ON S-PERMITABLY EMBEDDED AND WEAKLY C-NORMAL SUBGROUPS OF FINITE GROUPS

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Abstract. Let $G$ be a finite group, $p$ the smallest prime dividing the order of $G$ and $P$ a Sylow $p$-subgroup of $G$ with the smallest generator number $d$. We consider such a set $\mathcal{M}_d(P) = \{P_1, P_2, \ldots, P_d\}$ of maximal subgroups of $P$ such that $\cap_{i=1}^{d} P_i = \Phi(P)$. Groups with certain $s$-permutably embedded and weakly $c$-normal subgroups of prime power order are studied. We present some sufficient conditions for a group to be $p$-nilpotent or $p$-supersolvable.

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1. Introduction

All groups considered in this paper are finite. Terminology and notation employed agree with standard usage, as in Robinson [10].

In the present paper, we let $\mathcal{M}(G)$ be the set of all maximal subgroups of Sylow subgroups of a group $G$. An interesting problem in group theory is to study the influence of the elements of $\mathcal{M}(G)$ on the structure of $G$. A classical result in this direction is attributed to Srinivasan [12]. Srinivasan proved that $G$ is supersolvable provided that every member of $\mathcal{M}(G)$ is normal in $G$. This result has been extensively generalized.

In investigating structures in finite groups, normal subgroups often play an important role. Recently, several notions generalizing normality were introduced. Among them: two subgroups $H$ and $K$ of $G$ are said to be permutable if $HK = KH$. A subgroup $H$ of a group $G$ is said to be $s$-permutable (or $\pi$-quasinormal) in $G$ if $H$ permutes with every Sylow subgroups of $G$, i.e., $HP = PH$ for any Sylow subgroup $P$ of $G$. This concept was introduced by O.H.Kegel in [8] and has been studied widely by many authors, such as [4,11]. Recently, Ballester-Bolinches and Pedraza-Aquilera [3] generalized the notion of $s$-permutable subgroups to $s$-permutably embedded subgroups. A subgroup $H$ of $G$ is said to be $s$-permutably

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embedded in \( G \) provided every Sylow subgroup of \( H \) is a Sylow subgroup of some \( s \)-permutable subgroup of \( G \). On the other hand, Wang [15] introduced the concept of \( c \)-normal subgroups. A subgroup \( H \) of a group \( G \) is said to be \( c \)-normal in \( G \) if there exists a normal subgroup \( K \) of \( G \) such that \( G = HK \) and \( H \cap K \) is contained in \( H_G \), where \( H_G \) is the maximal normal subgroup of \( G \) contained in \( H \). In [6], Guo and Shum showed the following result: Let \( p \) be the smallest prime dividing \( |G| \) and let \( P \) be a Sylow \( p \)-subgroup of \( G \). If every member of \( M(P) \) is \( c \)-normal in \( G \), then \( G \) is \( p \)-nilpotent. More recently, Zhu [18] introduced the concept of weakly \( c \)-normal subgroups. A subgroup \( H \) of a group \( G \) is called a weakly \( c \)-normal subgroup of \( G \) if there exists a subnormal subgroup \( T \) of \( G \) such that \( G = HT \) and \( H \cap T \leq H_G \). It should be apparent from the summary above that there has been steady research in both the concepts of weakly \( c \)-normal subgroups and of \( s \)-permutable subgroups; however, the two concepts have been considered independently of each other.

In this paper, we restrict the set of maximal subgroups of Sylow subgroups by the following concept.

**Definition 1.1.** [9, Definition 1.1] Let \( d \) be the smallest generator number of a \( p \)-group \( P \), and let \( M(P) \) be the set of all maximal subgroups of \( P \). Then \( M_d(P) \) denotes a subset \( M_d(P) = \{P_1, P_2, \ldots, P_d\} \) of \( M(P) \) with the property that \( \cap_{i=1}^{d} P_i = \Phi(P) \), the Frattini subgroup of \( P \).

Observe that, then, there are \((p^d - 1)/(p-1)\) maximal subgroups of \( P \), and that \( \frac{1}{d}(p^d - 1)/(p-1) \) tends to infinity with \( d \). So \( M_d(P) \) usually (for large \( d \)) is much smaller than \( M(P) \). If \(|P| = 1\), then \( M_d(P) \) is empty; whereas if \(|P| = p\), then \( M_d(P) \) contains the trivial subgroup as its unique element. The latter will occur, for example, if \( G \) is any transitive permutation group of degree \( p \) (which may be non-soluble, hence 2-transitive by a theorem of Burnside [7, Satz V.21.3], implying that \( p - 1 \) is a divisor of \(|G|\)). For such a group, one cannot deduce much about the structure of \( G \) from that of \( M_d(P) \). Thus, we will impose some additional conditions on \( M_d(P) \) in our investigations.

We investigate the case in which, for \( P \in Syl_p(G) \), there is a choice of \( M_d(P) \) in which every element of \( M_d(P) \) is either weakly \( c \)-normal or \( s \)-permutable in \( G \); we will be able to unify and improve on known results.

2. Preliminaries

We first collect some properties of weakly \( c \)-normal and \( s \)-permutable embedded subgroup of a group.

**Lemma 2.1.** [18, Lemma 2.2] Let \( U \) be a weakly \( c \)-normal subgroup of \( G \) and \( N \) a normal subgroup of \( G \).
(1) If \( U \leq H \leq G \), then \( U \) is weakly \( c \)-normal in \( H \);
(2) If \( N \leq U \), then \( U/N \) is weakly \( c \)-normal in \( G/N \);
(3) Let \( \pi \) be a set of primes, \( U \) a \( \pi \)-subgroup and \( N \) a \( \pi' \)-subgroup. Then \( U/N \) is weakly \( c \)-normal in \( G/N \);
(4) \( U \) is weakly \( c \)-normal in \( G \) if and only if there exists a subnormal subgroup \( T \) of \( G \) such that \( G = UT \) and \( U \cap T = U_G \).

Lemma 2.2. [3, Lemma 1] Suppose that \( H \) is an \( s \)-permutably embedded subgroup of \( G \), \( K \leq G \) and \( N \) is a normal subgroup of \( G \). Then we have the following:

(1) If \( H \leq K \), then \( H \) is an \( s \)-permutably embedded subgroup of \( K \).
(2) \( HN/N \) is an \( s \)-permutably embedded subgroup of \( G/N \).

Lemma 2.3. [8] (1) If \( H \) is an \( s \)-permutable subgroup of a group \( G \), then \( H/H_G \) is nilpotent.
(2) Let \( K \triangleright G \) and \( K \leq H \). Then \( H \) is \( s \)-permutable in \( G \) if and only if \( H/K \) is \( s \)-permutable in \( G/K \).

The following lemmas play a crucial role in the proof of our results.

Lemma 2.4. [17, Lemma 2.8] Let \( G \) be a group and let \( p \) be a prime number dividing \( |G| \) with \( (|G|, p - 1) = 1 \).

(1) If \( N \) is normal in \( G \) of order \( p \), then \( N \) lies in \( Z(G) \);
(2) If \( G \) has cyclic Sylow \( p \)-subgroups, then \( G \) is \( p \)-nilpotent;
(3) If \( M \) is a subgroup of \( G \) with index \( p \), then \( M \) is normal in \( G \).

Lemma 2.5. [7, IV, Satz 4.7] If \( P \) is a Sylow \( p \)-subgroup of \( G \) and \( N \leq G \) such that \( P \cap N \leq \Phi(P) \), then \( N \) is \( p \)-nilpotent.

Lemma 2.6. [7, III, Satz 3.3] Let \( G \) be a group, and let \( N \) be a normal subgroup of \( G \) and \( H \leq G \). If \( N \leq \Phi(H) \), then \( N \leq \Phi(G) \).

Lemma 2.7. [16, Lemma 2.6] Let \( N \neq 1 \) be a normal subgroup of a group \( G \). If \( N \cap \Phi(G) = 1 \), then the Fitting subgroup \( F(N) \) of \( N \) is the direct product of minimal normal subgroups of \( G \) that are contained in \( F(N) \).

Lemma 2.8. [13, Lemma 2.5] (1) If \( A \) is subnormal in \( G \) and the index \( |G : A| \) is a \( p' \)-number, then \( A \) contains all Sylow \( p \)-subgroups of \( G \);
(2) If \( A \) is a subnormal Hall subgroup of \( G \), then \( A \) is normal in \( G \).

Lemma 2.9. [11] For a nilpotent subgroup \( H \) of \( G \), the following two statements are equivalent:

(1) \( H \) is \( s \)-permutable in \( G \).
(2) The Sylow subgroups of \( H \) are \( s \)-permutable in \( G \).
Lemma 2.10. [2] Let $P$ be a Sylow $p$-subgroup of $G$, and $P_1$ a maximal subgroup of $P$. Then the following two statements are equivalent:

1. $P_1$ is normal in $G$.
2. $P_1$ is $s$-permutable in $G$.

3. Main Results

We note that weakly $c$-normal subgroups and $s$-permutable embedded subgroups are two distinct concepts. Let us observe the symmetric group $S_4$. Let $H = \langle (12) \rangle$ and $K = \langle (123) \rangle$. Then $H$ is weakly $c$-normal but not $s$-permutable embedded in $S_4$, while $K$ is $s$-permutable embedded but not weakly $c$-normal in $S_4$.

Our first result is to unify and improve the results of [1] and [6] on the $p$-nilpotency of a group.

Theorem 3.1. Let $P$ be a Sylow $p$-subgroup of a group $G$, where $p$ is a prime divisor of $|G|$ with $(|G|, p-1) = 1$. If every member of some fixed $\mathcal{M}_d(P)$ is either weakly $c$-normal or $s$-permutable embedded in $G$, then $G$ is $p$-nilpotent.

Proof. Suppose that the theorem is false, and let $G$ be a counter-example of minimal order. We write $\mathcal{M}_d(P) = \{P_1, \ldots, P_d\}$. Then each $P_i$ is either weakly $c$-normal or $s$-permutable embedded in $G$. Without loss of generality, let $k, 1 \leq k \leq d$, be such that (i) each $P_i (1 \leq i \leq k)$ is weakly $c$-normal in $G$, and (ii) each $P_j (k+1 \leq j \leq d)$ is $s$-permutable embedded in $G$. Then for each $i, 1 \leq i \leq k$, there exists a subnormal subgroup $K_i$ of $G$ such that $G = P_i K_i$ and $P_i \cap K_i \leq (P_i)_G$; and for each $j, k+1 \leq j \leq d$, there exists an $s$-permutable subgroup $M_j \leq G$ such that $P_j$ is a Sylow $p$-subgroup of $M_j$.

Now we prove the theorem by the following several steps.

1. $O_{p'}(G) = 1$;

   Consider the quotient group $G/O_{p'}(G)$. Since $PO_{p'}(G)/O_{p'}(G)$ is a Sylow $p$-subgroup of $G/O_{p'}(G)$, which is isomorphic to $P$, $PO_{p'}(G)/O_{p'}(G)$ has the same smallest generator number $d$ as $P$. Set

   $$\mathcal{M}_d(PO_{p'}(G)/O_{p'}(G)) = \{P_1O_{p'}(G)/O_{p'}(G), \ldots, P_dO_{p'}(G)/O_{p'}(G)\}.$$ 

   Of course, each $P_sO_{p'}(G)/O_{p'}(G), s \in \{1, \ldots, d\}$

   is either $s$-permutable embedded or weakly $c$-normal in $G/O_{p'}(G)$ from Lemmas 2.2 and 2.1. As a result, $G/O_{p'}(G)$ satisfies the conditions of the theorem. If $O_{p'}(G) > 1$, then $G/O_{p'}(G)$ is $p$-nilpotent by the minimal choice of $G$. It follows that $G$ itself is $p$-nilpotent, a contradiction. Therefore, $O_{p'}(G) = 1$, as desired.

2. The quotient group $G/(P_i)_G$ is $p$-nilpotent for every $i \in \{1, 2, \ldots, k\}$.
By Lemma 2.1(4) we can assume $G = P_iK_i$ and $P_i \cap K_i = (P_i)G$. Then $G/(P_i)G = P_i/(P_i)G \cdot K_i/(P_i)G$. Therefore,

$$|K_i/(P_i)G|_p = |G : P_i|_p = |P : P_i| = p,$$

i.e., the factor group $K_i/(P_i)G$ possesses a cyclic Sylow subgroup of order $p$. By Lemma 2.4, we have that $K_i/(P_i)G$ is $p$-nilpotent. So $K_i/(P_i)G$ has a Hall normal $p'$-subgroup $H/(P_i)G$. Then $H/(P_i)G \triangleleft G/(P_i)G$ and $H/(P_i)G \in Hall(G/(P_i)G)$.

It follows from Lemma 2.8 that $H/(P_i)G$ is a normal $p$-complement of $G/(P_i)G$. Consequently, $G/(P_i)G$ is $p$-nilpotent, as desired.

(3) For every $j \in \{k + 1, k + 2, \ldots, d\}$, the factor group $G/(M_j)G$ is $p$-nilpotent:

It follows from Lemma 2.3 that $M_j/(M_j)G$ is $s$-permutable in $G/(M_j)G$ and $M_j/(M_j)G$ is nilpotent. Hence, we may apply Lemma 2.9 to see that every Sylow subgroup of $M_j/(M_j)G$ is $s$-permutable in $G/(M_j)G$. Thus, $P_j(M_j)G/(M_j)G$ is $s$-permutable in $G/(M_j)G$ because $P_j(M_j)G/(M_j)G$ is a Sylow $p$-subgroup of $M_j/(M_j)G$. It follows by Lemma 2.10 that $P_j(M_j)G/(M_j)G$ is normal in $G/(M_j)G$. So the core $(M_j)G$ of $M_j$ contains the Sylow $p$-subgroup $P_j$ of $M_j$ and we have $|G/(M_j)G|_p = p$. We conclude that $G/(M_j)G$ is $p$-nilpotent by Lemma 2.4. We have that (3) holds.

(4) Let $N = (\cap_{i=1}^k (P_i)G) \cap (\cap_{j=k+1}^d (M_j)G)$. We have $N \triangleleft G$. Now, we can show that $N$ is $p$-nilpotent. Consider the subgroup $P \cap N$. Recall that $P_j \in Syg_p((M_j)G)$ and $P_j$ is a maximal subgroup of $P$. We have

$$P \cap N = (\cap_{i=1}^k (P_i)G) \cap (\cap_{j=k+1}^d ((M_j)G \cap P))$$

$$= \cap_{i=1}^k (P_i)G \cap (\cap_{j=k+1}^d P_j) \leq \cap_{v=1}^d P_v = \Phi(P).$$

Thus $P \cap N \leq \Phi(P), N \triangleleft PN$. It is easy to see that $N$ is $p$-nilpotent by Lemma 2.5.

(5) $N \leq \Phi(G)$.

We know that $N$ possesses a normal Hall $p'$-subgroup $U$ such that $N = N_pU$, where $N_p \in Syg_p(N)$. Then $U$ is normal in $G$ and $U \leq O_{p'}(G) = 1$, so $U = 1$. Therefore, $N$ is a normal $p$-subgroup of $G$. Now, $N \leq P \cap N \leq \Phi(P)$. We see that $N \leq \Phi(G)$ by Lemma 2.6.

(6) The final contradiction:

By (2) and (3), $G/(P_i)G$ and $G/(M_j)G$ are $p$-nilpotent. Hence, $G/N$ is a $p$-nilpotent. Since $N \leq \Phi(G)$, it is easy to see that $G$ is $p$-nilpotent, the final contradiction. The proof of Theorem 3.1 is now complete.

Theorem 3.1 in [3] is a special case of Theorem 3.2.

Theorem 3.2. Let $G$ be a group and let $P$ be a Sylow $p$-subgroup of $G$ such that $N_G(P)$ is $p$-nilpotent, where $p$ is a prime divisor of $|G|$. If every member in some fixed $\mathcal{M}_d(P)$ is either weakly $c$-normal or $s$-permutable embedded in $G$, then $G$ is $p$-nilpotent.
Proof. By Theorem 3.1, it is easy to see that the theorem holds when \( p = 2 \). Assume that the theorem is false and let \( G \) be a counter-example of minimal order. By the hypotheses of the theorem, we can write \( M_d(P) = \{ P_1, P_2, \ldots, P_d \} \). Then each \( P_i \) is either weakly \( c \)-normal or \( s \)-permutably embedded in \( G \). With a similar argument as those used in the proof of Theorem 3.1, we first have the claim:

1. \( O_p(G) = 1 \).

Furthermore, we have:

2. If \( P \leq H < G \), then \( H \) is \( p \)-nilpotent:

Since \( N_H(P) \leq N_G(P) \), we have that \( N_H(P) \) is \( p \)-nilpotent. By Lemmas 2.1 and 2.2, \( H \) satisfies the hypotheses of the theorem. By the choice of \( G \), \( H \) is \( p \)-nilpotent, as desired.

3. \( G = PQ \), where \( Q \) is a Sylow \( q \)-subgroup of \( G \) with \( p \neq q \):

Since \( G \) is not \( p \)-nilpotent, by a result of Thompson [14, Corollary], there exists a non-trivial characteristic subgroup \( T \) of \( P \) such that \( N_G(T) \) is not \( p \)-nilpotent. Choose \( T \) such that the order of \( T \) is as large as possible. Since \( N_G(P) \) is \( p \)-nilpotent, we have \( N_G(K) \) is \( p \)-nilpotent for any characteristic subgroup \( K \) of \( P \) satisfying \( T < K \leq P \). Now, \( T \) is \( p \)-nilpotent, which gives \( T \not\leq N_G(P) \). So \( N_G(P) \leq N_G(T) \). By (2), we have that \( N_G(T) = G \) and \( T = O_p(G) \). Now, applying the result of Thompson again, we have that \( G/O_p(G) \) is \( p \)-nilpotent and therefore \( G \) is \( p \)-solvable. Then for any \( q \in \pi(G) \) with \( q \neq p \), there exists a Sylow \( q \)-subgroup of \( Q \) such that \( PQ \) is a subgroup of \( G \) [5, Theorem 6.3.5]. If \( PQ \not= G \), then \( PQ \) is \( p \)-nilpotent by (2), contrary to the choice of \( G \). Therefore, \( PQ = G \), as desired.

4. Every minimal normal subgroup of \( G \) contained in \( O_p(G) \) is of order \( p \):

As \( O_p(G) = 1 \), we get that \( O_p(G) > 1 \). Let \( N \) be a minimal normal subgroup of \( G \) contained in \( O_p(G) \). If \( N \leq \Phi(P) \), by Lemma 2.6, then \( N \leq \Phi(G) \), and \( G/N \) satisfies the hypotheses of the theorem. By the choice of \( G \), \( G/N \) is \( p \)-nilpotent. So \( G/\Phi(G) \) is \( p \)-nilpotent and it follows that \( G \) is \( p \)-nilpotent, a contradiction. Thus \( N \not\leq \Phi(P) \). Since \( \cap_{i=1}^d P_i = \Phi(P) \), where \( P_i \in M_d(P) \), we can assume \( N \not\leq P_i \) without loss of generality. By the conditions of the theorem, \( P_i \) is weakly \( c \)-normal in \( G \) or \( s \)-permutably embedded in \( G \). We claim that \( |N| = p \).

i) We first consider the case where \( P_i \) is weakly \( c \)-normal in \( G \). Then there exists \( K_i \triangleleft G \) such that \( G = P_iK_1 \) and \( P_i \cap K_1 \leq (P_i)_G \). Then \( (P_i)_G \cap N = 1 \) or \( N \). If \( (P_i)_G \cap N = 1 \), then \( N \leq (P_i)_G \leq P_1 \), a contradiction. So we have that \( (P_i)_G \cap N = 1 \), then \( P_1 \cap K_1 \cap N = 1 \). We consider \( (K_1)_G \cap N \). By the minimal normality of \( N \), we know that \( (K_1)_G \cap N = 1 \) or \( N \). If \( (K_1)_G \cap N = 1 \), then \( N \cong N(K_1)_G/(K_1)_G \) a minimal normal subgroup of \( G/(K_1)_G \), where \( G/(K_1)_G \) is a \( p \)-group since all Sylow \( q \)-subgroups of \( G \) is contained in \( K_1 \) by Lemma 2.8. Thus
we have that \(|N| = p\). If \((K_1)_G \cap N \neq 1\), we get that \(N \leq (K_1)_G \leq K_1\). Then \(1 = P_1 \cap K_1 \cap N = P_1 \cap N\) and so \(NP_1 = P\). We also get \(|N| = p\).

(ii) Next, we consider the case where \(P_1\) is \(s\)-permutably embedded in \(G\). Since \(P_1\) is \(s\)-permutably embedded in \(G\), then there exists an \(s\)-permutable subgroup \(H\) such that \(P_1 \in Syl_p(H)\). Hence, \(HQ\) is a subgroup of \(G\). Since \(N < G\), we have that \(N_1 = N \cap HQ < HQ\). It follows that \(N_1 < \langle HQ, N \rangle = G\). Moreover, by the minimal normality of \(N\), we get that \(N_1 = 1\) and so \(|N| = p\).

Now, we know that \(N \cap P_1 = 1\). By [7, I, 17.4], there exists a subgroup \(M\) of \(G\) such that \(G = NM\) and \(N \cap M = 1\). Certainly, \(N \nmid \Phi(G)\). From Lemma 2.8, we conclude \(O_p(G) = R_1 \times R_2 \times \cdots \times R_t\), where \(R_i (i = 1, \ldots, t)\) is a normal subgroup of order \(p\). It follows that \(P \leq \cap_{i=1}^t C_G(R_i) = C_G(O_p(G))\). Furthermore, according to [10, Theorem 9.31] and (3), we have that \(C_G(O_p(G)) \leq O_p(G)\) and so \(P = O_p(G)\). Thus \(G = N_G(P)\). Now, by the hypotheses that \(N_G(P)\) is \(p\)-nilpotent, we conclude that \(G\) is \(p\)-nilpotent. This is the final contradiction and the proof is complete. □

We observe the \(p\)-supersolvability of a \(p\)-solvable group by means of weakly \(c\)-normal and \(s\)-permutably embedded subgroups.

**Theorem 3.3.** Let \(G\) be a \(p\)-solvable group and let \(P\) be a Sylow \(p\)-subgroup of \(G\), where \(p\) is a prime divisor of \(|G|\). If every member in some fixed \(M_d(P)\) is either weakly \(c\)-normal or \(s\)-permutably embedded in \(G\), then \(G\) is \(p\)-supersolvable.

**Proof.** Assume that the theorem is false and let \(G\) be a counter-example of minimal order. We write \(M_d(P) = \{P_1, \ldots, P_d\}\). Then each \(P_i\) is either weakly \(c\)-normal or \(s\)-permutably embedded in \(G\). With the same arguments as those used in the proof of Theorem 3.1, we first have the claim:

1. \(O_{P'}(G) = 1\).
2. \(\Phi(P) = 1\), in particular, \(\Phi(O_p(G)) = 1\).

Otherwise, then let \(N = \Phi(P)G > 1\). We consider factor group \(G/N\). Obviously, \(M_d(P/N) = \{P_1/N, \ldots, P_d/N\}\). By Lemmas 2.1 and 2.2, \(P_i/N\) is either weakly \(c\)-normal or \(s\)-permutably embedded in \(G/N\) for any \(i \in \{1, \ldots, d\}\). Therefore, \(G/N\) satisfies the hypotheses of the theorem and consequently, \(G/N\) is \(p\)-supersolvable by the minimality of \(G\). Since \(N \leq \Phi(P), N \leq \Phi(G)\) by Lemma 2.6, it follows from \(G/N\) being \(p\)-supersolvable that \(G\) is \(p\)-supersolvable, which is contrary to the choice of \(G\).

3. Every minimal normal subgroup of \(G\) contained in \(O_p(G)\) is of order \(p\).

As \(O_{P'}(G) = 1\), we get that \(O_p(G) > 1\). Let \(N\) be a minimal normal subgroup of \(G\) contained in \(O_p(G)\). If \(N \leq \Phi(P)\), by Lemma 2.6, then \(N \leq \Phi(G)\), and \(G/N\) satisfies the hypotheses of the theorem. By the choice of \(G\), \(G/N\) is \(p\)-supersolvable. Since the class of \(p\)-supersolvable groups is a saturated formation, we have \(G\) is \(p\)-supersolvable, a contradiction. Thus \(N \nleq \Phi(P)\). Since \(\cap_{i=1}^d P_i = \Phi(P)\), where
$P_i \in \mathcal{M}_d(P)$, we can assume $N \not\leq P_i$ without loss of generality. By the conditions of the theorem, $P_i$ is weakly $c$-normal in $G$ or $s$-permutable embedded in $G$. We claim that $|N| = p$.

(i) We first consider the case where $P_i$ is weakly $c$-normal in $G$. Then there exists $K_1 \triangleleft G$ such that $G = P_1K_1$ and $P_1 \cap K_1 \leq (P_1)_G$. Then $(P_1)_G \cap N = 1$ or $N$. If $(P_1)_G \cap N = N$, then $N \leq (P_1)_G \leq P_1$, a contradiction. So we have that $(P_1)_G \cap N = 1$, then $P_1 \cap K_1 \cap N = 1$. We consider $(K_1)_G \cap N$. By the minimal normality of $N$, we know that $(K_1)_G \cap N = 1$ or $N$. If $(K_1)_G \cap N = 1$, then $N \cong N((K_1)_G)/(K_1)_G$ a minimal normal subgroup of $G/(K_1)_G$, where $G/(K_1)_G$ is a $p$-group since all Sylow $q$-subgroups of $G$ is contained in $K_1$ by Lemma 2.8. Thus we have that $|N| = p$. If $(K_1)_G \cap N \neq 1$, we get that $N \leq (K_1)_G \leq K_1$. Then $1 = P_1 \cap K_1 \cap N = P_1 \cap N$ and so $NP_1 = P$. We also get $|N| = p$.

(ii) Next, we consider the case where $P_i$ is $s$-permutable embedded in $G$. Since $P_i$ is $s$-permutable embedded in $G$, then there exists an $s$-permutable subgroup $H$ such that $P_i \in Syl_p(H)$. Hence, $HQ$ is a subgroup of $G$. Since $N \triangleleft G$, we have that $N_1 = N \cap HQ \triangleleft HQ$. It follows that $N_1 \triangleleft (HQ, N) = G$. Moreover, by the minimal normality of $N$, we get that $N_1 = 1$ and so $|N| = p$.

Therefore, $N \cap P_1 = 1$. By [7, I, 17.4], there exists a subgroup $M$ of $G$ such that $G = NM$ and $N \cap M = 1$. Certainly, $N \not\subseteq \Phi(G)$. Now, we can use Lemma 2.7 to derive that $O_p(G)$ is a direct product of normal subgroups of $G$ of order $p$. Hence, (3) holds.

(4) The counter-example does not exist.

Since $G/C_G(R_i)$ is a cyclic group of order $p - 1$, certainly, $G/ \cap_{i=1}^s C_G(R_i) = G/C_G(O_p(G))$ is $p$-supersolvable. On the other side, since $G$ is $p$-solvable and $O_p(G) = 1$, by [10, Theorem 9.3.1], $C_G(O_p(G)) \leq O_p(G)$. Hence, $G/O_p(G)$ is $p$-supersolvable. Now, claim (3) implies that $G$ is $p$-supersolvable. We are done. \hfill \Box

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