{2s}(x,y,z,w) & p_{2t}(x,y,z,w) & p_{2u}(x,y,z,w)\end{bmatrix}\begin{bmatrix} s \\ t \\ u\end{bmatrix} = \begin{bmatrix} 0 \\ 0\end{bmatrix},$$

have a unique solution up to scalar multiples

$$[s;t;u] = \left[\det\begin{pmatrix} p_{1t} & p_{1u} \\ p_{2t} & p_{2u}\end{pmatrix};\ -\det\begin{pmatrix} p_{1s} & p_{1u} \\ p_{2s} & p_{2u}\end{pmatrix};\ \det\begin{pmatrix} p_{1s} & p_{1t} \\ p_{2s} & p_{2t}\end{pmatrix}\right],$$

at the general point $[x; y; z; w] \in S$. Therefore, the map given by the minors of the matrix is the desired inversion map. $\square$

Example 3.2. *Given a quadratic parametrization $\phi = [t^2 - u^2; st; su; tu]$, we compute the moving planes of degree 1 and obtain*

$$P_1 = zt - yu, \quad P_2 = xs - yt + zu, \quad P_3 = ws - yu.$$

It is easy to check that the first two moving planes P_1 and P_2 are linearly independent. Then the inversion map is given by the determinant of the 2×2 minor of the matrix $\begin{bmatrix} 0 & z & -y \\ x & -y & z \end{bmatrix}$, that is

$$\psi : \; [x; y; z; w] = [z^2 - y^2; \; -xy; \; -xz].$$

Hence,

$$\psi(\phi[s; t; u]) = [(s^2u^2 - s^2t^2; -st(t^2 - u^2); -su(t^2 - u^2)] = s(u^2 - t^2)[s; t; u] = [s; t; u].$$

For the other composition, we need the implicit equation of the parametrization which is $F = xyz - y^2w + z^2w = 0$. Equivalently $w(y^2 - z^2) = xyz$. It follows that

$$\begin{aligned} \phi(\psi[x; y; z; w]) &= [x^2(y^2 - z^2); xy(y^2 - z^2); xz(y^2 - z^2); x^2yz] \\ &= [xy(y^2 - z^2); xz(y^2 - z^2); xw(y^2 - z^2)] \\ &= x(y^2 - z^2)[x; y; z; w] = [x; y; z; w]. \end{aligned}$$

Example 3.3. *Given a quadratic parametrization $\phi = [t^2; u^2; tu; su]$. The moving planes of degree 1 are*

$$P_1 = yt - zu, \quad P_2 = zt - xu, \quad P_3 = ys - wu, \quad P_4 = zs - wt.$$

The implicit equation of the surface is $z^2 - xy = 0$. Note also that P_1 and P_2 are linearly dependent over $\Bbbk[x, y, z, w]$ for all the points on the surface, since the rank of the matrix $\begin{bmatrix} 0 & y & -z \\ 0 & z & -x \end{bmatrix}$ is one. This is because the determinants of the 2×2 minor of the matrix are all zeros on the surface. We need to choose two moving planes P_1 and P_3 that are linearly independent over $\Bbbk[x, y, z, w]$ for all the points on the surface. Thus the inversion map is given by the determinant of the 2×2 minor of the matrix $\begin{bmatrix} 0 & y & -z \\ y & 0 & -w \end{bmatrix}$. That is,

$$\psi : \; [x; y; z; w] = [-yw; \; -yz; \; -y^2] = [w; z; y].$$

It is easy to check that $\psi(\phi[s;t;u]) = [s;t;u]$, *and*

$$\phi(\psi[x;y;z;w]) = y\left[\frac{z^2}{y};y;z;w\right] = [x;y;z;w].$$

The last equality is true because the implicit equation is $F = z^2 - xy = 0$, *thus on the surface* $x = \dfrac{z^2}{y}$.

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