COMPLETE HOMOMORPHISMS BETWEEN MODULE LATTICES

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For my good friend John Clark on his 70th birthday

Abstract. We examine the properties of certain mappings between the lattice $\mathcal{L}(R)$ of ideals of a commutative ring $R$ and the lattice $\mathcal{L}(R M)$ of submodules of an $R$-module $M$, in particular considering when these mappings are complete homomorphisms of the lattices. We prove that the mapping $\lambda$ from $\mathcal{L}(R)$ to $\mathcal{L}(R M)$ defined by $\lambda(B) = BM$ for every ideal $B$ of $R$ is a complete homomorphism if $M$ is a faithful multiplication module. A ring $R$ is semiperfect (respectively, a finite direct sum of chain rings) if and only if this mapping $\lambda: \mathcal{L}(R) \to \mathcal{L}(R M)$ is a complete homomorphism for every simple (respectively, cyclic) $R$-module $M$. A Noetherian ring $R$ is an Artinian principal ideal ring if and only if, for every $R$-module $M$, the mapping $\lambda: \mathcal{L}(R) \to \mathcal{L}(R M)$ is a complete homomorphism.

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1. Introduction

In this paper we continue the discussion in [7] concerning mappings, in particular homomorphisms, between the lattice of ideals of a commutative ring and the lattice of submodules of a module over that ring.

A lattice $L$ is called complete provided every non-empty subset $S$ has a least upper bound $\vee S$ and a greatest lower bound $\wedge S$. Given complete lattices $L$ and $L'$ we say that a mapping $\varphi: L \to L'$ is a complete homomorphism provided

$$\varphi(\vee S) = \vee \{\varphi(x) : x \in S\} \text{ and } \varphi(\wedge S) = \wedge \{\varphi(x) : x \in S\},$$

for every non-empty subset $S$ of $L$. A complete homomorphism which is a bijection (respectively, injection, surjection) will be called a complete isomorphism (respectively, complete monomorphism, complete epimorphism). The first result is standard and easy to prove.
Lemma 1.1. The following statements are equivalent for a bijection \( \varphi \) from a complete lattice \( L \) to a complete lattice \( L' \).

(i) \( \varphi \) is a complete isomorphism.

(ii) \( \varphi(\vee S) = \vee\{\varphi(x) : x \in S\} \) for every non-empty subset \( S \) of \( L \).

(iii) \( \varphi(\wedge S) = \wedge\{\varphi(x) : x \in S\} \) for every non-empty subset \( S \) of \( L \).

Moreover, in this case the inverse mapping \( \varphi^{-1} : L' \rightarrow L \) is also a complete isomorphism.

An element \( x \) of a complete lattice \( L \) is called compact in case whenever \( x \leq \vee S \), for some non-empty subset \( S \) of \( L \), there exists a finite subset \( F \) of \( S \) such that \( x \leq \vee F \). The next result is also easy to prove.

Lemma 1.2. Let \( \varphi : L \rightarrow L' \) be a complete isomorphism from a complete lattice \( L \) to a complete lattice \( L' \) and let \( x \) be a compact element of \( L \). Then \( \varphi(x) \) is a compact element of \( L' \).

A lattice \( L \) is called distributive in case

\[
x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z),
\]

for all elements \( x, y, z \) in \( L \). The next result is also well known and easy to prove. It states that a lattice is distributive if and only if its dual lattice is distributive.

Lemma 1.3. A lattice \( L \) is distributive if and only if \( x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z) \) for all \( x, y, z \) in \( L \).

Throughout this note all rings will be commutative with identity and all modules will be unital. Let \( R \) be a ring and \( M \) be any \( R \)-module. Let \( \mathcal{L}(R) \) denote the lattice of all ideals of the ring \( R \) and let \( \mathcal{L}(R M) \) denote the lattice of all submodules of the \( R \)-module \( M \). In [7] we investigate the mapping \( \lambda : \mathcal{L}(R) \rightarrow \mathcal{L}(R M) \) defined by \( \lambda(B) = BM \) for every ideal \( B \) of \( R \) and the mapping \( \mu : \mathcal{L}(R M) \rightarrow \mathcal{L}(R) \) defined by \( \mu(N) = (N :_R M) \) for every submodule \( N \) of \( M \), where \( (N :_R M) \) denotes the set of elements \( r \in R \) such that \( rM \subseteq N \). The module \( M \) is called a \( \lambda \)-module in [7] in case \( \lambda : \mathcal{L}(R) \rightarrow \mathcal{L}(R M) \) is a homomorphism. Similarly, in [7] the module \( M \) is called a \( \mu \)-module if the above mapping \( \mu \) is a homomorphism. For any unexplained terminology and notation, please see [7].

Note that the lattice \( \mathcal{L}(R M) \) is complete when we define

\[
\wedge S = \cap_{N \in S} N \quad \text{and} \quad \vee S = \sum_{N \in S} N,
\]
for every non-empty collection $S$ of submodules of $M$. In particular the lattice $\mathcal{L}(R)$ is complete. The module $M$ will be called $\lambda$-complete in case the above mapping $\lambda : \mathcal{L}(R) \to \mathcal{L}(R M)$ is a complete homomorphism. Similarly the module $M$ will be called $\mu$-complete if $\mu : \mathcal{L}(R M) \to \mathcal{L}(R)$ is a complete homomorphism. It is clear that every $\lambda$-complete module is a $\lambda$-module and every $\mu$-complete module is a $\mu$-module but, in each case, the converse is false in general, as we can easily show.

For example, let $\mathbb{Z}$ denote the ring of rational integers and let $p$ be any prime in $\mathbb{Z}$. Then the simple $\mathbb{Z}$-module $U = \mathbb{Z}/\mathbb{Z}p$ is a $\lambda$-module. Let $q$ be any prime in $\mathbb{Z}$ other than $p$ and let $S$ denote the collection of ideals of $\mathbb{Z}$ of the form $\mathbb{Z}q^n$ for all positive integers $n$. Then

$$\lambda(\wedge S) = \lambda(\cap_{n \geq 1} \mathbb{Z}q^n) = \lambda(0) = 0,$$

but

$$\wedge\{\lambda(B) : B \in S\} = \cap_{n \geq 1} q^n U = U.$$

Thus $U$ is not $\lambda$-complete.

Now let $\mathbb{Z}(p^\infty)$ denote the Prüfer $p$-group for any prime $p$ in $\mathbb{Z}$. Let $V = \mathbb{Z}(p^\infty)$. Then the $\mathbb{Z}$-module $V$ is a $\mu$-module (see [7, Example 3.11]). However $V$ contains an infinite collection $T$ of proper submodules $V_i (i \in I)$ such that $V = \cup_{i \in I} V_i$. Thus

$$\mu(\vee T) = \mu(V) = (V : \mathbb{Z} V) = \mathbb{Z},$$

but

$$\vee\{\mu(W) : W \in T\} = \sum_{i \in I} \mu(V_i) = \sum_{i \in I} (V_i : \mathbb{Z} V) = 0.$$

Thus the $\mathbb{Z}$-module $V$ is not $\mu$-complete.

**Proposition 1.4.** Given any ring $R$ and $R$-module $M$ the following statements are equivalent.

(i) The mapping $\lambda : \mathcal{L}(R) \to \mathcal{L}(R M)$ is a complete isomorphism.

(ii) The mapping $\mu : \mathcal{L}(R M) \to \mathcal{L}(R)$ is a complete isomorphism.

Moreover, in this case $M$ is a faithful $R$-module.

**Proof.** (i) $\Leftrightarrow$ (ii) By Lemma 1.1 and [7, Corollary 1.5].

Now suppose that (i) holds. Let $A = \text{ann}_R(M)$. Then $\lambda(A) = AM = 0 = 0M = \lambda(0)$ so that $A = 0$ and $M$ is faithful. $\square$

Again let $R$ be a ring and let $M$ be an $R$-module. Let $A = \text{ann}_R(M)$. By defining

$$(r + A)m = rm \quad (r \in R, m \in M),$$

$M$ becomes a faithful $(R/A)$-module with the property that a subset $X$ of $M$ is an $R$-submodule of $M$ if and only if $X$ is an $(R/A)$-submodule of $M$. Thus the lattice $L_{(R/A)M}$ is identical to the lattice $L_{(R/A)M}$. The mapping $\lambda : L(R/A) \rightarrow L(R/A)M$ will be denoted by $\overline{\lambda}$. Note that if $\overline{B}$ is any ideal of the ring $R/A$ then $\overline{B} = B/A$ for a unique ideal $B$ of $R$ containing $A$ and hence

$$\overline{\lambda}(\overline{B}) = \overline{\lambda}(B/A) = (B/A)M = BM.$$ 

In addition, the mapping $\mu : L_{(R/A)M} \rightarrow L(R/A)$ is denoted by $\overline{\mu}$ so that

$$\overline{\mu}(N) = (N :_{R/A} M) = (N :_{R} M)/A,$$

for every submodule $N$ of $M$, noting that, of course, $A \subseteq (N :_{R} M)$ for every submodule $N$ of $M$.

Let $R$ be any ring. An $R$-module $M$ is called a multiplication module in case for each submodule $N$ of $M$ there exists an ideal $B$ of $R$ such that $N = BM$. Cyclic modules are multiplication modules as are projective ideals of $R$ or ideals of $R$ generated by idempotent elements (see [2]). We prove that for any ring $R$ an $R$-module $M$ is $\mu$-complete if and only if $M$ is a finitely generated multiplication module (Theorem 2.2). An easy consequence is that the mapping $\mu$ (respectively, $\lambda$) is a complete isomorphism if and only if $M$ is a finitely generated faithful multiplication module (Corollary 2.4).

For any ring $R$, projective modules are $\lambda$-complete (Corollary 3.4) as are faithful multiplication modules (Theorem 3.6). We prove that a ring $R$ is arithmetical if and only if every $R$-module is a $\lambda$-module (Theorem 4.6). The ring $R$ is semiperfect if and only if every simple $R$-module is $\lambda$-complete (Theorem 4.2). On the other hand, $R$ is a direct sum of chain rings if and only if every cyclic $R$-module $M$ is $\lambda$-complete (Theorem 4.7). Note that we do not yet know which rings $R$ have the property that every $R$-module is $\lambda$-complete. It is proved that a Noetherian ring $R$ is an Artinian principal ideal ring if and only if every $R$-module is $\lambda$-complete (Theorem 4.12).

2. $\mu$-complete modules

Let $R$ be a ring and let $M$ be an $R$-module. In this section we shall investigate $\mu$-complete modules. We begin with the following basic result.

**Lemma 2.1.** Given any ring $R$, an $R$-module $M$ is $\mu$-complete if and only if

$$(\sum_{N \in T} N :_{R} M) = \sum_{N \in T} (N :_{R} M)$$

for any non-empty collection $T$ of submodules of $M$. 
Proof. Let \( \mathcal{T} \) be any non-empty collection of submodules of \( M \). Then
\[
\mu(\wedge \mathcal{T}) = \mu(\cap_{N \in \mathcal{T}} N) = (\cap_{N \in \mathcal{T}} N :_R M) = \wedge \{ \mu(N) : N \in \mathcal{T} \}.
\]
On the other hand
\[
\mu(\vee \mathcal{T}) = \mu(\sum_{N \in \mathcal{T}} N) = (\sum_{N \in \mathcal{T}} N :_R M),
\]
and
\[
\vee \{ \mu(N) : N \in \mathcal{T} \} = \sum_{N \in \mathcal{T}} (N :_R M).
\]
The result follows. \( \square \)

Note that, given any ring \( R \) and \( R \)-module \( M \), the mapping \( \mu \) is not a surjection in case \( M \) is not a faithful \( R \)-module because in this case no submodule \( N \) of \( M \) has the property that \( (N :_R M) = 0 \). The next result characterizes \( \mu \)-complete modules.

**Theorem 2.2.** Given any ring \( R \), the following statements are equivalent for an \( R \)-module \( M \) with annihilator \( A \) in \( R \).

(i) \( M \) is \( \mu \)-complete.

(ii) \( M \) is a finitely generated multiplication module.

(iii) The mapping \( \overline{\mu} : \mathcal{L}(R/A M) \to \mathcal{L}(R/A) \) is a complete isomorphism.

(iv) The mapping \( \overline{\lambda} : \mathcal{L}(R/A) \to \mathcal{L}(R/AM) \) is a complete isomorphism.

Moreover in this case the mapping \( \mu : \mathcal{L}(RM) \to \mathcal{L}(R) \) is a monomorphism.

**Proof.** (i) \( \Rightarrow \) (ii) Let \( \mathcal{T} \) denote the collection of all cyclic submodules of the \( \mu \)-complete module \( M \). Then \( M = \sum_{N \in \mathcal{T}} N \). By Lemma 2.1,
\[
R = (M :_R M) = (\sum_{N \in \mathcal{T}} N :_R M) = \sum_{N \in \mathcal{T}} (N :_R M),
\]
and hence \( R = (Rm_1 :_R M) + \cdots + (Rm_n :_R M) \) for some positive integer \( n \) and elements \( m_i \in M \) (1 \( \leq \) \( i \) \( \leq \) \( n \)). It follows that
\[
M = RM = (Rm_1 :_R M)M + \cdots + (Rm_n :_R M)M \subseteq Rm_1 + \cdots + Rm_n \subseteq M.
\]
Therefore \( M = Rm_1 + \cdots + Rm_n \). In other words, \( M \) is finitely generated. By [7, Theorem 3.8], \( M \) is also a multiplication module.

(ii) \( \Rightarrow \) (i) Suppose that \( M \) is a finitely generated multiplication module. By [7, Lemma 3.1 and Theorem 3.8] and induction,
\[
(K_1 + \cdots + K_n :_R M) = (K_1 :_R M) + \cdots + (K_n :_R M),
\]
for every positive integer $n$ and submodules $K_i \ (1 \leq i \leq n)$. Let $L_i \ (i \in I)$ be any non-empty collection of submodules of $M$. Clearly,

$$\sum_{i \in I} (L_i :_R M) \subseteq (\sum_{i \in I} L_i :_R M).$$

Let $r \in (\sum_{i \in I} L_i :_R M)$. Then $rM$ is a finitely generated submodule of $\sum_{i \in I} L_i$. There exists a finite subset $I'$ of $I$ such that $rM \subseteq \sum_{i \in I'} L_i$. Hence

$$r \in (\sum_{i \in I'} L_i :_R M) = \sum_{i \in I'} (L_i :_R M) \subseteq \sum_{i \in I} (L_i :_R M) = (\sum_{i \in I} L_i :_R M).$$

Thus $(\sum_{i \in I} L_i :_R M) \subseteq \sum_{i \in I'} (L_i :_R M)$ and we have proved that $(\sum_{i \in I} L_i :_R M) = \sum_{i \in I} (L_i :_R M)$. By Lemma 2.1, $M$ is $\mu$-complete.

(ii) $\Rightarrow$ (iii) By [7, Lemma 2.9], the $(R/A)$-module $M$ is a finitely generated faithful multiplication module and hence the mapping $\overline{\mu}$ is a bijection by [7, Theorem 4.3].

By the proof of (ii) $\Rightarrow$ (i), the mapping $\overline{\mu}$ is a complete isomorphism.

(iii) $\Leftrightarrow$ (iv) By Proposition 1.4.

(iii) $\Rightarrow$ (ii) By the proof of (i) $\Rightarrow$ (ii), the $(R/A)$-module $M$ is a finitely generated multiplication module and hence the $R$-module $M$ is a finitely generated multiplication module by [7, Lemma 2.9].

Finally, suppose that there exist submodules $N$ and $L$ of $M$ such that $\mu(N) = \mu(L)$. By [2, p. 756],

$$N = (N :_R M)M = \mu(N)M = \mu(L)M = (L :_R M)M = L.$$  

Thus $\mu$ is a monomorphism. $\square$

Given a ring $R$ and an $R$-module $M$, note that Theorem 2.2 shows that whenever the mapping $\mu : \mathcal{L}(R)M \to \mathcal{L}(R)$ is a complete homomorphism then it is a monomorphism. This is not true if $\mu$ is merely a homomorphism (see, for example, [7, Example 3.11 and Proposition 3.12]).

**Corollary 2.3.** Every homomorphic image of a $\mu$-complete module $M$ is $\mu$-complete.

**Proof.** By Theorem 2.2. $\square$

In contrast to Corollary 2.3 homomorphic images of $\lambda$-complete modules need not be $\lambda$-complete. For example, the $\mathbb{Z}$-module $\mathbb{Z}$ is $\lambda$-complete but we have already noted that the simple $\mathbb{Z}$-module $\mathbb{Z}/2p$ is not $\lambda$-complete for every prime $p$ in $\mathbb{Z}$. (Note that every homomorphic image of a $\lambda$-module over the ring $\mathbb{Z}$ is also a $\lambda$-module by [7, Theorem 2.3]).

**Corollary 2.4.** Given a ring $R$, the following statements are equivalent for an $R$-module $M$.
(i) The mapping $\lambda : \mathcal{L}(R) \rightarrow \mathcal{L}(R M)$ is a complete isomorphism.
(ii) The mapping $\mu : \mathcal{L}(R M) \rightarrow \mathcal{L}(R)$ is a complete isomorphism.
(iii) The $R$-module $M$ is a finitely generated faithful multiplication module.

**Proof.** By Proposition 1.4 and Theorem 2.2.

**Corollary 2.5.** Let $R$ be a ring and let $M$ be any $\mu$-complete $R$-module with $A = \text{ann}_R(M)$. Then the $(R/A)$-module $M$ is a $\lambda$-complete module.

**Proof.** By [7, Lemma 2.9], Theorem 2.2 and Corollary 2.4.

Note that in general $\mu$-complete modules are not $\lambda$-complete. For, let $R$ be a domain that is not Prüfer. By [7, Theorem 2.3], there exists a cyclic $R$-module $M$ which is not a $\lambda$-module and hence is not $\lambda$-complete. However, every cyclic module over any ring is a finitely generated multiplication module.

3. $\lambda$-complete modules

In contrast to the case of $\mu$-complete modules, the situation for (non-faithful) $\lambda$-complete modules is more complex. We already know that simple modules over $\mathbb{Z}$ are not $\lambda$-complete although they are clearly finitely generated multiplication modules. First we prove an elementary result characterizing $\lambda$-complete modules.

**Lemma 3.1.** Let $R$ be a ring. Then an $R$-module $M$ is $\lambda$-complete if and only if
\[ \lambda(\bigvee S) = \bigvee \{ \lambda(B) : B \in S \} \]
\[ \lambda(\bigwedge S) = \bigwedge \{ \lambda(B) : B \in S \} \]
for every non-empty collection $S$ of ideals of $R$.

**Proof.** Let $S$ be any non-empty collection of ideals of $R$. Then
\[ \lambda(\bigvee S) = (\bigvee_{B \in S} B)M = \sum_{B \in S} (BM) = \bigvee \{ \lambda(B) : B \in S \}. \]
In addition, \( \lambda(\bigwedge S) = (\bigwedge_{B \in S} B)M \) and $\bigwedge \{ \lambda(B) : B \in S \} = \bigwedge_{B \in S} (BM)$. The result follows.

**Corollary 3.2.** Let $A$ be any ideal of a ring $R$. Then the $R$-module $R/A$ is $\lambda$-complete if and only if $\bigcap_{B \in S} (A + B) = A + \bigcap_{B \in S} B$ for every non-empty collection $S$ of ideals of $R$.

**Proof.** Apply Lemma 3.1 to the module $M = R/A$.

**Lemma 3.3.** Let $R$ be any ring. Then
(a) Every direct summand of a $\lambda$-complete module is $\lambda$-complete.
(b) Every direct sum of $\lambda$-complete modules is also $\lambda$-complete.
By Lemma 3.1 \( K \) is a \( \lambda \)-complete module.

(b) Let \( L_i (i \in I) \) be any collection of \( \lambda \)-complete modules and let \( L = \oplus_{i \in I} L_i \).

Given any non-empty collection \( S \) of ideals of \( R \) we have:

\[
(\bigcap_{B \in S} B)L = \oplus_{i \in I} (\bigcap_{B \in S} B)L_i = \oplus_{i \in I} (\bigcap_{B \in S} (BL_i)) = \bigcap_{B \in S} (BL).
\]

By Lemma 3.1 \( L \) is \( \lambda \)-complete.

\[ \square \]

**Corollary 3.4.** Given any ring \( R \), every projective \( R \)-module is \( \lambda \)-complete.

\[ \square \]

**Proof.** Clearly the \( R \)-module \( R \) is \( \lambda \)-complete. Apply Lemma 3.3.

Recall the following result (see [2, Theorem 1.2] or [7, Lemma 2.10]).

**Lemma 3.5.** Let \( R \) be any ring. Then an \( R \)-module \( M \) is a multiplication module if and only if for each maximal ideal \( P \) of \( R \) either

(a) for each \( m \) in \( M \) there exists \( p \) in \( P \) such that \( (1 - p)m = 0 \), or

(b) there exist \( x \) in \( M \) and \( q \) in \( P \) such that \( (1 - q)M \subseteq Rx \).

We now strengthen [7, Theorem 2.12].

**Theorem 3.6.** Let \( R \) be any ring. Then every faithful multiplication \( R \)-module is a \( \lambda \)-complete module.

**Proof.** Let \( M \) be a faithful multiplication \( R \)-module. Let \( S \) be any non-empty collection of ideals of \( R \). Then \( (\bigcap_{B \in S} B)M \subseteq \bigcap_{B \in S} (BM) \). Suppose that there exists \( m \in \bigcap_{B \in S} (BM) \) with \( m \notin (\bigcap_{B \in S} B)M \). Let \( I = \{ r \in R : rm \in (\bigcap_{B \in S} B)M \} \). Then \( I \) is a proper ideal of \( R \). Let \( P \) be a maximal ideal of \( R \) such that \( I \subseteq P \).

Clearly \( (1 - p)m = 0 \) for some \( p \in P \) implies that \( 1 - p \in I \), a contradiction. By Lemma 3.5 there exist \( x \in M \) and \( q \in P \) such that \( (1 - q)M \subseteq Rx \). Note that for each ideal \( B \) in \( S \) \((1 - q)m \in (1 - q)BM = B(1 - q)M \subseteq Bx \). Thus \( (1 - q)m = r_Bx \) for some \( r_B \in B \) for each ideal \( B \) in \( S \). If \( B \) and \( C \) are ideals in \( S \) then \( (r_B - r_C)x = 0 \) and hence \( (1 - q)(r_B - r_C)M = (r_B - r_C)(1 - q)M \subseteq (r_B - r_C)Rx = 0 \). Because \( M \) is faithful we have \( (1 - q)(r_B - r_C) = 0 \) and \( (1 - q)r_B = (1 - q)r_C \). It follows that \( (1 - q)r_C \in \bigcap_{B \in S} B \). Thus \( (1 - q)^2m = (1 - q)r_Cx \in (\bigcap_{B \in S} B)M \). This implies that \( (1 - q)^2 \in I \subseteq P \), a contradiction. Thus \( \bigcap_{B \in S} (BM) = (\bigcap_{B \in S} B)M \) for every non-empty subset \( S \) of ideals of \( R \). By Lemma 3.1 \( M \) is \( \lambda \)-complete. \( \square \)
We have already noted that for any prime \( p \) in \( \mathbb{Z} \), the simple \( \mathbb{Z} \)-module \( \mathbb{Z}/\mathbb{Z}p \) is a multiplication module which is not \( \lambda \)-complete. Thus Theorem 3.6 requires that the module be faithful as well as a multiplication module.

If \( R \) is any ring and \( M \) the free \( R \)-module \( R \oplus R \), then it is not hard to check that the mapping \( \lambda : \mathcal{L}(R) \to \mathcal{L}(RM) \) is a complete monomorphism which is not an epimorphism. On the other hand, compare the following result with Theorem 2.2.

**Proposition 3.7.** Let \( R \) be a ring and let \( I \) be a proper ideal of \( R \) which is generated by idempotent elements such that \( \text{ann}_R(I) = 0 \). Then the \( R \)-module \( I \) is a faithful multiplication module and the mapping \( \lambda : \mathcal{L}(R) \to \mathcal{L}(RI) \) is a complete epimorphism but not a monomorphism.

**Proof.** By [7, Proposition 2.15] and Theorem 3.6. \( \square \)

### 4. Special rings

Let \( R \) be any ring. Then every cyclic \( R \)-module is \( \mu \)-complete by Theorem 2.2. However, the same theorem shows that the 2-generated \( R \)-module \( M = R \oplus R \) is not \( \mu \)-complete because \( M \) is not a multiplication module. Thus no non-zero ring \( R \) has the property that every finitely generated \( R \)-module is \( \mu \)-complete. We saw in Corollary 3.4 that for every ring \( R \) every projective \( R \)-module is \( \lambda \)-complete. In addition for every ring \( R \), every faithful multiplication module is \( \lambda \)-complete by Theorem 3.6. In this section we investigate rings \( R \) with the property that every module in a certain class of \( R \)-modules is \( \lambda \)-complete. The classes that we shall look at are the classes of simple \( R \)-modules, semisimple \( R \)-modules, cyclic \( R \)-modules, finitely generated \( R \)-modules and all \( R \)-modules.

First we investigate when simple modules are \( \lambda \)-complete. Following [1, p. 303] we call a ring \( R \) with Jacobson radical \( J \) a *semiperfect ring* in case \( R/J \) is semiprime Artinian and idempotents lift modulo \( J \). For properties of semiperfect rings see [1, Theorem 27.6] or [10, Theorem 42.6]. By a *local* ring we mean any (commutative) ring which contains only one maximal ideal. It is well known that a (commutative) ring \( R \) is semiperfect if and only if \( R \) is the (finite) direct sum of local rings (see, for example, [1, Theorem 27.6]). Given any ring \( R \), a submodule \( N \) of an \( R \)-module \( M \) *has a supplement \( K \) in case \( K \) is a submodule of \( M \) minimal with respect to the property that \( M = N + K \).

**Lemma 4.1.** Let \( R \) be a ring and let \( U \) be a simple \( R \)-module with annihilator \( P \). Then the \( R \)-module \( U \) is \( \lambda \)-complete if and only if \( P \) has a supplement in \( R \mathcal{R} \).
**Proof.** Suppose first that $U$ is $\lambda$-complete. Let $S$ denote the collection of ideals $B$ of $R$ such that $R = P + B$. By Corollary 3.2 $R = P + C$ where $C = \bigcap_{B \in S} B$. Clearly $C$ is a supplement of $P$ in $R$. Conversely, suppose that $P$ has a supplement $G$ in $R$. Let $T$ be any non-empty collection of ideals of $R$. Then

$$P + \left( \bigcap_{D \in T} D \right) = P = \bigcap_{D \in T} (P + D),$$

unless $D \not\subseteq P$ for all $D \in T$. Now suppose that $D \not\subseteq P$ for all $D \in T$. Let $D \in T$. Then $R = P + G = P + D$ implies that $R = P + (D \cap G)$ and hence $G = D \cap G \subseteq D$. It follows that

$$R = P + G \subseteq P + \left( \bigcap_{D \in T} D \right) \subseteq \bigcap_{D \in T} (P + D) \subseteq R.$$

Thus in any case $P + \left( \bigcap_{D \in T} D \right) = \bigcap_{D \in T} (P + D)$. By Corollary 3.2, the $R$-module $U$ is $\lambda$-complete.

**Theorem 4.2.** The following statements are equivalent for a ring $R$.

(i) Every semisimple $R$-module is $\lambda$-complete.

(ii) Every simple $R$-module is $\lambda$-complete.

(iii) The ring $R$ is semiperfect.

**Proof.** (i) $\Rightarrow$ (ii) Clear.

(ii) $\Rightarrow$ (iii) By Lemma 4.1 and [10, Theorem 42.6].

(iii) $\Rightarrow$ (i) By Lemma 4.1 and [10, Theorem 42.6] every simple $R$-module is $\lambda$-complete and by Lemma 3.3 every semisimple $R$-module is $\lambda$-complete.

Next we investigate rings $R$ with the property that every cyclic $R$-module is $\lambda$-complete. First we recall a result of Stephenson (see [9, Theorem 1.6]).

**Lemma 4.3.** The following statements are equivalent for a module $M$ over a ring $R$.

(i) The lattice $\mathcal{L}(R M)$ is distributive (i.e. $L \cap (K + N) = (L \cap K) + (L \cap N)$ for all submodules $K, L, N$ of $M$).

(ii) $K + \left( L \cap N \right) = (K + L) \cap (K + N)$ for all submodules $K, L, N$ of $M$.

(iii) $R = (Rx :_R Ry) + (Ry :_R Rx)$ for all $x, y \in M$.

**Corollary 4.4.** The following statements are equivalent for a module $M$ over a ring $R$.

(i) The lattice $\mathcal{L}(R M)$ is distributive.

(ii) Every finitely generated submodule of $M$ is a $\mu$-module.

(iii) Every 2-generated submodule of $M$ is a $\mu$-module.
(iv) \( R = (N :_R L) + (L :_R N) \) for all finitely generated submodules \( N \) and \( L \) of \( M \).

(v) Every finitely generated submodule of \( M \) is a multiplication module.

**Proof.** By Lemma 4.3 and [7, Corollary 3.9]. □

The next result is [7, Lemma 2.1].

**Lemma 4.5.** An \( R \)-module \( M \) is a \( \lambda \)-module if and only if \( (B \cap C)M = BM \cap CM \) for all (finitely generated) ideals \( B \) and \( C \) of \( R \).

We can now generalize [7, Theorems 2.3 and 3.13]. Recall that a ring \( R \) is called a **chain ring** in case the ideals of \( R \) form a chain, that is, for any ideals \( B \) and \( C \) of \( R \) either \( B \subseteq C \) or \( C \subseteq B \). For any ring \( R \) and prime ideal \( P \) of \( R \) the localization of \( R \) at \( P \) will be denoted by \( R_P \) as usual. (See [6, Chapter 5] for a good account of localization.) In 1949 Fuchs [3] called a ring \( R \) **arithmetical** provided the lattice \( \mathcal{L}(R) \) is distributive and Jensen [4, Lemma 1] showed that a ring \( R \) is arithmetical if and only if the local ring \( R_P \) is a chain ring for every prime ideal \( P \) of \( R \).

**Theorem 4.6.** The following statements are equivalent for a ring \( R \).

(i) \( R \) is an arithmetical ring.

(ii) Every \( R \)-module is a \( \lambda \)-module.

(iii) Every homomorphic image of a \( \lambda \)-module is a \( \lambda \)-module.

(iv) Every cyclic \( R \)-module is a \( \lambda \)-module.

(v) Every finitely generated ideal of \( R \) is a multiplication \( R \)-module.

(vi) Every finitely generated ideal of \( R \) is a \( \mu \)-module over the ring \( R \).

**Proof.** (i) ⇒ (ii) Let \( B \) and \( C \) be any finitely generated ideals of \( R \). By Corollary 4.4, \( R = (B :_R C) + (C :_R B) \). Then

\[
BM \cap CM = [(B :_R C) + (C :_R B)](BM \cap CM) \\
\subseteq (B :_R C)CM + (C :_R B)BM \subseteq (B \cap C)M.
\]

It follows that \( BM \cap CM = (B \cap C)M \). By Lemma 4.5 the \( R \)-module \( M \) is a \( \lambda \)-module.

(ii) ⇒ (iii) Clear.

(iii) ⇒ (iv) Because \( R \) is a \( \lambda \)-module.

(iv) ⇒ (i) Let \( A, B \) and \( C \) be any ideals of \( R \). Then the cyclic \( R \)-module \( R/A \) being a \( \lambda \)-module implies, by Lemma 4.5, \( (B \cap C)(R/A) = (B(R/A)) \cap (C(R/A)) \) and hence \( ((B \cap C) + A)/A = ((B + A)/A) \cap ((C + A)/A) \). It follows that \( (A + B) \cap (A + C) = A + (B \cap C) \). By Lemma 4.3, \( R \) is an arithmetical ring.
Theorem 4.6 applies to Prüfer domains because every finitely generated ideal is invertible and hence a multiplication module. More generally, if $R$ is a semihereditary ring (that is, every finitely generated ideal of $R$ is a projective $R$-module), then every finitely generated ideal of $R$ is a multiplication module by [8, Theorem 1] and hence Theorem 4.6 applies to $R$. It also applies to von Neumann regular rings because every ideal of such a ring is generated by idempotent elements and hence is a multiplication module (see [2, Corollary 1.3]).

**Corollary 4.7.** The following statements are equivalent for a ring $R$.

(i) Every cyclic $R$-module is $\lambda$-complete.

(ii) The ring $R = R_1 \oplus \cdots \oplus R_n$ is the direct sum of chain rings $R_i (1 \leq i \leq n)$ for some positive integer $n$.

**Proof.** (i) $\Rightarrow$ (ii) By Theorem 4.2 and [1, Theorem 27.6], the ring $R = R_1 \oplus \cdots \oplus R_n$ is the direct sum of local rings $R_i (1 \leq i \leq n)$ for some positive integer $n$. By Theorem 4.6 and [4, Lemma 1], $R_i$ is a chain ring for all $1 \leq i \leq n$.

(ii) $\Rightarrow$ (i) Without loss of generality we can suppose that $R$ is a chain ring. Let $A$ be any ideal of the chain ring $R$ and let $S$ be any non-empty collection of ideals of $R$. Then $A \subseteq \bigcap_{B \in S} B$ or $\bigcap_{B \in S} B \subseteq A$. Suppose first that $A \subseteq \bigcap_{B \in S} B$. Then

$$A + \left(\bigcap_{B \in S} B\right) = \bigcap_{B \in S} B = \bigcap_{B \in S} (A + B).$$

Now suppose that $\bigcap_{B \in S} B \subset A$. Then there exists an ideal $C$ in $S$ such that $A \not\subseteq C$ and hence $C \subseteq A$ because $R$ is a chain ring. In this case, it is easy to see that

$$A + \left(\bigcap_{B \in S} B\right) = A = \bigcap_{B \in S} (A + B).$$

In any case, we have proved that $A + \left(\bigcap_{B \in S} B\right) = \bigcap_{B \in S} (A + B)$. By Corollary 3.2 every cyclic $R$-module is $\lambda$-complete, as required.

Now we consider finitely generated modules and ask the question: Which rings $R$ have the property that every finitely generated module is $\lambda$-complete? Are these precisely the rings for which every cyclic module is $\lambda$-complete? This amounts to asking whether chain rings $R$ have the property that every finitely generated $R$-module is $\lambda$-complete. Some chain rings do have this property. Contrast the following result with Theorem 4.6.

**Theorem 4.8.** Let $R$ be a local principal ideal domain. Then $R$ is a chain ring such that every finitely generated $R$-module is $\lambda$-complete but no non-zero injective $R$-module is $\lambda$-complete.
Proof. It is well known that if \( P \) is the unique maximal ideal of \( R \) then the only ideals of \( R \) are the ideals \( R, P^n \ (n \geq 1) \) and \( 0 = \cap_{n \geq 1} P^n \). Thus \( R \) is a chain ring. Let \( M \) be any finitely generated \( R \)-module. Then \( M \) is a finite direct sum of cyclic \( R \)-modules (see, for example, [6, Theorem 10.30]) and hence \( M \) is \( \lambda \)-complete by Theorem 4.7 and Lemma 3.3. Now let \( X \) be any non-zero injective \( R \)-module. By [5, Proposition 2.6] and [6, Corollary 8.27],

\[
\cap_{n \geq 1}(P^nX) = X \neq 0 = (\cap_{n \geq 1} P^n)X.
\]

Thus \( X \) is not \( \lambda \)-complete by Lemma 3.1. □

Finally in this section we consider rings \( R \) with the property that every \( R \)-module is \( \lambda \)-complete. Note first the following simple fact which can be contrasted with Corollary 2.3.

**Proposition 4.9.** The following statements are equivalent for a ring \( R \).

(i) Every \( R \)-module is \( \lambda \)-complete.

(ii) Every homomorphic image of every \( \lambda \)-complete module is \( \lambda \)-complete.

**Proof.** (i) ⇒ (ii) Clear.

(ii) ⇒ (i) Let \( M \) be any \( R \)-module. There exist a free \( R \)-module \( F \) and a submodule \( K \) of \( F \) such that \( M \cong F/K \). By Corollary 3.4 the module \( F \) is \( \lambda \)-complete and hence so too is \( M \). □

In the case of Noetherian rings we can give a complete classification. We shall require the following two lemmas.

**Lemma 4.10.** Let \( R \) be a ring such that every \( R \)-module is \( \lambda \)-complete and let \( A \) be any ideal of \( R \). Then every \( (R/A) \)-module is \( \lambda \)-complete.

**Proof.** Let \( S \) be any non-empty collection of ideals of the ring \( R/A \). Then every ideal of \( S \) has the form \( B/A \) for some ideal \( B \) of \( R \). Let \( S' \) denote the collection of ideals \( B \) of \( R \) such that \( B/A \) belongs to \( S \). Let \( M \) be any \( (R/A) \)-module. Then \( M \) is an \( R \)-module in the usual way and we have

\[
(\cap_{C \in S} C)M = (\cap_{B \in S'} (B/A))M = ((\cap_{B \in S'} B)/A)M = (\cap_{B \in S'} B)M = \cap_{B \in S'} (B/M) = \cap_{B \in S'} ((B/A)M) = \cap_{C \in S} (CM).
\]

By Lemma 3.1, the \( (R/A) \)-module \( M \) is \( \lambda \)-complete. □

**Lemma 4.11.** The following statements are equivalent for a domain \( R \) with field of fractions \( F \).
(i) $R$ is a field.
(ii) Every $R$-module is $\lambda$-complete.
(iii) The $R$-module $F$ is $\lambda$-complete.

**Proof.** (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii) Clear by Lemma 3.1.
(iii) $\Rightarrow$ (i) Let $B_i(i \in I)$ denote the collection of all non-zero ideals of $R$. Then Lemma 3.1 gives that
\[ F = \cap_{i \in I} (B_iF) = (\cap_{i \in I} B_i)F. \]
Thus $\cap_{i \in I} B_i \neq 0$. It follows that $R$ has non-zero socle and hence $R = F$. $\Box$

Contrast the following result with Theorem 4.8.

**Theorem 4.12.** A Noetherian ring $R$ has the property that every $R$-module is $\lambda$-complete if and only if $R$ is an Artinian principal ideal ring.

**Proof.** Suppose first that every $R$-module is $\lambda$-complete. Let $P$ be any prime ideal of $R$. By Lemma 4.10, every $(R/P)$-module is $\lambda$-complete and hence the domain $R/P$ is a field by Lemma 4.11. Thus every prime ideal of $R$ is maximal. By [5, Theorem 4.6], the ring $R$ is Artinian. Next, by Theorem 4.6 every ideal of $R$ is a multiplication module and hence, by [2, Corollary 2.9] every ideal of $R$ is principal. Thus $R$ is a principal ideal ring.

Conversely, suppose that $R$ is an Artinian principal ideal ring. Let $M$ be any $R$-module. Let $S$ be any non-empty collection of ideals of $R$. Because $R$ is Artinian, there exists a finite subset $S'$ of $S$ such that $\cap_{B \in S} B = \cap_{B \in S'} B$. Noting that $R$ is a principal ideal ring and so every ideal of $R$ is a multiplication module, Theorem 4.6 and [7, Lemma 2.1] together give that $(\cap_{B \in S'} B)M = (\cap_{B \in S} B)M$. Thus,
\[ \cap_{B \in S} (BM) \subseteq \cap_{B \in S'} (BM) = (\cap_{B \in S'} B)M = (\cap_{B \in S} B)M, \]
and hence $(\cap_{B \in S} B)M = \cap_{B \in S} (BM)$. By Lemma 3.1 the $R$-module $M$ is $\lambda$-complete. $\Box$

5. Other homomorphisms

In general there will be many complete homomorphisms $\nu : \mathcal{L}(R) \to \mathcal{L}(RM)$ for a given ring $R$ and $R$-module $M$ (see [7, Section 5]). Note the following result.

**Proposition 5.1.** Let $R$ be a ring and let $M$ be an $R$-module such that there exists a complete isomorphism $\nu : \mathcal{L}(R) \to \mathcal{L}(RM)$. Then $M$ is a finitely generated $R$-module.
Proof. By Lemma 1.2 because $M$ is a finitely generated $R$-module if and only if $M$ is a compact element of $\mathcal{L}(RM)$. □

Recall that a ring $R$ is called *semilocal* provided it contains only a finite number of maximal ideals.

**Corollary 5.2.** Let $R$ be a ring and let $M$ be an $R$-module such that there exists a complete isomorphism $\nu : \mathcal{L}(R) \to \mathcal{L}(RM)$. Suppose further that either

(a) $R$ is a local ring, or
(b) $R$ is a semilocal ring and $M$ is a faithful $R$-module.

Then $M$ is a cyclic $R$-module.

**Proof.** By Proposition 5.1 and [7, Theorem 5.3]. □

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**References**

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