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# ON THE LEVITZKI RADICAL OF MODULES

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*Dedicated to the memory of Professor Efraim P. Armendariz*

Abstract. In [1] a Levitzki module which we here call an $l$-prime module was introduced. In this paper we define and characterize $l$-prime submodules. Let $N$ be a submodule of an $R$-module $M$. If

$$l.\sqrt{N} := \{m \in M \ : \ \text{every } l\text{- system of } M \text{ containing} m \text{ meets } N\},$$

we show that $l.\sqrt{N}$ coincides with the intersection $\mathcal{L}(N)$ of all $l$-prime submodules of $M$ containing $N$. We define the Levitzki radical of an $R$-module $M$ as $\mathcal{L}(M) = l.\sqrt{0}$. Let $\beta(M)$, $\mathcal{U}(M)$ and $\text{Rad}(M)$ be the prime radical, upper nil radical and Jacobson radical of $M$ respectively. In general $\beta(M) \subseteq \mathcal{L}(M) \subseteq \mathcal{U}(M) \subseteq \text{Rad}(M)$. If $R$ is commutative, $\beta(M) = \mathcal{L}(M) = \mathcal{U}(M)$ and if $R$ is left Artinian, $\beta(M) = \mathcal{L}(M) = \mathcal{U}(M) = \text{Rad}(M)$. Lastly, we show that the class of all $l$-prime modules ${}_RM$ with $RM \neq 0$ forms a special class of modules.

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## 1. Introduction

All modules are left modules, the rings are associative but not necessarily unital. By $I \lhd R$ and $N \leq M$ we respectively mean $I$ is an ideal of a ring $R$ and $N$ is a submodule of a module $M$. A submodule $P$ of an $R$-module $M$ with $RM \not\subseteq P$ is prime if for all $\mathcal{A} \lhd R$ and every $N \leq M$ such that $\mathcal{A}N \subseteq P$, we have $\mathcal{A}M \subseteq P$ or $N \subseteq P$. In all our definitions for "prime" submodules $P$ of ${}_RM$, we shall assume (without mention) that $RM \not\subseteq P$. In [10], a generalization of upper nil radical of rings to modules was done. A submodule $P$ of an $R$-module $M$ is $s$-prime if $P$ is prime and the ring $R/(P : M)$ has no nonzero nil ideals, i.e., $\mathcal{U}(R/(P : M)) = 0$ where $\mathcal{U}$ is the upper nil radical map. This definition generalizes that of $s$-prime ideals in [14, Definition 2.6.5, p.170] and in [16]. Since the upper nil radical of rings (sum of all nil ideals of a ring) coincides (see [16] ) with the intersection of all $s$-prime ideals of $R$, in [10] we defined the upper nil radical of a module $M$ denoted by $\mathcal{U}(M)$ as the intersection of all $s$-prime submodules of $M$. For any unital ring $R$, $\mathcal{U}({}_RR) = \mathcal{U}(R)$.

**Definition 1.1.** [17] An ideal $I$ of a ring $R$ is *l-prime* if it satisfies one of the following equivalent statements:

(1) $I$ is a prime ideal and $L(R/I) = 0$, where $L$ is the Levitzki radical map, i.e., the ring $R/I$ has no nonzero locally nilpotent ideals;
   (i) if $a, b \notin I$, then $(a)(b) \nsubseteq I$ where $(a)$ denotes principal ideal and
   (ii) if $a \notin I$, then $(a)$ is not locally nilpotent modulo $I$;

(2) given $a, b \notin I$, there exists elements $a_1, a_2, \cdots, a_n \in (a)$ and $b_1, b_2, \cdots, b_m \in (b)$ such that for every $p > 1$ there exists a product of $N \geq p$ factors, consisting of $a_i's$ and $b_j's$, which is not in $I$.

A set $L$ of elements of a ring $R$ is an $l$-system [17] if to every element $a \in L$ is assigned a finite number of elements $a_1, a_2, \cdots, a_{n(a)} \in (a)$, such that the following condition is satisfied: If $a, b \in L$, then for every $m > 1$ there exists a product of $N \geq m$ factors, consisting of $a_i$ and $b_j$'s, which is in $L$.

From [2], the intersection of all $l$-prime ideals coincides with the Levitzki radical of a ring (the sum of all locally nilpotent ideals of a ring). The notion of $l$-prime ideals and $l$-systems of near-rings was defined and studied by Groenewald and Potgieter in [9]. In this paper we generalize the Levitzki radical of rings to modules by defining $l$-prime submodules and having the Levitzki radical $\mathcal{L}(M)$ of a module $M$ as the intersection of all $l$-prime submodules of $M$.

## 2. $l$-prime submodules

**Definition 2.1.** A proper submodule $P$ of an $R$-module $M$ is an *$l$-prime submodule* if for any $\mathcal{A} \lhd R$, $N \leq M$ and for every $\{a_1, a_2, \cdots, a_n\} \subseteq \mathcal{A}$ there exists $m \in \mathbb{N}$ such that for any product $a_{1i}a_{2i} \cdots a_{mi}$ of elements from $\{a_1, a_2, \cdots, a_n\}$, $a_{1i}a_{2i} \cdots a_{mi}N \subseteq P$ implies $N \subseteq P$ or $\mathcal{A}M \subseteq P$.

**Proposition 2.2.** *If $P \leq M$, then the following statements are equivalent:*

(1) *$P$ is an $l$-prime submodule;*
(2) *$P$ is a prime submodule and for all $\mathcal{A} \lhd R$ with $\mathcal{A}M \nsubseteq P$ there exists $F = \{a_1, a_2, \cdots, a_n\} \subseteq \mathcal{A} \setminus (P : M)$ such that $a_{i1}a_{i2} \cdots a_{ii}M \nsubseteq P$ for all $1 \leq i \in \mathbb{N}$ where $a_{ij} \in F$;*
(3) *$P$ is a prime submodule and $\mathcal{L}(R/(P : M)) = 0$;*
(4) *$P$ is a prime submodule and $\mathcal{L}(R/(P : N)) = 0$ for all $N \leq M$ with $N \nsubseteq P$;*
(5) *for all $m \in M$, for each $a \in R$ and for every finite set $F = \{a_1, \cdots, a_n\} \subseteq (a)$ there exists $n \in \mathbb{N}$ such that for any product of elements $a_{i1}a_{i2} \cdots a_{in}$ from $F$, $a_{i1}a_{i2} \cdots a_{in} <m> \subseteq P$ implies $m \in P$ or $aM \subseteq P$, where $<m>$ denotes the submodule of $M$ generated by $m$;*
(6) *$P$ is a prime submodule of $M$ and $(P : M)$ is an $l$-prime ideal of $R$;*

(7) *$P$ is a prime submodule of $M$ and $(P:N)$ is an $l$-prime ideal of $R$ for all $N \leq M$ with $N \not\subseteq P$.*

**Proof.** (1) $\Rightarrow$ (2) Let $\mathcal{A} \lhd R$, $N \leq M$ such that $\mathcal{A}N \subseteq P$. For any finite subset $\{a_1, a_2, \cdots, a_n\} \subseteq \mathcal{A}$, and for any $t \in \mathbb{N}$, $a_{i1}a_{i2}\cdots a_{it}N \subseteq P$ for $a_{ij} \in \{a_1, a_2, \cdots a_n\}$. Since $P$ is $l$-prime we have $N \subseteq P$ or $\mathcal{A}M \subseteq P$. Hence $P$ is a prime submodule. Let $\mathcal{A} \lhd R$ such that $\mathcal{A}M \not\subseteq P$. $\mathcal{A}m \not\subseteq P$ for some $m \in M$. Since $P$ is $l$-prime, there exists $\{a_1, a_2, \cdots, a_n\} \subseteq \mathcal{A}$ such that $a_{i1}a_{i2}\cdots a_{ii}\mathcal{A}m \not\subseteq P$ for any $i \geq 1$, where $a_{ij} \in \mathcal{A}$. Hence, $a_{i1}a_{i2}\cdots a_{ii}M \not\subseteq P$ for any $i \geq 1$ and $a_{ij} \in \mathcal{A}$.

(2) $\Rightarrow$ (1) Let $\mathcal{A} \lhd R$ and $N \leq M$ such that $N \not\subseteq P$ and $\mathcal{A}M \not\subseteq P$. Since $P$ is prime, $(P:M) = (P:N)$. From our assumption, there exists a finite subset $F = \{a_1, a_2, \cdots, a_n\} \subseteq \mathcal{A}$ such that $a_{i1}a_{i2}\cdots a_{ii}M \not\subseteq P$ for every natural number $i \geq 1$ and $a_{ij} \in F$. Hence, $a_{i1}a_{i2}\cdots a_{ii}N \not\subseteq P$ for all $1 \leq i \in \mathbb{N}$ and $a_{ij} \in F$. Thus, $P$ is an $l$-prime submodule.

(2) $\Leftrightarrow$ (3) This is clear since $\mathcal{L}(R/(P:M)) = 0$ if and only if $R/(P:M)$ contains no nonzero locally nilpotent ideals.

(3) $\Rightarrow$ (4) Let $P$ is a prime submodule such that $\mathcal{L}(R/(P:N) = 0$. Now let $N \leq P$ such that $N \not\subseteq P$. Since $P$ is a prime submodule, we have $(P:N) = (P:M)$ and from our assumption $\mathcal{L}(R/(P:N)) = \mathcal{L}(R/(P:M)) = 0$.This proves 4.

(4) $\Rightarrow$ (3) Let $P$ is a prime submodule such that $\mathcal{L}(R/(P:N) = 0$ for all $N \leq P$ such that $N \not\subseteq P$. Since $P$ is a prime submodule and $N \not\subseteq P$, we have $(P:N) = (P:M)$ and from our assumption $\mathcal{L}(R/(P:N)) = \mathcal{L}(R/(P:M)) = 0$.This proves 3.

(5) $\Rightarrow$ (1) Let $\mathcal{A} \lhd R$, $N \leq M$ such that $\mathcal{A}M \not\subseteq P$ and $N \not\subseteq P$. There exists $a \in \mathcal{A}$ such that $aM \not\subseteq P$ and $m \in N \setminus P$. So, there exists $F = \{a_1, \cdots, a_n\} \subseteq (a) \subseteq \mathcal{A}$ such that $a_{i1}\cdots a_{ii} <m> \not\subseteq P$ for all $i \geq 1$. Hence $a_{i1}a_{i2}\cdots a_{ii}N \not\subseteq P$ for all $i \geq 1$. Therefore $P$ is $l$-prime.

(1) $\Rightarrow$ (5) Suppose $m \in M \setminus P$ and $a \in R$ such that $aM \not\subseteq P$. Then $(a)M \not\subseteq P$ and $<m> \not\subseteq P$. Since $P$ is $l$-prime there exists $F = \{a_1, \cdots a_n\} \subseteq (a)$ such that $a_{i1}a_{i2}\cdots a_{ii} <m> \not\subseteq P$ for all $i \geq 1$.

(6) $\Rightarrow$ (3) Suppose $P$ is a prime submodule and $(P:M)$ is an $l$ -prime ideal of $R$. By definition of $l$-prime ideals, $(P:M)$ is a prime ideal and $\mathcal{L}(R/(P:M)) = 0$. It follows that $P$ is a prime submodule and $\mathcal{L}(R/(P:M)) = 0$ which is 3.

(3) $\Rightarrow$ (6) Suppose that $P$ is a prime submodule of $M$ and $\mathcal{L}(R/(P:M)) = 0$. Then $(P:M)$ is a prime ideal of $R$ and $\mathcal{L}(R/(P:M)) = 0$. So, by definition $(P:M)$ is an $l$-prime ideal of $R$.

(6) $\Rightarrow$ (7) Let $P$ be a prime submodule of $M$ and $(P:M)$ an $l$-prime ideal of $R$. Now, let $N \leq M$ with $N \not\subseteq P$. Since $P$ is a prime submodule and $N \not\subseteq P$, we have $(P:N) = (P:M)$ and from our assumption $(P:N)$ is an $l$-prime ideal of $R$.

(7) $\Rightarrow$ (6). Let $P$ be a prime submodule of $M$ and $(P : N)$is an $l$-prime ideal of $R$ for all $N \leq M$ with $N \nsubseteq P$.Since $P$ is a prime submodule and $N \nsubseteq P$,we have $(P : N) = (P : M)$and from our assumption $(P : M)$is an $l$-prime ideal of $R$. □

$P$ is an $l$-prime submodule of $M$ if and only if $M/P$ is an $l$-prime module.

**Definition 2.3.** A module $M$ is *prime* if the zero submodule of $M$ is a prime submodule.

**Proposition 2.4.** *If $R$ is a unital ring, then $R$ is $l$-prime if and only if ${}_RR$ is an $l$ -prime module.*

**Proof.** We know that $R$ is a prime ring if and only if ${}_RR$ is a prime module. $R$ prime implies $(0 : R) = 0$. Hence, whenever $R$ is prime, $\mathcal{L}(R) = 0$ if and only if $\mathcal{L}(R/(0 : R)) = 0$. It follows that: $R$ is prime and $\mathcal{L}(R) = 0$ if and only if ${}_RR$ is prime and $\mathcal{L}(R/(0 : R)) = 0$, i.e., $R$ is $l$-prime if and only if ${}_RR$ is $l$-prime. □

**Corollary 2.5.** *For any unital ring $R$, $\mathcal{L}(R) = \mathcal{L}({}_RR)$.*

**Example 2.6.** *Any maximal submodule is $l$-prime, hence any simple module is $l$-prime.*

**Proposition 2.7.** *For any submodule $P$ of ${}_RM$,*

$$s\text{-prime} \Rightarrow l\text{-prime} \Rightarrow \text{prime}.$$

**Proof.** Suppose $P$ is prime and $\mathcal{U}(R/(P : M)) = 0$. Since for rings $\mathcal{L} \subseteq \mathcal{U}$, we have $\mathcal{L}(R/(P : M)) = 0$. So, $P$ is $l$-prime. The last implication is trivial. □

**Corollary 2.8.** *For any module $M$,*

$$\beta(M) \subseteq \mathcal{L}(M) \subseteq \mathcal{U}(M).$$

**Example 2.9.** *Any strictly prime submodule (as defined by Dauns in* [6]*) is $s$-prime (see* [11]*). Hence, it is $l$-prime by Proposition 2.7.*

**Example 2.10.** *In* [18, Section 2.2]*, an example of a ring $R$ which is prime and locally nilpotent was constructed. Hence $R$ is prime but not $l$-prime. Thus, the module $M = {}_RR$ is prime but not $l$-prime.*

**Example 2.11.** *In* [18, Section 2.3]*, an example of a prime nil ring $R$ which is not locally nilpotent was constructed. Hence $R$ is $l$-prime but not $s$-prime. Thus, the module $M = {}_RR$ is $l$-prime but not $s$-prime.*

**Theorem 2.12.** *For modules over commutative rings,*

$$s\text{-prime} \Leftrightarrow l\text{-prime} \Leftrightarrow \text{prime}.$$

*Hence,*

$$\beta(M) = \mathcal{L}(M) = \mathcal{U}(M).$$

**Proof.** Follows from Proposition 2.7 and the fact that prime and $s$-prime are the same for modules over commutative rings, see [10]. $\square$

## 3. Semi $l$-prime submodules

An ideal $\mathcal{I}$ of a ring $R$ is semi $s$-prime (resp. semi $l$ -prime) if $\mathcal{U}(R/I) = 0$ (resp. $\mathcal{L}(R/I) = 0$). A submodule $P$ of an $R$-module $M$ is a semi $s$-prime [15] submodule if for all $\mathcal{A} \lhd R$ and every $N \leq M$ with $a \in \mathcal{A}$ and $N \not\subseteq P$ such that $a^n N \subseteq P$ for some $n \in \mathbb{N}$, then $\mathcal{A}N \subseteq P$. It was shown in [15] that $P$ is a semi $s$-prime submodule if and only if $\mathcal{U}(R/(P:N)) = 0$ for all $N \leq M$ with $N \not\subseteq P$.

**Definition 3.1.** $P$ is a *semi $l$-prime* submodule of $M$, if for all $\mathcal{A} \lhd R$, for all $N \leq M$ such that $N \not\subseteq P$ and for every finite subset $F = \{a_1, a_2, \cdots, a_n\} \subseteq \mathcal{A}$, there exists $T = T(F) \in \mathbb{N}$ such that for any product of $m$ elements ($m$ less or equal to $T$) consisting of the $a_i$'s we have $a_{i1} \cdots a_{im} N \subseteq P$ implies $\mathcal{A}N \subseteq P$.

**Proposition 3.2.** *For any submodule $P$ of an $R$-module $M$, the following statements are equivalent:*

(1) *$P$ is a semi $l$-prime submodule of ${}_RM$;*
(2) *$\mathcal{L}(R/(P:N)) = 0$ for all $N \leq M$ with $N \not\subseteq P$.*

**Proof.** Follows from the definition of a semi $l$-prime submodule and the notion of a locally nilpotent ideal. $\square$

**Theorem 3.3.** *A submodule $P$ of ${}_RM$ is $l$-prime if and only if $P$ is prime and semi $l$-prime.*

**Proof.** Follows from Proposition 2.2 and Proposition 3.2. $\square$

**Proposition 3.4.** *For any module ${}_RM$,*

(1) *$\mathcal{L}(M) = \cap\{P : P \leq M,\ P$ $l$-prime submodule of $M\}$ is a semi $l$-prime submodule;*
(2) *$P$ is a semi $l$-prime submodule of ${}_RM$ if and only if $(P:N)$ is a semi $l$-prime ideal of $R$ for any $N \leq M$ with $N \not\subseteq P$.*

**Proof.** (1) Let $\mathcal{A} \lhd R$, $N \leq M$ such that $\mathcal{A}N \not\subseteq \mathcal{L}(M)$. Then there exists an $l$-prime submodule $P$ such that $\mathcal{A}N \not\subseteq P$. From Proposition 2.2, there exists a finite subset $F = \{a_1, a_2, \cdots, a_n\} \subseteq \mathcal{A} \setminus (P:N)$ such that $a_{i1}a_{i2} \cdots a_{ii} N \not\subseteq P$ for all $1 \leq i \in \mathbb{N}$ with $a_{ij} \in F$. Hence, $a_{i1}a_{i2} \cdots a_{ii} N \not\subseteq \mathcal{L}(M)$ for all $i \geq 1$ with $a_{ij} \in F$.

(2) Follows from Proposition 3.2 and the definition of semi $l$-prime ideals. $\square$

**Definition 3.5.** A submodule $P$ of an $R$-module $M$ is

(1) *semiprime* [6] if for all $a \in R$ and every $m \in M$, if $aRam \subseteq P$ then $am \in P$;
(2) *classical semiprime* [3] if for all $\mathcal{A} \lhd R$ and every $N \leq M$, if $\mathcal{A}^2 N \subseteq P$ then $\mathcal{A}N \subseteq P$.

**Remark 3.6.** *Classical semiprime submodules are called "semiprime" by Behboodi in* [3].

**Proposition 3.7.** *For any submodule $P$,*

$$\text{semi } s\text{-prime} \Rightarrow \text{semi } l\text{-prime} \Rightarrow \text{classical semiprime.}$$

**Proof.** Suppose $P$ is a semi $s$-prime submodule of $M$, then $\mathcal{U}(R/(P:N)) = 0$ for all $N \leq M$ with $N \nsubseteq P$. We know $\mathcal{L} \subseteq \mathcal{U}$, hence $\mathcal{L}(R/(P:N)) = 0$ for all $N \leq M$ with $N \nsubseteq P$, i.e., $P$ is semi $l$-prime. Let $P$ be a semi $l$-prime submodule of ${}_RM$. Suppose $\mathcal{A} \lhd R$ and $N \leq M$ such that $\mathcal{A}N \nsubseteq P$. Then there exists $F = \{a_1, \cdots a_n\} \subseteq \mathcal{A}$ such that for all $i \geq 1$, $a_{i1}a_{i2}\cdots, a_{ii}N \nsubseteq P$ for $a_{ij} \in F$. Hence for $i = 2$, there exists $a_{21}, a_{12} \in F$ such that $a_{21}a_{12}N \nsubseteq P$, i.e., $\mathcal{A}^2N \nsubseteq P$ and hence $P$ is classical semiprime. □

**Proposition 3.8.** *For modules over a commutative ring,*

$$\text{semi } s\text{-prime} \Leftrightarrow \text{semi } l\text{-prime} \Leftrightarrow \text{ classical semiprime} \Leftrightarrow \text{ semiprime.}$$

**Proof.** Suppose $R$ is commutative and $P$ is classical semiprime. Let $\mathcal{A} \lhd R$ and $\mathcal{A}N \nsubseteq P$ for some $N \leq M$. Then there is $a \in \mathcal{A}$ such that $aN \nsubseteq P$. $P$ semiprime implies $(P:N)$ is semiprime and because $R$ is commutative $(P:N)$ is completely semiprime, hence $a^n \notin (P:N)$ for all $n \in \mathbb{N}$, i.e., $a^nN \nsubseteq P$ for all $n \in \mathbb{N}$ which shows that $P$ is semi $s$-prime. The rest follows from Proposition 3.7 and the fact that for commutative rings the notions of semiprime and classical semiprime are the same. □

**Remark 3.9.** *We have seen in Proposition 3.4 that any intersection of $l$-prime submodules is a semi $l$-prime submodule. The converse does not hold in general. For over a commutative ring, prime is the same as $l$-prime (see Theorem 2.12) and semiprime is the same as semi $l$-prime, (see Proposition 3.8). Now, let $R = \mathbb{Z}[x]$ and $F = R \oplus R$ . If $f := (2, x) \in F$ and $P = 2R + Rx$ which is a maximal ideal of $R$, then $N = Pf$ is a semiprime submodule of $F$ which is not an intersection of prime submodules, see* [12, p.3600].

**Definition 3.10.** Let $R$ be a ring and $M$ an $R$-module. A nonempty set $L \subseteq M\backslash\{0\}$ is called an *$l$-system* if, for each $\mathcal{A} \lhd R$ and for all $J, K \leq M$, if $(K+J) \cap L \neq \emptyset$ and $(K + \mathcal{A}M) \cap L \neq \emptyset$, then there exists a finite subset $F = \{a_1, a_2, \cdots a_n\} \subseteq \mathcal{A}$ such that $K + (a_{i1}a_{i2}\cdots a_{ii}J) \cap L \neq \emptyset$ for any $i \geq 1$ and $a_{ij} \in F$.

It is easy to see that every $s$-system as defined in [10] is an $l$-system and any $l$-system is an $m$-system as defined in [4].

**Corollary 3.11.** *Let $M$ be an $R$-module. A submodule $P$ of $M$ is $l$-prime if and only if $M \setminus P$ is an $l$-system of $M$.*

**Proof.** ($\Rightarrow$) Suppose $L := M\backslash P$. Let $\mathcal{A}\lhd R$ and $K, J \leq M$ such that $(K+J)\cap L \neq \emptyset$ and $(K+\mathcal{A}M)\cap L \neq \emptyset$. Suppose that for every finite subset $F = \{a_1, a_2, \cdots a_n\} \subseteq \mathcal{A}$ there exists $t = t(F) \in \mathbb{N}$ such that for any product of $t$ elements $a_{i1}a_{i2}\cdots a_{it}$ from $F$ we have $(K + a_{i1}a_{i2}\cdots a_{it}J) \cap L = \emptyset$. Hence $a_{i1}a_{i2}\cdots a_{it}J \subseteq P$. Since $P$ is $l$-prime, $J \subseteq P$ or $\mathcal{A}M \subseteq P$. It follows that $(J + K)\cap L = \emptyset$ or $(\mathcal{A}M + K)\cap L = \emptyset$, a contradiction.

($\Leftarrow$) Suppose $\mathcal{A}\lhd R$, $N \leq M$ and for every finite subset $F = \{a_1, a_2, \cdots , a_n\} \subseteq \mathcal{A}$ there exists $t = t(F) \in \mathbb{N}$ such that for any product of $t$ elements $a_{i1}, \cdots , a_{it}$ from $F$, $a_{i1}a_{i2}\cdots a_{it}J \subseteq P$. If $J \not\subseteq P$ and $\mathcal{A}M \not\subseteq P$, $J \cap L \neq \emptyset$ and $\mathcal{A}M \cap L \neq \emptyset$. Since $M \setminus P$ is an $l$-system, there exists a finite subset $\{b_1, b_2, \cdots , b_m\} \subseteq \mathcal{A}$ such that $b_{i1}b_{i2}\cdots b_{ii}J \cap L \neq \emptyset$ for every $i \geq 1$. This leads to a contradiction. Hence $J \subseteq P$ or $\mathcal{A}M \subseteq P$ and therefore $P$ is an $l$-prime submodule of $M$. □

**Lemma 3.12.** *Let $M$ be an $R$-module, $L \subseteq M$ an $l$-system and $P$ a submodule of $M$ maximal with respect to the property that $P \cap L = \emptyset$. Then, $P$ is an $l$-prime submodule of $M$.*

**Proof.** Let $\mathcal{A}\lhd R$ and $J \leq M$. Suppose that for any finite subset $F = \{a_1, a_2, \cdots , a_n\}$ $\subseteq \mathcal{A}$ there exists a natural number $n$ such that for the product of any $n$ elements $a_{i1}, \cdots , a_{in}$ from $F$ we have $a_{i1}a_{i2}\cdots a_{in}J \subseteq P$. If $J \not\subseteq P$ and $\mathcal{A}M \not\subseteq P$ then $(J + P) \cap L \neq \emptyset$ and $(\mathcal{A}M + P) \cap L \neq \emptyset$. Since $L$ is an $l$-system, there exists $\{b_1, b_2, \cdots b_m\} \subseteq \mathcal{A}$ such that $(b_{i1}b_{i2}\cdots b_{ii}J + P) \cap L \neq \emptyset$ for every $i \geq 1$ and $b_{ij} \in \{b_1, \cdots b_m\}$. But for this finite subset $\{b_1, \cdots b_m\}$ it follows from above that there exists a natural number $n$ such that for the product of any $n$ elements from the set $b_{i1}b_{i2}\cdots b_{in}J \subseteq P$. Hence $P \cap L \neq \emptyset$. Thus, we must have $J \subseteq P$ or $\mathcal{A}M \subseteq P$ and therefore $P$ must be an $l$-prime submodule. □

**Definition 3.13.** Let $R$ be a ring and $M$ an $R$-module. For $N \leq M$, if there is an $l$-prime submodule containing $N$, then we define

$$l.\sqrt{N} := \{m \in M \ : \ \text{every } l\text{-system of } M \text{ containing } m \text{ meets } N\}.$$

We write $l.\sqrt{N} = M$ whenever there are no $l$-prime submodules of $M$ containing $N$.

**Theorem 3.14.** *Let $M$ be an $R$-module and $N \leq M$. Then, either $l.\sqrt{N} = M$ or $l.\sqrt{N}$ equals the intersection of all $l$-prime submodules of $M$ containing $N$.*

**Proof.** Suppose $l.\sqrt{N} \neq M$. This means

$$\beta^l(N) := \cap\{P \ : \ P \text{ is an } l\text{-prime submodule of } M \text{ and } N \subseteq P\} \neq \emptyset.$$

Both $l.\sqrt{N}$ and $N$ are contained in the same $l$-prime submodules. By definition of $l.\sqrt{N}$ it is clear that $N \subseteq l.\sqrt{N}$. Hence, any $l$-prime submodule of $M$ which contains $l.\sqrt{N}$ must necessarily contain $N$. Suppose $P$ is an $l$-prime submodule of $M$ such that $N \subseteq P$, and let $t \in l.\sqrt{N}$. If $t \notin P$, then the complement of $P$, $C(P)$ in $M$ is an $l$-system containing $t$ and therefore we would have $C(P) \cap N \neq \emptyset$. However, since $N \subseteq P$, $C(P) \cap P = \emptyset$ and this contradiction shows that $t \in P$. Hence $l.\sqrt{N} \subseteq P$ as we wished to show. From this we have $l.\sqrt{N} \subseteq \beta^l(N)$. Conversely, assume $s \notin l.\sqrt{N}$. Then there exists an $l$-system $L$ such that $l \in L$ and $L \cap N = \emptyset$. From Zorn's lemma, there exists a submodule $P \supseteq N$ which is maximal with respect to $P \cap L = \emptyset$. From Lemma 3.12, $P$ is an $l$-prime submodule of $M$ and $l \notin P$, as desired. □

**Proposition 3.15.** *If $\mathcal{P} \lhd R$, then there is an $l$-prime $R$ -module $M$ with $\mathcal{P} = (0 : M)$ if and only if $\mathcal{P}$ is an $l$ -prime ideal of $R$.*

**Proof.** Suppose $M$ is an $l$-prime module. Then by Proposition 2.2, $\mathcal{P} = (0 : M)$ is an $l$-prime ideal of $R$. For the converse, let $\mathcal{P}$ be an $l$-prime ideal of $R$. $M = R/\mathcal{P}$ is an $R$-module with the usual operation and $\mathcal{P} = (0 : M)$. $(0 : M)$ $l$-prime implies $(0 : M)$ is prime and $\mathcal{L}(R/(0 : M)) = 0$. From [8, Proposition 3.14.16] $(0 : M)$ prime implies $M$ is a prime module. Thus $M$ is a prime module and $\mathcal{L}(R/(0 : M)) = 0$ which proves that $M$ is an $l$-prime module. □

**Corollary 3.16.** *A ring $R$ is an $l$-prime ring if and only if there exists a faithful $l$ -prime $R$-module.*

**Example 3.17.** *If $R$ is a domain, then ${}_RR$ is a faithful $l$-prime module since every domain is an $l$-prime ring.*

Throughout the remaining part of this section rings have unity and all modules are unital left modules.

For any module $M$, we define the Levitzki radical $\mathcal{L}(M)$ as $\mathcal{L}(0)$, i.e.,

$$\mathcal{L}(0) := \{m \in M, \text{ every } l\text{-system in } M \text{ which contains } m \text{ also contains } 0\}.$$

From Theorem 3.14, we have

$$\mathcal{L}(M) = \cap\{K \ : \ K \leq M, M/K \text{ is } l\text{-prime}\}$$

which is a radical by [13, Proposition 1] since $l$-prime modules are closed under taking non-zero submodules.

**Proposition 3.18.** *For any $R$-module $M$,*

(1) *$\mathcal{L}(\mathcal{L}(M)) = \mathcal{L}(M)$, i.e., $\mathcal{L}$ is idempotent;*
(2) *$\mathcal{L}(M)$ is a characteristic submodule of $M$;*
(3) *If $M$ is projective then $\mathcal{L}(R)M = \mathcal{L}(M)$.*

**Proof.** Follows from [5, Proposition 1.1.3]. □

**Proposition 3.19.** *For any $M \in R$-mod,*

(1) *if $M = \bigoplus_{\Lambda} M_\lambda$ is a direct sum of submodules $M_\lambda (\lambda \in \Lambda)$, then*

$$\mathcal{L}(M) = \bigoplus_{\Lambda} \mathcal{L}(M_\lambda)$$

(2) *if $M = \prod_{\Lambda} M_\lambda$ is a direct product of submodules $M_\lambda (\lambda \in \Lambda)$, then*

$$\mathcal{L}(M) \subseteq \prod_{\Lambda} \mathcal{L}(M_\lambda).$$

**Proof.** Follows from [5, Proposition 1.1.2]. □

## 4. The radicals $\mathcal{L}({}_RR)$ and $\mathcal{L}(R)$

**Lemma 4.1.** *For any associative ring $R$, $\mathcal{L}({}_RR) \subseteq \mathcal{L}(R)$.*

**Proof.** Let $x \in \mathcal{L}({}_RR)$ and $I$ be an $l$-prime ideal of $R$. From Proposition 3.15, we have $R/I$ is an $l$-prime $R$-module. Hence, $x \in I$ and we have $x \in \mathcal{L}(R)$, i.e., $\mathcal{L}({}_RR) \subseteq \mathcal{L}(R)$. □

**Remark 4.2.** *In general the containment in Lemma 4.1 is strict.*

**Example 4.3.** *Let $R = \{\begin{pmatrix} x & y \\ 0 & 0 \end{pmatrix} : x, y \in \mathbb{Z}_2\}$ and $M = {}_RR$. It is easy to check that $(0)$ is an $l$-prime submodule of ${}_RR$. Hence, $\mathcal{L}({}_RR) = 0$. Now, we have $(0 : R)_R$ is an $l$-prime ideal of $R$, $(0 : R)_R \neq (0)$. For if $b \neq 0$, $b \in \mathbb{Z}_2$, then $\begin{pmatrix} 0 & b \\ 0 & 0 \end{pmatrix} R = 0$. Hence, $\mathcal{L}(R) \subseteq (0 : R)_R$. But since $(0 : R)_R(0 : R)_R = 0 \subseteq \mathcal{L}(R)$ and $\mathcal{L}(R)$ is a semiprime ideal, we have $(0 : R)_R \subseteq \mathcal{L}(R)$. Hence, $\mathcal{L}(R) = (0 : R)_R \neq 0$.*

**Lemma 4.4.** *For any ring $R$ and any $R$-module $M$,*

$$\mathcal{L}(R) \subseteq (\mathcal{L}(M) : M).$$

**Proof.** We have $(\mathcal{L}(M) : M) = (\bigcap_{S \leq M} S : M) = \bigcap_{S \leq M} (S : M)$, where $S$ is an $l$-prime submodule of $M$. Since $(S : M)$ is an $l$-prime ideal of $R$ for each $l$-prime submodule $S$ of $M$, we get $\mathcal{L}(R) \subseteq (\mathcal{L}(M) : M)$. □

**Remark 4.5.** *In general, $\mathcal{L}(R)M \subset \mathcal{L}(M)$ even over a commutative ring. For let $R = \mathbb{Z}$ and $M = \mathbb{Z}_{p^\infty} \oplus \mathbb{Z}$. Since $R$ is commutative, $\mathcal{L}(M) = \beta(M)$. From* [4, Example 3.4]*, we have $\beta(M) = \mathbb{Z}_{p^\infty}$. But $\beta(R) = \mathcal{L}(R) = 0$. Hence, $0 = \mathcal{L}(R)M \subset \mathcal{L}(M) = \mathbb{Z}_{p^\infty}$.*

We recall that, the Jacobson radical $\text{Rad}(M)$ of a module $M$ is the intersection of all maximal submodules of $M$.

**Theorem 4.6.** *If $M$ is a module over a left Artinian ring $R$, then*

$$\beta(M) = \mathcal{L}(M) = \mathcal{U}(M) = Rad(M).$$

**Proof.** Since every maximal submodule is $s$-prime, we have

$$\mathcal{U}(M) \subseteq \text{Rad}(M) = \text{Jac}(R)M.$$

Since $R$ is left Artinian $\mathcal{L}(R) = \text{Jac}(R)$. Hence,

$$\mathcal{L}(M) \subseteq \mathcal{U}(M) \subseteq \text{Rad}(M) = \text{Jac}(R)M = \mathcal{L}(R)M \subseteq \mathcal{L}(M).$$

□

**Proposition 4.7.** *For any ring $R$, $\mathcal{L}(R) = (\mathcal{L}(_RR) : R)$.*

**Proof.** From Lemma 4.4, $\mathcal{L}(R) \subseteq (\mathcal{L}(_RR) : R)$. Since $\mathcal{L}(_RR) \subseteq \mathcal{L}(R)$ we have $\mathcal{L}(R) \subseteq (\mathcal{L}(_RR) : R) \subseteq (\mathcal{L}(R) : R)$. Let $x \in (\mathcal{L}(R) : R)$. Hence $xR \subseteq \mathcal{L}(R) = \bigcap_{\mathcal{P}\ l\text{-prime in } R} \mathcal{P} \subseteq \mathcal{P}$ for all $l$-prime ideals $\mathcal{P}$ of $R$. Since $xR \subseteq \mathcal{P}$ for $\mathcal{P}$ $l$-prime, we have $x \in \mathcal{P}$ and $x \in \mathcal{L}(R)$. Hence, $(\mathcal{L}(R) : R) \subseteq \mathcal{L}(R)$. □

**Proposition 4.8.** *For all $R$-modules $M$,*

(1) $\mathcal{L}(M) = \{x \in M \ : \ Rx \subseteq \mathcal{L}(M)\}$*;*
(2) *if* $\mathcal{L}(R) = R$*, then* $\mathcal{L}(M) = M$*.*

**Proof.** (1) Since $\mathcal{L}(M) \leq M$, we have $R\mathcal{L}(M) \subseteq \mathcal{L}(M)$. Conversely, let $x \in M$ with $Rx \subseteq \mathcal{L}(M)$. Hence $Rx \subseteq P$ for all $l$-prime submodules $P$ of $M$. Since $P$ is also a prime submodule, we have $x \in P$ and hence $x \in \mathcal{L}(M)$.

(2) $R = \mathcal{L}(R)$ gives $R \subseteq (\mathcal{L}(M) : M)$ from Lemma 4.4. Hence $RM \subseteq \mathcal{L}(M)$ and from (1), we have $M \subseteq \mathcal{L}(M)$, i.e., $M = \mathcal{L}(M)$. □

**Proposition 4.9.** *Let $R$ be any ring. Then, any of the following conditions implies $\mathcal{L}(R) = \mathcal{L}(_RR)$.*

(1) *$R$ is commutative;*
(2) *$x \in xR$ for all $x \in R$, e.g., if $R$ has an identity or $R$ is Von Neumann regular.*

**Proof.** (1) Since $R$ is commutative, it follows from Proposition 4.7 and Proposition 4.8 that $\mathcal{L}(R) \subseteq \mathcal{L}(_RR) \subseteq \mathcal{L}(R)$ and $\mathcal{L}(R) = \mathcal{L}(_RR)$.

(2) Let $x \in \mathcal{L}(R)$, then from Proposition 4.7, $xR \subseteq \mathcal{L}(_RR)$ and since $x \in xR$, we get $x \in \mathcal{L}(_RR)$ such that $\mathcal{L}(_RR) = \mathcal{L}(R)$. □

## 5. A special class of $l$-prime modules

A class $\rho$ of associative rings is called a special class if $\rho$ is hereditary, consists of prime rings and is closed under essential extensions, cf., [8, p.80]. Andrunakievich and Rjabuhin in [1] extended this notion to modules and showed that prime modules, irreducible modules, simple modules, modules without zero divisors, etc form special classes of modules. De La Rosa and Veldsman in [7] defined a weakly special class of modules. We follow the definition in [7] of a weakly special class of modules to define a special class of modules.

**Definition 5.1.** For a ring $R$, let $\mathcal{K}_R$ be a (possibly empty) class of $R$-modules. Let $\mathcal{K} = \cup\{\mathcal{K}_R \ : \ R \text{ a ring}\}$. $\mathcal{K}$ is a *special class of modules* if it satisfies:

**S1**. $M \in \mathcal{K}_R$ and $I \triangleleft R$ with $I \subseteq (0 : M)_R$ implies $M \in \mathcal{K}_{R/I}$.
**S2**. If $I \triangleleft R$ and $M \in \mathcal{K}_{R/I}$ , then $M \in \mathcal{K}_R$.
**S3**. $M \in \mathcal{K}_R$ and $I \triangleleft R$ with $IM \neq 0$ implies $M \in \mathcal{K}_I$.
**S4**. $M \in \mathcal{K}_R$ implies $RM \neq 0$ and $R/(0 : M)_R$ is a prime ring.
**S5**. If $I \triangleleft R$ and $M \in \mathcal{K}_I$, then there exists $N \in \mathcal{K}_R$ such that $(0 : N)_I \subseteq (0 : M)_I$.

**Remark 5.2.** *It is known that the class of all prime $R$-modules $M$ with $RM \neq 0$ is special hence satisfies the conditions S1 through S5.*

**Theorem 5.3.** *Let $R$ be any ring and*

$$\mathcal{M}_R := \{M \ : \ M \textit{ is an } l\textit{-prime } R\textit{-module with } RM \neq 0\}.$$

*If $\mathcal{M} = \cup \mathcal{M}_R$, then $\mathcal{M}$ is a special class of $R$-modules.*

**Proof. S1.** Let $M \in \mathcal{M}_R$ and $I \triangleleft R$ with $IM = 0$. $M$ is an $R/I$-module via $(r + I)m = rm$. Since $M \in \mathcal{M}_R$, $M$ is a prime $R$-module and $\mathcal{L}(R/(0 : M)_R) = 0$. Since $M$ is also a prime $R/I$-module we only need to show that $\mathcal{L}((R/I)/(0 : M)_{R/I}) = 0$. Because

$$(R/I)/(0 : M)_{R/I} = (R/I)/((0 : M)_R/I) \cong R/(0 : M)_R,$$

we have $\mathcal{L}((R/I)/(0 : M)_{R/I}) = 0$ and therefore $M \in \mathcal{M}_{R/I}$.

**S2.** Let $I \triangleleft R$ and $M \in \mathcal{M}_{R/I}$. Then $M$ is a prime $R/I$-module and $\mathcal{L}((R/I)/(0 : M)_{R/I}) = 0$. From

$$(R/I)/(0 : M)_{R/I} = (R/I)/((0 : M)_R/I) \cong R/(0 : M)_R,$$

we get $\mathcal{L}(R/(0 : M)_R) = 0$. Thus, $M \in \mathcal{M}_R$.

**S3.** Suppose $M \in \mathcal{M}_R$ and $I \triangleleft R$ with $IM \neq 0$. Then $M$ is a prime $R$-module and $\mathcal{L}(R/(0 : M)_R) = 0$. Since

$$I/(0 : M)_I = I/((0 : M)_R \cap I) \cong (I + (0 : M)_R)/(0 : M)_R \triangleleft R/(0 : M)_R$$

and a Levitzki semisimple class is hereditary, we have $\mathcal{L}(I/(0:M)_I) = 0$. Hence, $M \in \mathcal{M}_I$. Therefore, $M \in \mathcal{M}_I$.

**S4.** Let $M \in \mathcal{M}_R$. Hence $RM \neq 0$. Since $(0:M)_R$ is an $l$-prime ideal of $R$, $R/(0:M)_R$ is an $l$-prime ring and hence a prime ring.

**S5.** Let $I \triangleleft R$ and $M \in \mathcal{M}_I$. Since $M$ is an $l$-prime $I$-module, $(0:M)_I$ is an $l$-prime ideal of $I$. Now, $(0:M)_I \triangleleft I \triangleleft R$ and $I/(0:M)_R$ an $l$-prime ring implies $(0:M)_I \triangleleft R$. Choose $K/(0:M)_I \triangleleft R/(0:M)_I$ maximal with respect to $I/(0:M)_I \cap K/(0:M)_I = 0$. Then, $I/(0:M)_I \cong (I+K)/K \triangleleft \cdot R/K$ by [8, Lemma 3.2.5]. Since $I/(0:M)_I \triangleleft \cdot R/K$ and $I/(0:M)_I$ an $l$-prime ring $R/K$ is $l$-prime. Let $N = R/K$. $N$ is an $R$-module. Clearly, $RN \neq 0$. From Proposition 3.15, we have $(0:N)_R = K$. We show $(0:N)_I \subseteq (0:M)_I$. Let $x \in (0:N)_I$. Then $xR/K = 0$, i.e., $xR \subseteq K$. Now, $xR \subseteq I \cap K$ and from definition of $K/(0:M)_I$, we have $xR \subseteq I \cap K \subseteq (0:M)_I$. Hence $xRM = 0$ and since $xIM \subseteq xRM$ we have $xI \subseteq (0:M)_I$ and $(0:M)_I$ is a prime ideal of $I$ implies $x \in (0:M)_I$. Hence, $(0:N)_I \subseteq (0:M)_I$. □

**Proposition 5.4.** *If $\mathcal{M}_s$ is the special class of $l$-prime modules, then the special radical induced by $\mathcal{M}_s$ on a ring $R$ is $\mathcal{L}$.*

**Proof.** Let $R$ be a ring. From Proposition 3.15, we have $\mathcal{L}(R) = \cap\{(0:M)_R : M \text{ is an } l\text{-prime } R\text{-module}\} = \cap\{I \triangleleft R : I \text{ is an } l\text{-prime ideal}\}$. □

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