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INTERSECTION GRAPH OF A SIMPLICIAL COMPLEX

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Abstract. In this note, firstly we introduce the intersection graph $G(\Delta)$ of a simplicial complex Δ, as a graph whose vertices are all facets of Δ and two distinct vertices are adjacent if they have non-empty intersection. We investigate some properties of this graph and simplicial complexes. Moreover, we apply this graph for finding a couple of upper and lower bounds for the vertex covering number of Δ. Also, we introduce and study the intersection ideal of a simplicial complex.

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1. Introduction

In the past ten or so years, there has been considerable researches done on associating graphs with mathematical structures (e.g. [1], [2], [3], [5], [8] and [9]).

On the other hand, an intersection graph is an undirected graph formed from a family of sets, by creating one vertex for each set and connecting two vertices by an edge whenever the corresponding two sets have a non-empty intersection. Any undirected graph G may be represented as an intersection graph: for each vertex v_i of G, form a set S_i consisting of the edges which pass from v_i; then two such sets have a non-empty intersection if and only if the corresponding vertices share an edge. For an overview of the theory of intersection graphs, and of important special classes of intersection graphs, see [17]. There has been a couple of papers devoted to study of the intersection graph of algebraic structures (see [8], [10], [15], [16] and [19]).

Simplicial complexes are some algebraic and topological tools which are useful in algebraic topology, commutative algebra and combinatorics. They can be considered as some generalizations of graphs. There has been a couple of graphs associated to these objects. For instance (r, s)-adjacency graph is defined and studied in [15], which is a special kind of intersection graph. In this paper, we are also going to study the intersection graph of a simplicial complex, in some sense.

The main purpose of this paper is studying the intersection of facets in a simplicial complex using graph theoretic concepts. In fact, firstly, we introduce the total graph of a simplicial complex, $T(\Delta)$, as the (undirected) graph whose vertices are all non-empty faces of Δ and two distinct vertices are adjacent if they are contained in the same facet or they have a non-empty intersection. Recall that when G is a graph, the total graph $T(G)$ of G is a graph whose vertices are all the vertices and edges of G and two vertices are adjacent if they are adjacent in G, two edges are adjacent if they are passes from the same vertex and a vertex and an edge are adjacent if the edge passes through the vertex (see [7]). Also, the simplicial complex $\Delta(G)$ is a simplicial complex associated to G, with the edges of G as its facets.

Assume that Δ is a simplicial complex with facets $F_1, \ldots, F_m$. Then it is easy to see that $T(\Delta)$ is a natural generalization of the known total graph of a graph. In fact, if G is a graph, then $T(\Delta(G))$ is the total graph of G. Also, if we concentrate on the induced subgraph of $T(\Delta)$ whose vertices are all faces of Δ contained in F_i, then this subgraph is a complete graph of order $2^{|F_i|} - 1$. Moreover, if we concentrate on the induced subgraph $G(\Delta)$ of $T(\Delta)$ whose vertices are all facets of Δ, then $G(\Delta)$ is the intersection graph with vertices $F_1, \ldots, F_m$. Furthermore, if G is a graph, then $G(\Delta(G))$ is the line graph of G. So, studying $G(\Delta)$, which is a kind of intersection graph, can help us for obtaining characterizations of total graphs and line graphs. In this regard, we limit our scope on $G(\Delta)$.

In section two, among the other things, we study the diameter and radius of the graph $G(\Delta)$. Also, we find some lower and upper bounds for vertex covering number $\alpha(\Delta)$ of Δ by applying the intersection graph $G(\Delta)$. In the third section, we introduce and study the intersection ideal J_Δ of a simplicial complex. We investigate the interplay between the algebraic properties of the intersection ideal and graph-theoretic properties of $G(\Delta)$. Finally, in Theorem 3.12, we study the graph $G(\Delta)$ when Δ is an order complex.

Now, we start to remind a brief necessary background of graph theory from [4]. In a graph G, $V(G)$ and $E(G)$ are the sets of vertices and edges of G, respectively and for two distinct vertices a and b in G, the notation $a - b$ means that a and b are adjacent. A graph G is said to be *connected* if there exists a path between any two distinct vertices, and it is *complete* if each pair of distinct vertices is joined by an edge. For a positive integer n, we use K_n to denote the complete graph with n vertices, which is a graph that any two distinct vertices are adjacent. We also use C_n for a cycle graph with n vertices. In a graph G any edge in a cycle of G is called a chord of G. Any complete subgraph of G is called a *clique* in G and the size of the largest clique in G is called the *clique number* of G and denoted by $w(G)$. Also, we say that G is *totally disconnected* if no two vertices of G are adjacent. The size of the largest subgraph of G, which is totally disconnected is denoted by $\mathrm{Coclique}(G)$.

The *distance* between two distinct vertices a and b in G, denoted by $d_G(a,b)$, is the length of a shortest path connecting a and b, if such a path exists; otherwise, we set $d_G(a,b) := \infty$. The *diameter* of a connected graph G is

$$\operatorname{diam}(G) = \sup\{d_G(a,b) \mid a \text{ and } b \text{ are distinct vertices of } G\}.$$

The *girth* of G, denoted by $\operatorname{girth}(G)$, is the length of the shortest cycle in G, if G contains a cycle; otherwise, $\operatorname{girth}(G) := \infty$. For any vertex x of a connected graph G, the *eccentricity* of x, denoted by $e(x)$, is the maximum of the distances from x to the other vertices of G, and the minimum value of the eccentricity of the vertices of G is called the *radius* of G, which is denoted by $\mathrm{r}(G)$. We shall say that the diameter and radius of G are zero if G has no edges. Let $\chi(G)$ denote the *chromatic number* of the graph G, that is the minimum number of colors which can be assigned to the vertices of G in such a way that every two adjacent vertices have different colors. A *bipartite graph* is one whose vertex-set can be partitioned into two subsets so that no edge has both ends in any one subset. A *complete bipartite graph* is a bipartite graph in which, each vertex is joined to every vertex that is not in the same subset. The complete bipartite graph with part sizes m and n is denoted by $K_{m,n}$. The complete bipartite graph $K_{1,n}$ is called a *star graph*.

2. Total Graph of A Simplicial Complex

Recall that a *simplicial complex* Δ over a finite set of vertices $V = \{x_1, \ldots, x_n\}$ is a collection of subsets of V containing all singletons $\{x_i\}$ for each $1 \leq i \leq n$, with the property that if $F \in \Delta$, then all subsets of F are also in Δ. An element of Δ is called a *face* of Δ and the maximal faces of Δ are called *facets* of Δ. Since every simplicial complex can be uniquely determined by its facets, if $F_1, \ldots, F_k$ are all of the facets of Δ, Δ is denoted by $\langle F_1, \ldots, F_k\rangle$. Also, for each face F of Δ, dimension of F, which is denoted by $\dim(F)$ equals to $|F| - 1$ and the *dimension* and *codimension* of Δ is defined as follows:

$$\dim(\Delta) = \max\{\dim F \mid F \text{ is a facet in } \Delta\};$$

$$\operatorname{codim}(\Delta) = \min\{\dim F \mid F \text{ is a facet in } \Delta\}.$$

Definition 2.1. Let $\Delta = \langle F_1, \ldots, F_m\rangle$ be a simplicial complex. We define the *total graph* of Δ, denoted by $T(\Delta)$, as a graph whose vertices are all non-empty faces of Δ and two distinct vertices are adjacent if they are contained in the same facet or they have a non-empty intersection. We set

$$E_\Delta = \{F_i \cap F_j \mid 1 \leq i,j \leq m, i \neq j \text{ and } F_i \cap F_j \neq \emptyset\},$$

and for each $1 \leq i \leq m$, we set

$$N_E(F_i) = \{F_i \cap F_j \mid 1 \leq j \leq m, j \neq i \text{ and } F_i \cap F_j \neq \emptyset\},$$

and

$$N_V(F_i) = \{F_j \mid 1 \leq j \leq m, j \neq i \text{ and } F_i \cap F_j \neq \emptyset\}.$$

A facet F of Δ is called an *isolated facet* of Δ if $N_E(F) = \emptyset$ or equivalently $N_V(F) = \emptyset$.

In the reminder of our note, $\Delta = \langle F_1, \ldots, F_m \rangle$ is a simplicial complex over a finite set of vertices $V = \{x_1, \ldots, x_n\}$. Also, we use F_i for both of the facet F_i of Δ and the vertex of $G(\Delta)$ corresponding to F_i.

Recall that a facet F of Δ is called a *leaf* of Δ if either it is an isolated facet of Δ or there is a facet G of Δ distinct from F such that for each facet F' of Δ with $F' \neq F$, we have $F \cap F' \subseteq F \cap G$ (see [12]). Let Δ_1 and Δ_2 be two simplicial complexes. Then we say that Δ_1 is a subcollection of Δ_2 if every facet in Δ_1 is a facet in Δ_2. For other concepts in the context of simplicial complexes we refer the reader to [12] and [14].

The following remarks can be immediately gained.

Remarks 2.2. (1) *$G(\Delta)$ is connected if and only if Δ is connected.*

(2) *$G(\Delta)$ is totally disconnected if and only if $\sum_{i=1}^{m} \dim F_i = n - m$.*

(3) *$\beta(\Delta) = \text{Coclique}(G(\Delta))$, where $\beta(\Delta)$ is the independence number of Δ.*

(4) *The facet F of Δ is a leaf if and only if it is an isolated vertex in $G(\Delta)$ or $N_E(F)$ has a unique maximal element.*

(5) *If Δ is a cycle, then $G(\Delta)$ is either C_m or K_m such that all chords are $\bigcap_{i=1}^{m} F_i$ (See [14].)*

(6) *If Δ' is a subcollection of Δ, then $G(\Delta')$ is an induced subgraph of $G(\Delta)$.*

In the following examples, we exhibit the intersection graph $G(\Delta)$ for special values of n, m or $\dim(\Delta)$.

Examples 2.3. (i) *If $m = 1$, then $G(\Delta) = K_1$.*

(ii) *If $n = 2$, then $G(\Delta)$ is K_1 or the totally disconnected graph with two vertices.*

(iii) *If $n = 3$, then $G(\Delta)$ is K_1 or K_2 or totally disconnected graph with two or three vertices.*

(iv) *If $m = 2$, then $G(\Delta)$ is connected if and only if $\dim F_1 + \dim F_2 \geq n - 1$.*

(v) *If $m = 3$ and $G(\Delta)$ is connected, then $\dim F_1 + \dim F_2 + \dim F_3 \geq n - 1$.*

(vi) *If $\dim \Delta = 0$, then Δ is a tatally disconnected graph with n vertices.*

(vii) *If $\dim \Delta = n - 1$, then $G(\Delta) = K_1$.*

(viii) *If $\dim \Delta = n - 2$, then $G(\Delta) = K_m$ or $G(\Delta)$ is totally disconnected.*

In the following result, we gain an upper bound for distance between two distinct vertices of $G(\Delta)$.

Theorem 2.4. *Assume that* $\Delta = \langle F_1, \ldots, F_m \rangle$ *is a connected simplicial complex over the set of vertices* V *and* i *and* j *are two distinct positive integers with* $1 \leq i, j \leq m$. *Then* $d(F_i, F_j) = 1$ *if* $n - t \leq 1$ *and else* $d(F_i, F_j) \leq n - t$, *where* $t = \dim F_i + \dim F_j$.

Proof. Since $\dim F_i + \dim F_j = t$, we have $|F_i| + |F_j| = t + 2$. If $n - t \leq 1$, we have $t + 2 > n$ which implies that $F_i \cap F_j \neq \emptyset$ and hence there is nothing to prove in this situation. So, if we set $s = n - t - 2$, we may assume that s is a non-negative integer. If $F_i \cap F_j \neq \emptyset$, then $d(F_i, F_j) = 1 \leq n - t$ as required. So, we suppose that $F_i \cap F_j = \emptyset$. Hence, we may assume that $V \setminus (F_i \cup F_j)$ has exactly s elements and we use induction on s. If $s = 0$, since Δ is connected, every other facet of Δ has non-empty intersection with both of F_i and F_j. So, $d(F_i, F_j) = 2 \leq n - t$ as desired. Now, assume inductively that s is a positive integer and the result has been proved for any two facets F'_i and F'_j in a connected simplicial complex Δ', with $|V \setminus (F'_i \cup F'_j)| = s'$, when s' is a non-negative integer smaller than s. Since Δ is connected, there is a path between F_i and F_j. Suppose that $F_i - F_{k_1} - \cdots - F_{k_{r-1}} - F_j$ is the shortest path from F_i to F_j of length r with $r \geq 2$. Now, if $F_{k_1} \cap F_j \neq \emptyset$, then $d(F_i, F_j) = 2 \leq n - t$. So, assume that $F_{k_1} \neq F_i$, $F_{k_1} \neq F_j$ and $F_{k_1} \cap F_j = \emptyset$. We set $F'_i := F_i \cup F_{k_1}$ and $\Delta' := \langle \{F_l \mid 1 \leq l \leq m, l \neq i, k_1\} \cup \{F'_i\} \rangle$. It is obvious that Δ' is connected, $|F'_i| > |F_i|$ and so $|V \setminus (F'_i \cup F_j)| = s'$, for some non-negative integer s' with $s' < s$. Therefore, inductive hypothesis insures that $d(F'_i, F_j) \leq s' + 2$ and so $d(F'_i, F_j) \leq s + 1$ in the graph $G(\Delta')$. Now, $d(F_i, F_j) = d(F'_i, F_j) + 1$ implies that $d(F_i, F_j) \leq n - t$ as desired. □

The following corollary, which presents some upper bounds for the eccentricity of a vertex, radius and diameter of $G(\Delta)$, immediately follows from Theorem 2.4 and their definitions.

Corollary 2.5. *Let* Δ *be a connected simplicial complex over the set of vertices* V *with* $|V| = n$ *and* F *be a facet of* Δ. *Then*

$$e(F) \leq n - \dim F - \operatorname{codim}\Delta,$$

$$r(G(\Delta)) \leq n - \dim \Delta - \operatorname{codim}\Delta,$$

and

$$\operatorname{diam}G(\Delta) \leq n - 2\operatorname{codim}\Delta.$$

The next corollary is a direct consequence of Corollary 2.5. Recall that a simplicial complex is called *pure* if all of its facets have the same dimensions.

Corollary 2.6. *Let* Δ *be a connected pure simplicial complex with dimension* d. *If* $d \geq \frac{n-1}{2}$, *then* $G(\Delta)$ *is a complete graph and else for each vertex* F *in* $G(\Delta)$, *we have* $e(F) \leq n - 2d$ *and so* $n - 2d$ *is an upper bound for diameter and radius of* $G(\Delta)$ *in this case.*

In the following result we find another upper bound for the diameter of $G(\Delta)$.

Theorem 2.7. *Let Δ be a connected simplicial complex. Then $G(\Delta)$ is complete or*

$$\mathrm{diam}(G(\Delta)) \leq n - \mathrm{dim}\Delta - 1.$$

Proof. Suppose that $\dim \Delta = n - s - 1$, where $0 \leq s \leq n-1$. We are supposed to show that $\mathrm{diam}(G(\Delta)) \leq s$. We use induction on s. If $s = 0$, then $\dim \Delta = n - 1$, which means that Δ is a simplicial complex with only one facet. Hence $G(\Delta) = K_1$ in this case. Therefore, inductively suppose that s is a positive integer and the result has been proved for smaller values of s. Since $\dim \Delta = n - s - 1$, there is a facet, say F_i, of Δ with $n - s$ elements. Hence, $V \setminus F_i$ has s elements. Since Δ is connected, there is a facet F_k in Δ intersecting with F_i. We set $F'_i := F_i \cup F_k$ and $\Delta' := \langle \{F_r \mid 1 \leq r \leq m, r \neq i, k\} \cup \{F'_i\}\rangle$. It is obvious that $\dim \Delta' = |F'_i| - 1 = n - s' - 1$, where s' is a non-negative integer smaller than s. Also, one can observe that Δ' is connected. In fact, every path in Δ' is the corresponding path in Δ, where F'_i is inserted instead of F_i and F_k if necessary. Hence, for each $1 \leq r, r' \leq m$ with $r \neq r'$ and $\{r, r'\} \cap \{i, k\} = \emptyset$ we also have $d_{G(\Delta)}(F_r, F_{r'}) \leq d_{G(\Delta')}(F_r, F_{r'}) + 1$. Since $s' \leq s-1$, inductive hypothesis implies that $\mathrm{diam}(G(\Delta')) \leq s-1$. Therefore, for each $1 \leq r, r' \leq m$ with $r \neq r'$ and $\{r, r'\} \cap \{i, k\} = \emptyset$ we have $d_{G(\Delta)}(F_r, F_{r'}) \leq s$. Moreover, for each $1 \leq r, r' \leq m$ with $r, r' \neq i, k$ we have $d_{G(\Delta)}(F_i, F_r) \leq d_{G(\Delta')}(F'_i, F_r) + 1 \leq s$ and $d_{G(\Delta)}(F_k, F_{r'}) \leq d_{G(\Delta')}(F'_i, F_{r'}) + 1 \leq s$. These complete the proof. □

Recall that a *vertex cover* for Δ, over a finite set of vertices V, is a subset A of V that intersects every facet of Δ. If A is a minimal element (under inclusion) of the set of vertex covers of Δ, it is called a *minimal vertex cover*. The smallest of the cardinalities of the vertex covers of Δ is called the *vertex covering number* of Δ and is denoted by $\alpha(\Delta)$ (see [14]).

In the next result, we find some lower and upper bounds for $\alpha(\Delta)$.

Theorem 2.8. *Let Δ be a simplicial complex such that Δ has t isolated facets. Set*

$$T = \{C \subseteq E_\Delta \mid \text{for all facets } F \in \Delta, \text{ there are } e \in N_E(F) \text{ and } e' \in C \text{ such that } e \cap e' \neq \emptyset\}.$$

Then

$$\min\{|C| \mid C \in T\} + t \leq \alpha(\Delta) \leq \min\{|A| \mid A = \bigcup_{e \in C} e \text{ when } C \in T\} + t.$$

Proof. Without loss of generality, one can assume that $t = 0$. To prove the first inequality, suppose that A is a minimal vertex cover of Δ. Then it is enough to show that there is an element C in T with $|C| = |A|$. Since A is a vertex cover of Δ, for each facet F of Δ, there is an element $x \in A \cap F$. Now, assume that for each facet G of Δ with $G \neq F$, we have $x \notin G$. Since F is not an isolated facet, there is a facet G of Δ with $G \neq F$ such that $F \cap G \neq \emptyset$. So, there is an element $x' \in F \cap G$. Now, set $A' = A \cup \{x'\} \setminus \{x\}$. Hence, A' is also a minimal vertex cover of Δ with $|A| = |A'|$. Therefore, we may assume that each element of A belongs to at least two facets of Δ. Hence, for each $x \in A$, there are distinct facets F_x and G_x with $x \in F_x \cap G_x$. Also, minimality of A shows that for two distinct elements $x, y \in A$, we can choose F_x, G_x, F_y and G_y such that $F_x \cap G_x \neq F_y \cap G_y$. Now, we set

$$C = \{F_x \cap G_x \mid x \in A\}.$$

It is clear that $C \subseteq E_\Delta$ and $|C| = |A|$. Moreover, for each facet F of Δ, there are $x \in A$ and a facet G of Δ with $G \neq F$ such that $x \in F \cap G$. If we set $e = F \cap G$ and $e' = F_x \cap G_x$, then $x \in e \cap e'$. So, $C \in T$ and $|C| = |A|$ as desired.

To prove the second inequality, we show that for each $C \in T$, the set $\bigcup_{e \in C} e$ is a vertex cover of Δ. Let $C \in T$ and F be a facet of Δ. Then there are $e \in N_E(F)$ and $e' \in C$ such that $e \cap e' \neq \emptyset$. So, by choosing $x \in e \cap e'$, we have $x \in F \cap \bigcup_{e \in C} e$ as required. □

The following corollary is immediately gained from Theorem 2.8.

Corollary 2.9. *By the notions that was used in Theorem 2.8, if all elements in each minimal member of T are of zero dimension, then*

$$\alpha(\Delta) = \min\{|C| \mid C \in T\} + t.$$

3. The Intersection Ideal of A Simplicial Complex

Assume that $\Delta = \langle F_1, \ldots, F_m \rangle$ is a simplicial complex over a finite set of vertices $V = \{x_1, \ldots, x_n\}$ and k is a ring. Hereafter, we use the same notion $x_{i_1} \ldots x_{i_s}$ for the face $\{x_{i_1}, \ldots, x_{i_s}\}$ of Δ and the monomial $x_{i_1} \ldots x_{i_s}$ in the polynomial ring $R = k[x_1, \ldots, x_n]$. Let

$$J_\Delta = \langle e \mid e \in E_\Delta \rangle.$$

We call J_Δ as the *intersection ideal* of Δ.

As we promised in the introduction, in this section, we are going to study the relations between algebraic properties of J_Δ and graph theoretic concepts concerning $G(\Delta)$. We begin with the following result.

Proposition 3.1. *Let Δ be a simplicial complex without any isolated facet such that any two generators of J_Δ are not coprime. Then $G(\Delta)$ is complete. In particular, if Δ is a simplicial complex with no isolated facet such that its intersection ideal is principal, then $G(\Delta)$ is complete.*

Proof. Assume that F and G are two distinct facets of Δ. Since they are not isolated, there are two facets $F' \neq F$ and $G' \neq G$ of Δ such that $F \cap F' \neq \emptyset$ and $G \cap G' \neq \emptyset$. Hence, $F \cap F', G \cap G' \in J_\Delta$. Now, there are two generators e and f

of J_Δ such that $e|F \cap F'$ and $f|G \cap G'$ and by our assumption, e and f are not coprime. So, the non-empty face $e \cap f$ is contained in $F \cap G$, which insures that F and G are adjacent in $G(\Delta)$. □

Note that the converse of Proposition 3.1 is not generally true. For example, if $F_1 = \{a, b, c\}, F_2 = \{b, d, e\}, F_3 = \{a, d, f\}$ and $F_4 = \{c, e, f\}$, then $J_\Delta = \langle a, b, c, d, e, f\rangle$ and so all of the generators of J_Δ are mutually coprime.

Proposition 3.2. *Assume that $G(\Delta)$ is a star graph with m vertices. Then the intersection ideal J_Δ has $m - 1$ generators which are mutually coprime.*

Proof. Without loss of generality one may assume that $G(\Delta)$ is a star graph with center F_1. Hence, for each $2 \leq i \leq m$, $F_1 \cap F_i \neq \emptyset$ and for every two distinct positive integers $2 \leq i, j \leq m$, $F_i \cap F_j = \emptyset$. Hence, $J_\Delta = \langle F_1 \cap F_i \mid 2 \leq i \leq m\rangle$. Also, suppose in contrary that for some integers i and j with $2 \leq i, j \leq m$, the monomials $F_1 \cap F_i$ and $F_1 \cap F_j$ are not coprime. So, $(F_1 \cap F_i) \cap (F_1 \cap F_j) \neq \emptyset$. Now, since $F_i \cap F_j$ includes the non-empty set $F_1 \cap F_i \cap F_j$, we have that $F_i \cap F_j \neq \emptyset$, which is a contradiction. □

Recall that a *subdivision* of a graph is any graph that can be obtained from the original graph by replacing edges by paths. If G and H are two graphs, we say that G is a *refinement* of H if $E(H) \subseteq E(G)$. Also, a graph is said to be *planar* if it can be drawn in the plane, so that its edges intersect only at their ends. A remarkable simple characterization of the planar graphs was given by Kuratowski in 1930. Kuratowski's Theorem says that a graph is planar if and only if it contains no subdivision of K_5 or $K_{3,3}$ (cf. [4, p. 153]).

In the following result, by using a minimal generating set of the intersection ideal J_Δ, we obtain some properties of $G(\Delta)$.

Proposition 3.3. *Let $\{e_1, \ldots, e_s\}$ be a minimal generating set for J_Δ and $s > 1$. Also, suppose that $G(\Delta)$ has no isolated vertex. Then the following statements hold.*

(i) *If there exists a facet F such that $e_i \cap F \neq \emptyset$, for all i with $1 \leq i \leq s$, then $G(\Delta)$ is a refinement of a star graph.*

(ii) *Suppose that for all facets F, $|N_E(F)| = 1$. Then $G(\Delta)$ is the union of complete graphs.*

(iii) *If there exists a facet F such that $|N_E(F)| = 1$ and $|N_V(F)| > 3$, then $G(\Delta)$ is not planar.*

Proof. (i) We show that F is adjacent to all other vertices in $G(\Delta)$. To this end, let F' be an arbitrary vertex with $F' \neq F$. Since $G(\Delta)$ has no isolated vertex, we have that $F' \cap H \neq \emptyset$, for some facet H in Δ. So $e_i \subseteq F' \cap H$, for some i with $1 \leq i \leq s$. Thus $e_i \subseteq F'$ and hence $e_i \cap F \subseteq F' \cap F$. Therefore, we have that

$F' \cap F \neq \emptyset$ and so F' is adjacent to F, which insures that $G(\Delta)$ is a refinement of a star graph with center F.

(ii) For any facet F in Δ, suppose that $N_V(F) = \{F_1, \dots, F_k\}$. Since $|N_E(F)| = 1$, we have $F \cap F_i = F \cap F_j$, for all i, j with $1 \leq i, j \leq k$. This implies that $F_i \cap F_j \neq \emptyset$. So a subgraph of $G(\Delta)$ with vertex-set $\{F, F_1, \dots, F_k\}$ forms a complete subgraph. Now, one can easily see that $G(\Delta)$ is the union of complete graphs.

(iii) In view of the proof of part (ii), it is easy to check that K_5 is isomorphic to a subgraph of $G(\Delta)$ and so by Kuratowski's Theorem $G(\Delta)$ is not planar. □

In the next two results, we present some circumstances under which the girth of $G(\Delta)$ is three.

Proposition 3.4. *If J_Δ has two distinct generators which are not coprime, then the girth of $G(\Delta)$ is three.*

Proof. Let $F_i \cap F_j$ and $F_r \cap F_s$ are two generators of J_Δ which are not coprime. Then $(F_i \cap F_j) \cap (F_r \cap F_s) \neq \emptyset$. So, all the sets $F_i \cap F_r$, $F_i \cap F_s$, $F_j \cap F_r$ and $F_j \cap F_s$ are non-empty. This shows that the girth of $G(\Delta)$ is three. □

Theorem 3.5. *Let Δ be a simplicial complex without any isolated facet. If $2\mathrm{ara}(J_\Delta) < m$, then the girth of $G(\Delta)$ is three.*

Proof. Suppose that $\mathrm{ara}(J_\Delta) = s$ and $J_\Delta = \langle F_{i_k} \cap F_{j_k} \mid 1 \leq k \leq s, i_k \neq j_k, 1 \leq i_k, j_k \leq m \rangle$. Now, since $m > 2s$, there is an integer $1 \leq r \leq m$ such that for each $1 \leq k \leq s$, $r \neq i_k$ and $r \neq j_k$. Also, we know that F_r is not an isolated facet. So, there is a facet G of Δ so that $F_r \cap G \in J_\Delta$. Therefore, there exists an integer $1 \leq k \leq s$ such that $F_{i_k} \cap F_{j_k} | F_r \cap G$ and hence $F_{i_k} \cap F_{j_k} \subseteq F_r$. This implies that F_r is adjacent to both of F_{i_k} and F_{j_k} , which implies that $\mathrm{girth}(G(\Delta)) = 3$. □

We recall that for a graph G, a subset S of the vertex-set of G is called a *dominating set* if every vertex not in S is adjacent to a vertex in S.

Proposition 3.6. *Let Δ be a connected simplicial complex such that $\mathrm{ara}(J_\Delta) = s$. Then we can find a dominating set with s elements for $G(\Delta)$.*

Proof. Assume that $J_\Delta = \langle F_{i_k} \cap F_{j_k} \mid 1 \leq k \leq s, i_k \neq j_k, 1 \leq i_k, j_k \leq m \rangle$ and F is a facet of Δ. Then since Δ is connected, there is a facet G in Δ such that $F \cap G \in J_\Delta$. Therefore, $F_{i_k} \cap F_{j_k} \subseteq F \cap G$ for some $1 \leq k \leq s$. This implies that F is adjacent to both of F_{i_k} and F_{j_k}. So, $\{F_{i_1}, \dots, F_{i_s}\}$ is a dominating set for $G(\Delta)$. □

Theorem 3.7. *If Δ is a connected simplicial complex, then either $G(\Delta)$ is a refinement of a star graph, or there is a dominating set with $|V| - \dim \Delta - 2$ elements.*

Proof. Let $\dim \Delta = s$. Then there is a facet F such that $|F| = s+1$. Without loss of generality one may assume that $V \setminus F = \{x_1, \dots, x_{n-s-1}\}$. For each $1 \leq i \leq n-s-1$, there is a facet F_{j_i} containing x_i. If for every choice of $F_{j_1}, \dots, F_{j_{n-s-1}}$, we have $F_{j_i} \cap \{x_1, \dots, x_{n-s-1}\} = \{x_i\}$, then we should have $F \cap F' \neq \emptyset$ for all facets F' of Δ and so $G(\Delta)$ is a refinement of a star graph in this case. Otherwise, without loss of generality we may assume that $F_{j_{n-s-1}} = F_{j_{n-s-2}}$. Now, for every facet F' of Δ, we have $F' \cap F_{j_i} \neq \emptyset$, for some $1 \leq i \leq n-s-2$, which implies that $\{F_{j_1}, \dots, F_{j_{n-s-2}}\}$ is a dominating set for $G(\Delta)$. □

In the next two propositions, by applying the intersection ideal, we find some relations between connectedness of a simplicial complex and its subcollections.

Proposition 3.8. *Let Δ_1 be a subcollection of the simplicial complex Δ_2 such that $I_{\Delta_1} = I_{\Delta_2}$ and Δ_2 doesn't have any isolated facet. Then if Δ_1 is connected, then Δ_2 is also connected.*

Proof. By Remarks 2.2(1), it is enough to show that if $G(\Delta_1)$ is connected, then $G(\Delta_2)$ is also connected. So suppose that $G(\Delta_1)$ is connected and let F and G be two distinct vertices in $G(\Delta_2)$. Since they are not isolated, there are two vertices F' and G' in $G(\Delta_2)$ such that $F \cap F' \neq \emptyset$ and $G \cap G' \neq \emptyset$. Hence, $F \cap F'$ and $G \cap G'$ belong to I_{Δ_2} and so to I_{Δ_1}. Therefore, there are elements $e, f \in I_{\Delta_1}$ such that $e | F \cap F'$ and $f | G \cap G'$. Hence, there are vertices H, H', K, and K' in $G(\Delta_1)$ such that $e = H \cap H' \subseteq F \cap F'$ and $f = K \cap K' \subseteq G \cap G'$, which insures that $e \subseteq F \cap H$ and $f \subseteq G \cap K$. This implies that F is adjacent to H and also G is adjacent to K in $G(\Delta_2)$. Now, since H and K are two vertices in $G(\Delta_1)$ and $G(\Delta_1)$ is connected, F and G are connected in $G(\Delta_2)$. So, $G(\Delta_2)$ is also connected as desired. □

Proposition 3.9. *Let Δ be a simplicial complex without any isolated facet such that*

$$J_\Delta = \langle H_i \cap K_i \mid 1 \leq i \leq k \rangle$$

and $\Delta' = \langle H_1, \dots, H_k \rangle$. Then if $G(\Delta')$ is connected, then so is $G(\Delta)$.

Proof. Let F and G be two distinct vertices in $G(\Delta)$. Since they are not isolated, there are two vertices F' and G' in $G(\Delta)$ such that $F \cap F' \neq \emptyset$ and $G \cap G' \neq \emptyset$. Hence, there are integers i and j with $1 \leq i, j \leq k$ such that $H_i \cap K_i \subseteq F \cap F'$ and $H_j \cap K_j \subseteq G \cap G'$. Therefore, we have $H_i \cap K_i \subseteq F \cap H_i$ and $H_j \cap K_j \subseteq G \cap H_j$. So, F is adjacent to H_i and also G is adjacent to H_j in $G(\Delta)$. Now, if $i = j$, it is clear that $F - H_i - G$ is a path from F to G in $G(\Delta)$. Otherwise, since $G(\Delta')$ is connected, there is a path from H_i to H_j which completes our proof. □

In the following result, for a simplicial complex Δ, we present some relations between the intersection ideal J_Δ and $\alpha(\Delta)$.

Theorem 3.10. (i) *If Δ is a tree (or forest), then*

$$\alpha(\Delta) = \mathrm{Coclique}(G(\Delta)).$$

(ii) *Let Δ be a simplicial complex without any isolated facet. Then*

$$\alpha(\Delta) \leq \min\{\sum_{e \in T} |e| \mid T \text{ is a minimal generating set of } J_\Delta\}.$$

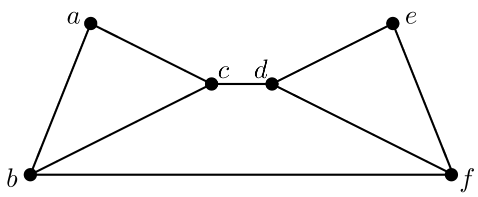
(iii) *Let Δ be a simplicial complex without any isolated facet. Then*

$$\alpha(\Delta) \leq \mathrm{height}_R(J_\Delta).$$

Proof. (i) The result follows from [13, Theorem 5.3] and part (3) in Remarks 2.2.
(ii) Let $T = \{e_1, \ldots, e_k\}$ be a minimal generating set for J_Δ and F be a facet of Δ. Since F is not an isolated facet, there is a facet G in Δ such that $F \cap G \in J_\Delta$. Hence, there is an integer $1 \leq i \leq k$ such that $e_i | F \cap G$. So, e_i is contained in F. Now, choose an element x_F from e_i. It is clear that the set $\{x_F \mid F \text{ is a facet in } \Delta\}$ is a vertex cover of Δ and it has at most $\sum_{e \in T} |e|$ elements. This completes the proof.
(iii) Let $\mathrm{height}_R(J_\Delta) = k$. Then there is a minimal prime ideal $\mathfrak{p} = \langle x_{i_1}, \ldots, x_{i_k} \rangle$ of J_Δ, where $\{x_{i_1}, \ldots, x_{i_k}\}$ is contained in $\{x_1, \ldots, x_n\}$. Now, for each facet F of Δ there is a facet G of Δ such that $F \cap G$ is contained in J_Δ and so in $\mathfrak{p}$. Therefore, there exists an integer $1 \leq j \leq k$ such that $x_{i_j} | F \cap G$, that is $x_{i_j} \in F$. This implies that $\{x_{i_1}, \ldots, x_{i_k}\}$ is a vertex cover for Δ. □

In the following example, we state that the inequality in part (iii) of Theorem 3.10, may be strict.

Example 3.11. *Consider the simplicial complex Δ in Figure (1), with facets $F_1 = \{a, b, c\}$, $F_2 = \{c, d\}$, $F_3 = \{d, e, f\}$ and $F_4 = \{b, f\}$. It is easy to see that $\alpha(\Delta) = 2$ and* $\mathrm{height}_R(J_\Delta) = 4$.



(1)

Here, we recall some definitions and notations on partially ordered sets. We use the standard terminology of partially ordered sets in [11].

In a partially ordered set $(P, \leq)$ (poset, briefly) an element m in P is *minimal* if $x \leq m$ for some $x \in P$, implies that $x = m$ and it is called the *least element* if $m \leq x$, for all $x \in P$. Also, an element m in P is *maximal* if $m \leq x$ for some $x \in P$, implies that $x = m$ and it is said to be the *greatest element* if $x \leq m$, for all $x \in P$. $(P, \leq)$ is called *bounded* if it has the least and the greatest elements. Assume that S is a subset of P. Then an element x in P is a *lower bound* of S if $x \leq s$ for all $s \in S$. An *upper bound* is defined in a dual manner. The set of all lower bounds of S is denoted by S^ℓ and the set of all upper bounds of S is denoted by S^u, i.e.,

$$S^\ell := \{x \in P \mid x \leq s, \text{ for all } s \in S\}$$

and

$$S^u := \{x \in P \mid s \leq x, \text{ for all } s \in S\}.$$

If $S = \{s\}$, we denote S^u and S^ℓ by $[s]^u$ and $[s]^\ell$, respectively. If for any a and b in P, either $a \leq b$ or $b \leq a$, then the partial order is called a *total order.* If a subset of P is totally ordered, it is called a *chain.* An *antichain* is a set of elements that are pairwise incomparable.

Recall that the *order complex* $\Delta(\Pi)$ of a poset $(\Pi, \leq)$ is the set of chains of Π (see [6]). In the sequel, we study $G(\Pi)$ which is the intersection graph of the order complex of the poset $(\Pi, \leq)$.

Theorem 3.12. *Let Π be a finite poset with minimal elements $a_1, \ldots, a_n$. Then the following statements hold.*

(i) *If t is the maximum number of elements in an antichain of Π, then $|V(G(\Pi))| \geqslant t$.*

(ii) *If Π is bounded, then $G(\Pi)$ is complete.*

(iii) *Suppose that there exists a minimal element a in Π such that $[a]^u$ has more than four maximal elements, then $G(\Pi)$ is not planar.*

(iv) *Let t_i be the number of maximal elements in $[a_i]^u$. Then*

$$\chi(G(\Pi)) \geq \omega(G(\Pi)) \geq \max\{t_i \mid i = 1, \ldots, n\}.$$

In particular, if $[a_i]^u \cap [a_j]^u = \emptyset$, for all $1 \leq i, j \leq n$ with $i \neq j$, then $G(\Pi)$ is the union of complete graphs and we also have

$$\chi(G(\Pi)) = \omega(G(\Pi)) = \max\{t_i \mid i = 1, \ldots, n\}.$$

Proof. (i) The vertices of $G(\Pi)$ are the maximal chains in Π and one can easily see that $|V(G(\Pi))|$ is equal to or greater than the minimum number of disjoint chains which together contain all elements of Π. Now, by Dilworth's Theorem [18], we have that the minimum number of disjoint chains which together contain all elements of Π is equal to the maximum number of elements in an antichain of π. Thus the result holds.

(ii) If Π is bounded, then all maximal chains in Π contain the least element and so they have non-empty intersection. So, $G(\Pi)$ is a complete graph.

(iii) Consider $m_1, \ldots, m_5$ in $[a]^u$. For $1 \leq i \leq 5$, let F_i be the maximal chain descending from m_i and contains a. Therefore, the set of vertices $\{F_1, \ldots, F_5\}$ forms a complete subgraph isomorphic to K_5 and so by Kuratowski's Theorem, $G(\Pi)$ is not planar.

(iv) Clearly $\chi(G(\Pi)) \geq \omega(G(\Pi))$. Let $m_{i1}, \ldots, m_{it_i}$ be distinct maximal elements in $[a_i]^u$. Also, for each $j = 1, \ldots, t_i$, let F_{ij} be a maximal chain descending from m_{ij} and containing a_i. Now, it is easy to check that the set of vertices $F_{i1}, \ldots, F_{it_i}$ forms a clique for $G(\Pi)$. Thus we have

$$\chi(G(\Pi)) \geq \omega(G(\Pi)) \geq \max\{t_i \mid i = 1, \ldots, n\}.$$

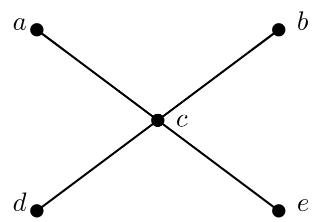
Also, since $[a_i]^u \cap [a_j]^u = \emptyset$, for all $1 \leq i, j \leq n$ with $i \neq j$, one can easily see that $G(\Pi)$ is the union of n complete graphs which are isomorphic to $K_{t_1}, \ldots, K_{t_n}$ and so we have

$$\chi(G(\Pi)) = \omega(G(\Pi)) = \max\{t_i \mid i = 1, \ldots, n\}.$$

□

We end this note by the following example which states that the inequality in part (i) of Theorem 3.12, may be strict.

Example 3.13. *Consider the order complex* $\Delta(\Pi)$ *of the poset* $(\Pi, \leq)$ *in Figure 2, with chains* $\{a, c, d\}, \{a, c, e\}, \{b, c, d\}$ *and* $\{b, c, e\}$*. It is easy to see that* $|V(G(\Pi))| = 4$ *and the maximum number of elements in an antichain of* π *is equal to 2.*



(2)

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