SOME QUANTITATIVE CHARACTERIZATIONS OF CERTAIN SYMPLECTIC GROUPS OVER THE BINARY FIELD

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Abstract. Given a finite group $G$, denote by $D(G)$ the degree pattern of $G$ and by $OC(G)$ the set of all order components of $G$. Denote by $h_{OD}(G)$ (resp. $h_{OC}(G)$) the number of isomorphism classes of finite groups $H$ satisfying conditions $|H| = |G|$ and $D(H) = D(G)$ (resp. $OC(H) = OC(G)$). A finite group $G$ is called $OD$-characterizable (resp. $OC$-characterizable) if $h_{OD}(G) = 1$ (resp. $h_{OC}(G) = 1$). Let $C = C_p(2)$ be a symplectic group over the binary field, for which $2^p - 1 > 7$ is a Mersenne prime. The aim of this article is to prove that $h_{OD}(C) = 1 = h_{OC}(C)$.

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1. Introduction

Only finite groups will be considered. Let $G$ be a group, $\pi(G)$ the set of all prime divisors of its order and $\omega(G)$ be the spectrum of $G$, that is the set of its element orders. The prime graph $GK(G)$ (or Gruenberg-Kegel graph) of $G$ is a simple graph whose vertex set is $\pi(G)$ and two distinct vertices $p$ and $q$ are joined by an edge if and only if $pq \in \omega(G)$. Let $t(G)$ be the number of connected components of $GK(G)$. The vertex set of the $i$th connected component of $GK(G)$ is denoted by $\pi_i(G)$ for each $i = 1, 2, \ldots, t(G)$. In the case when $2 \in \pi(G)$, we assume that $2 \in \pi_1(G)$. The classification of finite simple groups with disconnected prime graph was obtained by Williams [13] and Kondrat’év [4]. Recall that a clique in a graph is a set of pairwise adjacent vertices. Note that for all non-abelian simple groups $S$ with disconnected prime graph, all connected components $\pi_i(S)$ for $2 \leq i \leq t(S)$ are cliques, for instance, see [13]. The degree $\deg_G(p)$ of a vertex $p \in \pi(G)$ in $GK(G)$ is the number of edges incident on $p$. If $\pi(G) = \{p_1, p_2, \ldots, p_n\}$ with $p_1 < p_2 < \cdots < p_n$, then we define

$$D(G) = (\deg_G(p_1), \deg_G(p_2), \ldots, \deg_G(p_n)),$$
which is called the degree pattern of $G$. Given a group $G$, denote by $h_{OD}(G)$ the number of isomorphism classes of groups with the same order and degree pattern as $G$. All finite groups, in terms of the function $h_{OD}()$, are classified as follows:

**Definition 1.1.** A group $G$ is called $k$-fold OD-characterizable if $h_{OD}(G) = k$. Usually, a 1-fold OD-characterizable group is simply called OD-characterizable.

There are scattered results in the literature showing that certain simple groups are $k$-fold OD-characterizable for $k \in \{1, 2\}$. The most recent version of the list of such simple groups is presented in [8, Tables 2 and 3]. Until now, no examples of simple groups $S$ with $h_{OD}(S) \geq 3$ were known. Therefore, we posed the following question:

**Problem 1.2.** Is there a non-abelian simple group $S$ with $h_{OD}(S) \geq 3$?

In this article, we focus our attention on the symplectic groups $C_p(2) \cong S_{2p}(2)$, where $p$ is an odd prime. Recall that $C_2(2)$ is not a simple group, in fact, the derived subgroup $C_2(2)'$ is a simple group which is isomorphic with $A_6 \cong \text{L}_2(9)$. In addition, we recall that $B_2(3) \cong \text{A}_4(2^2)$, $B_n(2^m) \cong C_n(2^m)$ and $B_2(q) \cong C_2(q)$ (see [2]). Previously, it was determined the values of $h_{OD}(\cdot)$ for some symplectic and orthogonal groups (see [1,6,9]). In the table below, $\pi(n)$ is the set of all prime divisors of $n$, where $n$ is a natural number.

<table>
<thead>
<tr>
<th>$G$</th>
<th>Restrictions on $G$</th>
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<td>$B_3(4) \cong C_3(4)$</td>
<td></td>
<td>1</td>
<td>[6]</td>
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<tr>
<td>$B_2(q) \cong C_2(q)$</td>
<td>$\pi\left(\frac{q^2+1}{2}\right)=1$</td>
<td>1</td>
<td>[1]</td>
</tr>
<tr>
<td>$B_2^{2m}(q) \cong C_2^{2m}(q)$</td>
<td>$\pi\left(\frac{q^2+1}{2}\right)=1$</td>
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<tr>
<td>$B_3(5)$, $C_3(5)$</td>
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<td>2</td>
<td>[1]</td>
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<tr>
<td>$B_n(q)$, $C_n(q)$, $n = 2^m &gt; 2$, $\pi\left(\frac{q^{2m}+1}{2}\right)=1$, $q$ is even</td>
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<tr>
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<td></td>
<td>2</td>
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Given a group $G$, the order of $G$ can be expressed as a product of some coprime natural numbers $m_i(G)$, $i = 1, 2, \ldots, t(G)$, with $\pi(m_i(G)) = \pi_i(G)$. The numbers $m_1(G), m_2(G), \ldots, m_t(G)(G)$ are called the order components of $G$. We set $OC(G) = \{m_1(G), m_2(G), \ldots, m_t(G)(G)\}$.
In a similar manner, we define $h_{OC}(G)$ as the number of isomorphism classes of finite groups with the same set $OC(G)$ of order components. Again, in terms of function $h_{OC}(\cdot)$, the groups $G$ are classified as follows:

**Definition 1.3.** A finite group $G$ is called $k$-fold OC-characterizable if $h_{OC}(G) = k$. In the case when $k = 1$ the group $G$ is simply called OC-characterizable.

A Mersenne prime is a prime that can be written as $2^p - 1$ for some prime $p$.

The purpose of this article is to prove the following theorem.

**Main Theorem.** Let $C = C_{p^r}(2)$ be the symplectic group over the binary field, for which $2^p - 1 > 7$ is a Mersenne prime. Then $h_{OD}(C) = 1 = h_{OC}(C)$.

It is worth noting that the values of functions $h_{OD}(\cdot)$ and $h_{OC}(\cdot)$ may be different. For instance, suppose $M \in \{B_3(5), C_3(5)\}$. By [13], the prime graph associated with $M$ is connected and so $OC(M) = \{|M|\} = \{2^9 \cdot 3^4 \cdot 5^9 \cdot 7 \cdot 13 \cdot 31\}$. On the other hand, it is easy to see that the prime graph associated with a nilpotent group is always a clique, hence, we have

$$h_{OC}(M) > \nu_{nil}(|M|) \geq \nu_{a}(|M|) = Par(9)^2 \cdot Par(4) = 30^2 \times 5 = 4500,$$

where $\nu_{nil}(n)$ (resp. $\nu_{a}(n)$) signifies the number of non-isomorphic nilpotent (resp. abelian) groups of order $n$ and $Par(n)$ denotes the number of partitions of $n$. However, by Theorem 1.3 in [1], we know that $h_{OD}(M) = 2$.

**2. Preliminaries**

If $a$ is a natural number, $r$ is an odd prime and $(r, a) = 1$, then by $e(r,a)$ we denote the multiplicative order of $a$ modulo $r$, that is the minimal natural number $n$ with $a^n \equiv 1 \pmod{r}$. If $a$ is odd, we put $e(2,a) = 1$ if $a \equiv 1 \pmod{4}$, and $e(2,a) = 2$ if $a \equiv -1 \pmod{4}$. The following lemma is a consequence of Zsigmondy’s Theorem (see [14]).

**Lemma 2.1.** Let $a > 1$ be an integer. Then for every natural number $n$ there exists a prime $r$ with $e(r,a) = n$ except for the cases $(n,a) \in \{(1,2), (1,3), (6,2)\}$.

A prime $r$ with $e(r,a) = n$ is called a primitive prime divisor of $a^n - 1$. By Lemma 2.1, such a prime exists except for the cases mentioned in the lemma. We denote by $\text{ppd}(a^n - 1)$ the set of all primitive prime divisors of $a^n - 1$. By our definition, we have $\pi(a - 1) = \text{ppd}(a - 1)$ but for the following sole exception, namely, $2 \notin \text{ppd}(a - 1)$ if $e(2,a) = 2$. In this case, we assume that $2 \in \text{ppd}(a^2 - 1)$. 
From the definition it is easy to conclude that: Let \( p > 2 \) be an integer. Then
\[
\pi(a^p - 1) = \text{ppd}(a^p - 1) \text{ if and only if } p \text{ is a prime.}
\]

In the following results, we will consider the function \( \eta : \mathbb{N} \to \mathbb{N} \), which is defined as follows
\[
\eta(m) = \begin{cases} 
m & \text{if } m \equiv 1 \pmod{2}, \\
m/2 & \text{if } m \equiv 0 \pmod{2}.
\end{cases}
\]

**Lemma 2.2.** ([11]) Let \( M \) be one of the simple groups of Lie type \( B_n(q) \) or \( C_n(q) \) over a field of characteristic \( p \), and let \( r \in \pi(M) \setminus \{p\} \) and \( r \in \text{ppd}(q^k - 1) \). Then \( r \) and \( p \) are non-adjacent if and only if \( \eta(k) > n - 1 \).

**Lemma 2.3.** ([12]) Let \( M \) be one of the simple groups of Lie type \( B_n(q) \) or \( C_n(q) \) over a field of characteristic \( p \). Let \( r, s \) be odd primes with \( r, s \in \pi(M) \setminus \{p\} \). Suppose that \( r \in \text{ppd}(q^k - 1) \), \( s \in \text{ppd}(q^l - 1) \) and \( 1 \leq \eta(k) \leq \eta(l) \). Then \( r \) and \( s \) are non-adjacent if and only if \( \eta(k) + \eta(l) > n \) and \( l/k \) is not an odd natural number.

Using Lemmas 2.2 and 2.3, we conclude that the prime graphs \( \Gamma(B_n(q)) \) and \( \Gamma(C_n(q)) \) coincide (see also [11, Proposition 7.5]), and hence
\[
D(B_n(q)) = D(C_n(q)).
\]

**Corollary 2.4.** Let \( p > 3 \) be a prime and \( C = C_p(2) \). Then \( \deg C(3) = |\pi_1(C)| - 1 \).

**Proof.** Recall that, by [4], we have
\[
\pi_1(C) = \pi(2(2^p + 1) \prod_{i=1}^{p-1} (2^{2i} - 1)) \quad \text{and} \quad \pi_2(C) = \pi(2^p - 1).
\]

Now, it follows from Lemma 2.3 that all primitive prime divisors of \( 2^p - 1 \) (and so all primes in \( \pi(2^p - 1) \)) are non-adjacent to 3. On the other hand, by Lemmas 2.2 and 2.3, we deduce that \( \deg_C(3) = |\pi_1(C)| - 1 \), as desired.

The following lemma is crucial to the study of characterizability of symplectic groups \( C_p(2) \) by order components.

**Lemma 2.5.** ([3]) Let \( G \) be a group whose prime graph has more than one component. If \( H \) is a normal \( \pi_k(G) \)-subgroup of \( G \), then \( |H| - 1 \) is divisible by \( m_k(G) \), for all \( i \neq k \).

A group \( G \) is called 2-Frobenius if there exists a normal series \( 1 \leq H \leq K \leq G \) of \( G \) such that \( H \) is the Frobenius kernel of \( K \) and \( K/H \) is the Frobenius kernel of \( G/H \).
Lemma 2.6. ([7]) Let $S$ be a simple group with disconnected prime graph $GK(S)$, except $U_4(2)$ and $U_5(2)$. If $G$ is a finite group with $OC(G) = OC(S)$, then $G$ is neither a Frobenius group nor a $2$-Frobenius group.

Lemma 2.7. ([13]) Let $G$ be a group with $t(G) \geq 2$. Then one of the following hold:

1. $G$ is a Frobenius group;
2. $G$ is a $2$-Frobenius group; or
3. $G$ has a normal series $1 \leq H < K \leq G$ such that $H$ is a nilpotent $\pi_1$-group, $K/H$ is a non-abelian simple group, $G/K$ is a $\pi_1$-group, $|G/K|$ divides $|\text{Out}(K/H)|$ and any odd order component of $G$ is equal to one of the odd order components of $K/H$.

Lemma 2.8. ([5]) The only solution of the equation $p^m - q^n = 1$, where $p, q$ are primes and $m, n > 1$ are integers, is $(p, q, m, n) = (3, 2, 2, 3)$.

Given a natural number $B$ and a prime number $t$, we denote by $B_t$ the $t$-part of $B$, that is the largest power of $t$ dividing $B$.

Lemma 2.9. ([10]) Let $B = (2^2 - 1)(2^4 - 1) \cdots (2^{2n} - 1)$. If $t$ is a prime divisor of $B$, then $B_t < 2^{3n}$. Furthermore, if $t \geq 5$ then $B_t < 2^{2n}$.

3. Proof of the main theorem

Throughout this section, we will assume that $2^p - 1 > 7$ is a Mersenne prime and $C = C_p(2)$. Suppose that $G$ is a group with the same order and degree pattern as $C$, that is

$$|G| = |C| = 2^{2p} \prod_{i=1}^{p} (2^{2i} - 1) \quad \text{and} \quad \text{D}(G) = \text{D}(C).$$

Note that, according to the results summarized in [4], we have $t(C) = 2$, and

$$\pi_1(C) = \pi \left( \frac{2(2^p + 1) \prod_{i=1}^{p-1} (2^{2i} - 1)}{2^{p-1}} \right) \quad \text{and} \quad \pi_2(C) = \{2^p - 1\}.$$

By our hypothesis, it is easy to see that

$$\pi_2(G) = \pi_2(C) = \{2^p - 1\} \quad \text{and} \quad \pi(G) = \pi(C) = \pi_1(C) \cup \{2^p - 1\}.$$

First of all, we notice that $2^p - 1$ is the largest prime in $\pi(G) = \pi(C)$. Moreover, it follows from Corollary 2.4 that

$$\deg_G(3) = \deg_C(3) = |\pi_1(C)| - 1,$$
and this forces $\pi_1(G) = \pi_1(C)$, and so $t(G) = 2$. Hence, we have

$$\text{OC}(G) = \text{OC}(C) = \left\{ 2^{2^i}(2^{2^i} + 1) \prod_{i=1}^{p-1} (2^{2^i} - 1), \quad 2^p - 1 \right\},$$

and from Lemma 2.6, the group $G$ is neither a Frobenius group nor a 2-Frobenius group. Finally, Lemma 2.7, reduces the problem to the study of the simple groups. Indeed, by Lemma 2.7, there is a normal series $1 \triangleright H \triangleleft K \triangleright G$ of $G$ such that:

1. $H$ is a nilpotent $\pi_1(G)$-group, $K/H$ is a non-abelian simple group and $G/K$ is a $\pi_1(G)$-group. Moreover, we have $K/H \leq G/H \leq \text{Aut}(K/H)$, and $t(K/H) \geq t(G) \geq 2$,

2. $2^p - 1$ is the only odd order component of $G$ which is equal to one of those of the quotient $K/H$,

3. $|G/K|$ divides $|\text{Out}(K/H)|$.

For odd order components of $K/H$ see [4,13]. Now, we will continue the proof step by step.

**Step 3.1.** $K/H \nmid 2A_3(2), 2F_4(2)^{\prime}, 2A_5(2), E_7(2), E_7(2), A_2(4), 2E_6(2)$ nor one of the sporadic simple groups.

Note that either the odd order components of above groups are not equal to a Mersenne prime $2^p - 1 > 7$ or their orders do not divide the order of $G$.

In the following, $\mathfrak{A}_n$ denotes the alternating group on $n$ letters.

**Step 3.2.** $K/H \nmid \mathfrak{A}_n$, where $n$ and $n - 2$ are both prime numbers.

In this case, it follows that $n = 2^p - 1$. Now, simple computations show that

$$|\mathfrak{A}_n|_2 = \frac{n!}{2} = 2^\left[\frac{n}{2}\right] + \left[\frac{n}{2}\right] + \cdots - 1 = 2^{2^p - p - 2}.$$

If $p > 5$, then $2^p - p - 2 > p^2$ and hence the 2-part of $|\mathfrak{A}_n|$ does not divide the 2-part of $|G|$, i.e. $2^{p^2}$, which is a contradiction. In the case when $p = 5$, then $n = 31$ and $|K/H| = (31!)/2$, which does not divide $|G| = |C_5(2)| = 2^{25} \cdot 3^9 \cdot 5^2 \cdot 7 \cdot 11 \cdot 17 \cdot 31$, which is again a contradiction.

**Step 3.3.** $K/H \nmid \mathfrak{A}_n$, where $n = q$, $q + 1$, or $q + 2$ ($q$ is a prime), and one of $n$, $n - 2$ is not prime.

Here, $q$ is the only odd order component of $K/H$, and so $q = 2^p - 1$. We now consider the alternating group $\mathfrak{A}_q$ which is a subgroup of $K/H \cong \mathfrak{A}_n$. Similar arguments as those in the previous step, on the subgroup $\mathfrak{A}_q$ instead of $\mathfrak{A}_n$, lead us a contradiction.
Step 3.4. $K/H$ is isomorphic to neither $^2E_6(q)$, $q > 2$, nor $E_6(q)$.

We deal with $^2E_6(q)$, $q > 2$, the proof for $E_6(q)$ being quite similar. Suppose that $K/H \cong ^2E_6(q)$. First of all, we recall that

$$\left| ^2E_6(q) \right| = \frac{1}{(3, q + 1)} q^{36} (q^{12} - 1)(q^9 + 1)(q^6 - 1)(q^6 - 1)(q^5 + 1)(q^2 - 1).$$

Considering the only odd order component of $^2E_6(q)$, that is $(q^6 - q^3 + 1)/(3, q + 1)$, we must have $(q^6 - q^3 + 1)/(3, q + 1) = 2^p - 1$, which implies that $q^9 > 2^p$, or equivalently $q^{36} > 2^{4p}$. Let $q = r^f$. If $r$ is an odd prime, then from Lemma 2.9, we get

$$q^{36} = r^{36f} = |K/H|_r \leq |G|_r < 2^{3p},$$

which is a contradiction. Therefore we may assume that $r = 2$. In this case, we have

$$(2^{6f} - 2^{3f} + 1)/(3, 2^f + 1) = 2^p - 1.$$ Now, if $(3, 2^f + 1) = 1$, then we obtain $2^{3f}(2^{3f} - 1) = 2(2^{p-1} - 1)$, from which we deduce that $3f = 1$, a contradiction. In the case where $(3, 2^f + 1) = 3$, an easy calculation shows that

$$2^{3f}(2^{3f} - 1) = 2^2(3 \cdot 2^{p-2} - 1),$$

and so $3f = 2$, which is again a contradiction.

Step 3.5. $K/H \ncong F_4(q)$, where $q$ is an odd prime power.

We remark that $q^4 - q^2 + 1$ is the only odd order component of $F_4(q)$, and clearly this forces $q^4 - q^2 + 1 = 2^p - 1$. Then $q^2(q^2 - 1) = 2(2^{p-1} - 1)$, which shows that $2(2^{p-1} - 1)$ is divisible by 4, a contradiction.

Step 3.6. $K/H \ncong ^2B_2(q)$, where $q = 2^{2m+1} > 2$.

Recall that $|^2B_2(q)| = q^2(q^2 + 1)(q - 1)$ and the odd order components of $^2B_2(q)$ are:

$$q - 1, \quad q - \sqrt{2q} + 1, \quad q + \sqrt{2q} + 1.$$ If $q - 1 = 2^p - 1$, then $q = 2^p$. Now, we consider the primitive prime divisor $r \in \text{ppd}(2^{4p} - 1)$. Clearly $r \in \pi(2^{4p} - 1)$, and so $r \in \pi(^2B_2(q)) \subseteq \pi(G)$. This is a contradiction.

In the case when

$$q - \sqrt{2q} + 1 = 2^p - 1 \quad \text{(resp. } q + \sqrt{2q} + 1 = 2^p - 1),$$
by simple computations we obtain
\[ 2^{m+1}(2^m - 1) = 2(2^p - 1) \quad \text{(resp.} \quad 2^{m+1}(2^m + 1) = 2(2^p - 1)), \]
a contradiction.

**Step 3.7.** \( K/H \nmid E_8(q), \) where \( q \equiv 2, 3 \pmod{5} \).

The odd order components of \( E_8(q) \) in this case are
\[ q^8 - q^4 + 1, \quad \frac{q^{10} + q^5 + 1}{q^2 + q + 1}, \quad \frac{q^{10} - q^5 + 1}{q^2 - q + 1}. \]

If \( q^8 - q^4 + 1 = 2^p - 1 \), then we obtain \( q^4(q - 1)(q + 1)(q^2 + 1) = 2(2^p - 1). \) However, the left hand side is divisible by 16, while the right hand side is not divisible by 4, which is impossible.

If \( \frac{q^{10} + q^5 + 1}{q^2 + q + 1} = 2^p - 1 \), then after subtracting 1 from both sides of this equation and some simple computations, we obtain
\[ q(q - 1)(q + 1)(q^2 + 1)(q^3 - q^2 + 1) = 2(2^p - 1). \]

Now, if \( q \) is odd, then the left hand side is divisible by 16, a contradiction. Moreover, if \( q \) is even, then it follows that \( q = 2 \), and if this is substituted in above equation we get \( 76 = 2^p - 1 \), a contradiction.

The case \( \frac{q^{10} - q^5 + 1}{q^2 - q + 1} = 2^p - 1 \) is quite similar to the previous case and it is omitted.

**Step 3.8.** \( K/H \nmid E_8(q), \) where \( q \equiv 0, 1, 4 \pmod{5} \).

The odd order components of \( E_8(q) \) in this case are
\[ \frac{q^{10} + 1}{q^2 + 1}, \quad q^8 - q^4 + 1, \quad \frac{q^{10} + q^5 + 1}{q^2 + q + 1}, \quad \frac{q^{10} - q^5 + 1}{q^2 - q + 1}. \]

Consider the first case. Let \( \frac{q^{10} + 1}{q^2 + 1} = 2^p - 1. \) Subtracting 1 from both sides of this equality, we get
\[ q^2(q^2 - 1)(q^4 + 1) = 2(2^p - 1), \]
which implies \( 2(2^p - 1) \) is divisible by 4, a contradiction.

Similarly, if \( q^8 - q^4 + 1 = 2^p - 1, \) we obtain \( q^4(q - 1)(q + 1)(q^2 + 1) = 2(2^p - 1), \) which shows that \( 2(2^p - 1) \) is divisible by 16, a contradiction. Similar arguments work if \( \frac{q^{10} + q^5 + 1}{q^2 + q + 1} = 2^p - 1 \) or \( \frac{q^{10} - q^5 + 1}{q^2 - q + 1} = 2^p - 1, \) and we omit the details.

**Step 3.9.** \( K/H \nmid 2F_4(q), \) where \( q = 2^{2m+1} > 2. \)
The odd order components of \(2F_4(q)\) are:
\[
q^2 + \sqrt{2}q^3 + q + \sqrt{2}q + 1 \quad \text{and} \quad q^2 - \sqrt{2}q^3 + q - \sqrt{2}q + 1.
\]
Therefore, we have
\[
q^2 + \sqrt{2}q^3 + q + \sqrt{2}q + 1 = 2^p - 1 \quad \text{or} \quad q^2 - \sqrt{2}q^3 + q - \sqrt{2}q + 1 = 2^p - 1.
\]
However, if \(2^{2m+1}\) is substituted in these equations we obtain
\[
2^{m+1}(2^{3m+1} \pm 2^{2m+1} + 2^{m} \pm 1) = 2(2^{p-1} - 1),
\]
which is a contradiction.

**Step 3.10.** \(K/H \not\cong F_4(q)\), where \(q = 2^m\).

The odd order components of \(F_4(q)\) are \(q^4 + 1\) and \(q^4 - q^2 + 1\), hence \(q^4 + 1 = 2^p - 1\) or \(q^4 - q^2 + 1 = 2^p - 1\). Now, it is easy to see that in both cases, \(2^{2m}\) divides \(2(2^{p-1} - 1)\), a contradiction.

**Step 3.11.** \(K/H \not\cong 2G_2(q)\), where \(q = 3^{2m+1} > 3\).

The odd order components of \(2G_2(q)\) are \(q + \sqrt{3}q + 1\) and \(q - \sqrt{3}q + 1\). If \(q - \sqrt{3}q + 1 = 2^p - 1\), then \(q^3 > 2^{3p}\), while Lemma 2.9 shows that \(q^3 < 2^{3p}\), which is a contradiction. If \(q + \sqrt{3}q + 1 = 2^p - 1\), then
\[
2^p - 2 = 2(2^{(p-1)/2} - 1)(2^{(p-1)/2} + 1) = 3^{m+1}(3^m + 1). \tag{1}
\]
First of all, we recall that \(2^{(p-1)/2} - 1, 2^{(p-1)/2} + 1 = 1\). Now we consider two cases separately:

(i) If \(3^{m+1}\) divides \(2^{(p-1)/2} - 1\), then
\[
3^m + 1 < 3^{m+1} \leq 2^{(p-1)/2} - 1 < 2^{(p-1)/2} + 1.
\]
Hence, we obtain
\[
3^{m+1}(3^m + 1) < 2(2^{(p-1)/2} - 1)(2^{(p-1)/2} + 1),
\]
a contradiction.

(ii) If \(3^{m+1}\) divides \(2^{(p-1)/2} + 1\), then \(2^{(p-1)/2} + 1 = k \cdot 3^{m+1}\) where \(k\) is a natural number. Now, from Eq. (1), it follows that
\[
2k(2^{(p-1)/2} - 1) = 3^m + 1,
\]
and consequently \(3^m \geq 2^{(p-1)/2} - 1\). Therefore we have
\[
2^{(p+1)/2} - 1 \leq 3^m < 3^{m+1} \leq 2^{(p-1)/2} + 1,
\]
a contradiction.
Step 3.12. \( K/H \cong G_2(q), \) where \( q = 3^m. \)

Recall that the odd order components of \( G_2(q) \) are \( q^2 - q + 1 \) and \( q^2 + q + 1. \) If \( q^2 - q + 1 = 2^p - 1 \) then \( q^6 > 2^{3p}, \) while one can follow from Lemma 2.9 that \( q^6 < 2^{3p}, \) which is a contradiction. If \( q^2 + q + 1 = 2^p - 1, \) then \( q(q+1) \equiv 2 \pmod{4}, \) which forces \( m \) is even. But then, it is obvious that \( 2^p - 2 = q(q+1) \equiv 2 \pmod{8}, \) a contradiction.

Step 3.13. \( K/H \cong 2D_r(3), \) where \( r = 2^m + 1 \) is a prime number and \( m \geq 1. \)

Recall that 

\[
|2D_r(3)| = \frac{1}{(4,3^r+1)} 3^{r(r-1)}(3^r+1) \prod_{i=1}^{r-1} (3^{2i} - 1),
\]

and the odd order components of \( 2D_r(3) \) are

\[
(3^{r-1}+1)/2 \quad \text{and} \quad (3^r+1)/4.
\]

In the case when \( (3^{r-1}+1)/2 = 2^p - 1, \) adding 1 to both sides of this equality, we obtain

\[
3(3^{r-2}+1) = 2^{p+1},
\]

which is a contradiction. If \( (3^r+1)/4 = 2^p - 1, \) then \( r \geq 5 \) because \( p \geq 5. \) Moreover, on the one hand, from last equation we obtain \( 3^r = 2^{p+2} - 5 > 2^{p+1}, \) which implies that

\[
3^{r(r-1)} > 2^{(p+1)(r-1)} > 2^{4(p+1)}.
\]

On the other hand, it follows from Lemma 2.9 that

\[
3^{r(r-1)} = |K/H|_3 \leq |G|_3 < 2^{3p},
\]

which is a contradiction.

Step 3.14. \( K/H \cong B_n(q), \) where \( n = 2^m \geq 4 \) and \( q = r^f \) is an odd prime power.

Note that 

\[
|B_n(q)| = \frac{1}{(2, q-1)} q^n \prod_{i=1}^{n} (q^{2i} - 1),
\]

and the only odd order component of \( B_n(q) \) is \((q^n + 1)/2.\) If \((q^n + 1)/2 = 2^p - 1, \) then \( q^n = 2^{p+1} - 3 > 2^p \) and clearly \( q \) is not divisible by 2 and 3. Since \( p \geq 5 \) and \( n \geq 4, \) it is easy to see that

\[
q^n > q^{3n} > 2^{3p} > 2^{2p}.
\]

On the other hand, by Lemma 2.9, we obtain

\[
q^n = |K/H|_r \leq |G|_r < 2^{2p},
\]
which is a contradiction.

**Step 3.15.** \(K/H \not\cong B_r(3)\).

The only odd order component of \(B_r(3)\) is \((3^r - 1)/2\). If \((3^r - 1)/2 = 2^p - 1\), then \(2^{p+1} - 3^r = 1\). However, this equation has no solution by Lemma 2.8, which is impossible.

**Step 3.16.** \(K/H \not\cong 3D_4(q)\).

We recall that \(q^4 - q^2 + 1\) is the only odd order component of \(3D_4(q)\), and so \(q^4 - q^2 + 1 = 2^p - 1\). But then, \(q^2(q^2 - 1) = 2(2^{p-1} - 1)\), which shows that \(2(2^{p-1} - 1)\) is divisible by 4, a contradiction.

**Step 3.17.** \(K/H \not\cong G_2(q)\), where \(2 < q \equiv \pm 1 \pmod{3}\).

In this case, the odd order components of \(G_2(q)\) are \(q^2 + q + 1\) and \(q^2 - q + 1\). Let \(q = r^f\). If \(q^2 + q + 1 = 2^p - 1\), then \(q(q + 1) = 2(2^{p-1} - 1)\), which shows that \(q > 2\) is not a power of 2. Moreover, since \(q - 1 \geq 2\), we obtain
\[q^3 - 1 = (q - 1)(q^2 + q + 1) \geq 2(2^p - 1),\]
and so \(q^3 > 2^{p+1} - 1 > 2^p\), which yields that \(q^6 > 2^{2p}\). However, since
\[|G_2(q)| = q^6(q^2 - 1)(q^6 - 1),\]
from Lemma 2.9, we conclude that
\[q^6 = |K/H|_r \leq |G|_r < 2^{2p},\]
which is a contradiction.

The case when \(q^2 - q + 1 = 2^p - 1\) is similar and left to the reader.

**Step 3.18.** \(K/H \not\cong 2D_n(3)\), where \(n = 2^m + 1\) which is not a prime and \(m \geq 2\).

The odd order component of \(2D_n(3)\) is \((3^{n-1} + 1)/2\). If \((3^{n-1} + 1)/2 = 2^p - 1\), then \(2^{p+1} = 3(3^{n-2} + 1)\), a contradiction.

**Step 3.19.** \(K/H \not\cong 2D_r(3)\), where \(r \geq 5\) is a prime and \(r \neq 2^m + 1\).

Here, we have
\[|2D_r(3)| = \frac{1}{(4, 3^r + 1)} 3^{r(r-1)}(3^r + 1) \prod_{i=1}^{r-1} (3^{2i} - 1).\]
The only odd order component of $2D_r(3)$ is $(3^r + 1)/4$, and so $(3^r + 1)/4 = 2^p - 1$. An easy computation shows that $3^r = 2^{p+2} - 5 > 2^{p+1}$. Moreover, we note that $r - 1 \geq 4$, and so

$$3^{(r-1)} > 3^{4r} > 2^{4(p+1)}.$$ 

On the other hand, by Lemma 2.9, we obtain

$$3^{r(r-1)} = |K/H|_3 \leq |G|_3 < 2^{3p},$$

which is a contradiction.

**Step 3.20.** $K/H \not\cong 2D_n(2)$, where $n = 2^m + 1$, $m \geq 2$.

The only odd order component of $2D_n(2)$ is $2^{n-1} + 1$. Therefore, we obtain $2^{n-1} + 1 = 2^p - 1$, which is impossible.

**Step 3.21.** $K/H \not\cong 2D_n(q)$, where $n = 2^m \geq 4$ and $q = r^f$.

Recall that

$$|2D_n(q)| = \frac{1}{(4, q^n + 1)} q^{n(n-1)}(q^n + 1) \prod_{i=1}^{n-1} (q^{2i} - 1),$$

and the only odd order component of $2D_n(q)$ is $(q^n + 1)/(2, q + 1)$. Therefore, $(q^n + 1)/(2, q + 1) = 2^p - 1$. Assume first that $(2, q + 1) = 1$. In this case, we obtain $q^n = 2(2^{p-1} - 1)$, a contradiction. Assume next that $(2, q + 1) = 2$. Again, using simple calculations we obtain $q^n = 2^{p+1} - 3 > 2^p$ and so $q$ cannot be a power of 2. Moreover, since $n - 1 \geq 3$, $q^{n(n-1)} \geq q^{3n} > 2^{3p}$. Now, Lemma 2.9 shows that

$$q^{n(n-1)} = |K/H|_r \leq |G|_r < 2^{3p},$$

which is a contradiction.

**Step 3.22.** $K/H \not\cong D_{r+1}(q)$, where $q = 2, 3$.

Since, the only odd order component of $D_{r+1}(q)$ is $(q^r - 1)/(2, q - 1)$, we have $(q^r - 1)/(2, q - 1) = 2^p - 1$. If $(2, q - 1) = 1$, then $r = p$ and $q = 2$, and we have

$$|K/H| = |D_{p+1}(2)| = \frac{1}{(4, 2^{p+1} - 1)} 2^{p(p+1)}(2^{p+1} - 1) \prod_{i=1}^{p} (2^{2i} - 1),$$

this shows that $|K/H|_2 = 2^{p(p+1)}/(4, 2^{p+1} - 1)$ does not divide $|G|_2 = 2^{p^2}$, which is a contradiction. In the case when $(2, q - 1) = 2$, we have the equation $2^{p+1} - 3^r = 1$, which has no solution for $p \geq 5$, by Lemma 2.8. This is again a contradiction.

**Step 3.23.** $K/H \not\cong D_r(q)$, where $q = 2, 3, 5$ and $r \geq 5$. 
We recall that the only odd order component of $D_r(q)$ is $(q^r - 1)/(q - 1)$. We distinguish three cases separately.

(i) $q = 2$. In this case, we have $2^r - 1 = 2^p - 1$, and so $r = p$ and

$$|K/H| = |D_p(2)| = 2^{p(p-1)}(2^p - 1)\prod_{i=1}^{p-1}(2^{2i} - 1).$$

Note that $|\text{Out}(D_p(2))| = 2$ and $D_p(2) \leq G/H \leq \text{Aut}(D_p(2))$. Now, considering the order of groups, we get $|H| = 2^n(2^p + 1)$ where $p - 1 \leq \alpha \leq p$.

Let $r \in \text{ppd}(2^{2p} - 1)$ and $Q \in \text{Syl}_r(H)$. Clearly $r \in \pi(2^p + 1)$, $Q$ is a normal $\pi_1(G)$-subgroup of $G$ and $|Q|$ divides $2^p + 1$. Now, from Lemma 2.5, it follows that $|Q| - 1$ is divisible by $m_2(G) = 2^p - 1$, and so $|Q| - 1 > 2^{p-1}$ or equivalently $|Q| > 2^p$. This forces $|Q| = 2^p + 1$. But then $m_2(G) = 2^p - 1$ does not divide the value $|Q| - 1 = 2^p$, which is a contradiction.

(ii) $q = 3$. In this case, from the equality $(3^r - 1)/2 = 2^p - 1$, we deduce that $2^{p+1} - 3^r = 1$. However, this equation has no solution when $p > 5$. It is easy to verify that it has no solution when $p = 5$ by Lemma 2.8, a contradiction.

(iii) $q = 5$. Here $(5^r - 1)/4 = 2^p - 1$, and so $5^r = 2^{p+2} - 3 > 2^{p+1}$.

As before, since $r - 1 > 4$, we obtain $5^{r(r-1)} > 5^{4^r} > 2^{4^r(r+1)}$. On the other hand, by Lemma 2.9, we have

$$5^{r(r-1)} = |K/H|/|G| < 2^{2p},$$

which is a contradiction.

**Step 3.24.** $K/H \nmid C_r(3)$.

The only odd order component of $C_r(3)$ is $(3^r - 1)/2$. Thus, if $(3^r - 1)/2 = 2^p - 1$, then $2^{p+1} - 3^r = 1$. However, this equation has no solution by Lemma 2.8, which is impossible.

**Step 3.25.** $K/H \nmid C_n(q)$, where $n = 2^m \geq 2$.

Note that

$$|C_n(q)| = \frac{1}{(2, q - 1)}q^{n^2}\prod_{i=1}^{n}(q^{2i} - 1),$$

and the only odd order component of $C_n(q)$ is $(q^n + 1)/(2, q - 1)$. Therefore, $(q^n + 1)/(2, q - 1) = 2^p - 1$. If $(2, q - 1) = 1$, then $q^n = 2(2^p - 1) - 1$, which yields $q = p = 2$ and $n = 1$, a contradiction. If $(2, q - 1) = 2$, then $q^n = 2^{p+1} - 3 > 2^p$, which implies that $q$ is not a power of 2 and 3. Let $q = r^f$. When $n \geq 4$, it is easy to see that

$$q^{n^2} > q^{3n} > 2^{3p} > 2^{2(p+1)}.$$
But, from Lemma 2.9, we obtain
\[ q^{n^2} = |K/H| \leq |G| < 2^{2p}, \]
a contradiction. Assume now that \( n = 2 \). In this case, we have \( q^2 = 2^{p+1} - 3 \), or equivalently
\[ (q - 1)(q + 1) = 2^2(2^{p-1} - 1). \]
However, the left hand side is divisible by 8, while the right hand side is divisible by 4, a contradiction.

**Step 3.26.** \( K/H \not\cong A_1(q) \), where \( q = 2^m > 2 \).

The odd order components of \( A_1(q) \) are \( q + 1 \) and \( q - 1 \). If \( q + 1 = 2p - 1 \), then \( q = 2(2^{p-1} - 1) \), a contradiction. If \( q - 1 = 2p - 1 \), then \( q = 2^p \). Moreover, since \( A_1(q) \leq G/H \leq \text{Aut}(A_1(q)) \), it is easy to see that the order of \( H \) is divisible by \((2^2 - 1)(2^4 - 1)\cdots(2^{2p-1} - 1)\). Let \( r \in \text{ppd}(2^{2(p-1)} - 1) \) and \( Q \in \text{Syl}_r(H) \). Clearly \( Q \) is a normal \( \pi_1(G) \)-subgroup of \( G \) and \( |Q| \) divides \( 2^p - 1 \). On the other hand, from Lemma 2.5, \( |Q| - 1 \) is divisible by \( 2^p - 1 \) which implies that \( |Q| \geq 2^p \). This is a contradiction.

**Step 3.27.** \( K/H \not\cong A_1(q) \), where \( 3 \leq q \equiv \pm 1 \pmod{4} \) and \( q = r^f \).

Assume first that \( 3 \leq q \equiv 1 \pmod{4} \). In this case, the odd order components of \( A_1(q) \) are \((q + 1)/2\) and \( q \). If \((q + 1)/2 = 2^p - 1\), then \( r^f = q = 2^{p+1} - 3 \). First of all, we claim that \( f \) is an odd number. Otherwise, we have
\[ (r^{f/2} - 1)r^{f/2} + 1 = 2^2(2^{p-1} - 1). \]
But then, the left hand side is divisible by 8, while the right hand side is divisible by 4, which is a contradiction. Furthermore, by easy computations we observe that
\[ |A_1(q)| = \frac{1}{2}q(q^2 - 1) = 2^2(2^{p+1} - 3)(2^{p-1} - 1)(2^p - 1). \]
On the other hand, we have \(|G/K| \cdot |H| = |G|/|A_1(q)|\), from which we deduce that
\[ |G/K|_2 \cdot |H|_2 = \frac{|G|_2}{|A_1(q)|_2} = 2^{p^2 - 2}. \]
But since \(|G/K|\) divides \(|\text{Out}(A_1(q))| = 2f\) and \( f \) is odd, \(|G/K|_2\) is at most 2. Hence, if \( S_2 \subset \text{Syl}_2(H) \), then \(|S_2| = 2^{p^2-2} \) or \(|S_2| = 2^{p^2-3}\). We notice that \( S_2 \) is a normal subgroup of \( G \), because \( H \) is nilpotent. Now, it follows from Lemma 2.5 that \( 2^p - 1 \) divides \( 2^{p^2-2} - 1 \) or \( 2^{p^2-3} - 1 \), which is a contradiction. If \( q = 2^p - 1 \), we get a contradiction by Lemma 2.8.

Assume next that \( 3 \leq q \equiv -1 \pmod{4} \). In this case, the odd order components of \( A_1(q) \) are \((q - 1)/2\) and \( q \). If \((q - 1)/2 = 2^p - 1\), then \( 2^{p+1} - r^f = 1 \). Noting
Lemma 2.8, we deduce that \( f = 1 \), and hence \( r = 2^{p+1} - 1 \) is a Mersenne prime, which is a contradiction because \( p + 1 \) is not a prime.

The case when \( q = 2^p - 1 \) is similar to the previous paragraph.

**Step 3.28.** \( K/H \ncong A_r(q) \), where \( (q - 1)|(r + 1) \).

Recall that

\[
|K/H| = |A_r(q)| = \frac{1}{(r + 1, q - 1)} q^{(r+1)/2} \prod_{i=2}^{r+1} (q^i - 1).
\]

The only odd order component of \( A_r(q) \) is \( (q^r - 1)/(q - 1) \), and so

\[
(q^r - 1)/(q - 1) = 2^p - 1.
\]

As a simple observation we see that \( q^r - 1 \geq (q^r - 1)/(q - 1) = 2^p - 1 \) and so \( q^r \geq 2^p \).

(i) Suppose first that \( r \geq 7 \). Then \( q^{(r+1)/2} > q^{3(r+1)} \geq 2^3 q^3 \geq 2^{2(p+1)} \).

Now, if \( t \) is an odd prime, then by Lemma 2.9 we obtain

\[
q^{(r+1)/2} = |K/H|_t \leq |G|_t < 2^{2p},
\]

which is a contradiction. Therefore, we may assume that \( t = 2 \). In this case, we have

\[
(2^{fr} - 1)/(2^f - 1) = 2^p - 1,
\]

from which one can deduce that \( f = 1 \) and \( r = p \). Thus

\[
|G/K| \cdot |H| = \frac{2^p 2^p \prod_{i=1}^{p} (2^{2i} - 1)}{2^{p(p+1)} \prod_{i=2}^{p+1} (2^i - 1)}.
\]

Since \( |G/K| \) divides \( |\text{Out}(K/H)| = |\text{Out}(A_p(2))| = 2 \), we conclude that \( |H| \) is divisible by \( 2^p + 1 \). Let \( s \in \text{ppd}(2^p - 1) \subseteq \pi(2^p + 1) \) and \( Q \in \text{Syl}_s(H) \).

Clearly \( |Q||2^p + 1 \). Since \( H \) is a normal \( \pi_1(G) \)-subgroup of \( G \) which is nilpotent, \( Q \) is also a normal \( \pi_1(G) \)-subgroup of \( G \). Now, by Lemma 2.5, \( m_2(G) = 2^p - 1 \) divides \( |Q| - 1 \), and so \( |Q| \geq 2^p \). But, this forces \( |Q| = 2^p + 1 \).

However, this contradicts the fact that \( m_2(G)||Q| - 1 \).

(ii) Suppose next that \( r = 5 \). If \( q \) is even, then from \( (q^5 - 1)/(q - 1) = 2^p - 1 \), we obtain \( q(q^3 + q^2 + q + 1) = 2(2^{p-1} - 1) \), which implies that \( q = 2 \) and \( r = p = 5 \). Therefore, by easy calculations we see that

\[
|G/K| \cdot |H| = \frac{2^{10} \prod_{i=1}^{5} (2^i + 1)}{2^p - 1},
\]

which is not a natural number, a contradiction. If \( q \) is odd, then we get

\[
q(q + 1)(q^2 + 1) = q^4 + q^3 + q^2 + q = 2^p - 2,
\]
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however \( q(q + 1)(q^2 + 1) \equiv 0 \pmod{4} \), while \( 2^p - 2 \equiv 2 \pmod{4} \), a contradiction.

(iii) Finally suppose that \( r = 3 \). Then \( q(q + 1) = 2(2^p - 1) \). First of all, we note that \( q \) is not even, otherwise \( p = 3 \), which is impossible. In addition, we have

\[
q(q + 1) = 2(2^{(p-1)/2} - 1)(2^{(p-1)/2} + 1).
\]

(2)

Now we consider two cases separately:

(a) If \( q \) divides \( 2^{(p-1)/2} - 1 \), then

\[
q \leq 2^{(p-1)/2} - 1, \quad q + 1 < 2^{(p-1)/2} + 1.
\]

Hence, we obtain

\[
q(q + 1) < 2(2^{(p-1)/2} - 1)(2^{(p-1)/2} + 1),
\]

a contradiction.

(b) If \( q \) divides \( 2^{(p-1)/2} + 1 \), then \( 2^{(p-1)/2} + 1 = kq \) for some natural number \( k \). Now from Eq. (2), it follows that

\[
2k(2^{(p-1)/2} - 1) = q + 1.
\]

If \( k = 1 \), then \( p = q = 5 \). Hence \( 13 \in \pi(K/H) = \pi(A_3(5)) \), however \( 13 \notin \pi(G) = \pi(C_5(2)) \), a contradiction. Thus, \( k \geq 2 \) and we obtain

\[
2(2^{(p+1)/2} - 2) - 1 \leq q < q + 1 \leq kq = 2^{(p-1)/2} + 1,
\]

which is a contradiction.

**Step 3.29.** \( K/H \not\cong A_{r-1}(q) \), where \( (r, q) \neq (3, 2), (3, 4) \).

Again, we recall that

\[
|K/H| = |A_{r-1}(q)| = \frac{1}{(r, q - 1)^r} q^{r(r-1)/2} \prod_{i=2}^{r} (q^i - 1),
\]

and the only odd order component of \( A_{r-1}(q) \) is \( (q^r - 1)/(q - 1)(r, q - 1) \). Hence, we must have

\[
(q^r - 1)/(q - 1)(r, q - 1) = 2^p - 1,
\]

which implies that

\[
q^r - 1 \geq (q^r - 1)/(q - 1)(r, q - 1) = 2^p - 1,
\]

or equivalently \( q^r \geq 2^p \). Let \( q = t^f \), where \( t \) is a prime and \( f \) is a natural number.

In what follows, we consider several cases separately.
(i) \( r \geq 7 \). In this case, we obtain
\[
q^{r(r-1)/2} \geq q^3r \geq 2^{3p},
\]
and Lemma 2.9 implies that \( t = 2 \). Now, Lemma 2.1 shows that \( q = 2 \) and \( r = p \), and hence we obtain
\[
|G/K| \cdot |H| = \frac{2p^2 \prod_{i=1}^p (2^{2i} - 1)}{2^{(p)} \prod_{i=2}^p (2^i - 1)} = \frac{2^{p(p+1)}}{2} \prod_{i=1}^p (2^i + 1).
\]
On the other hand, \( |G/K| \) divides \( |\text{Out}(K/H)| = 2 \). From this we deduce that \( |H| \) is divisible by \( 2^p + 1 \). Let \( s \in \text{ppd}(2^{2p} - 1) \subseteq \pi(2^p + 1) \) and \( Q \in \text{Syl}_s(H) \). Evidently \( Q \) is a normal subgroup of \( G \) and \( |Q| \) divides \( 2^p + 1 \). Now, it follows from Lemma 2.5 that \( m_2(G) = 2^p - 1 ||Q| - 1 \), which is impossible.

(ii) \( r = 5 \). Assume first that \((5, q-1) = 1\). In this case, we have
\[
\frac{q^5 - 1}{q - 1} = q^4 + q^3 + q^2 + q + 1 = 2^p - 1,
\]
or equivalently
\[
q(q + 1)(q^2 + 1) = 2(2^{p-1} - 1). \tag{3}
\]
If \( q \) is even, then we conclude that \( q = 2 \) and \( r = p = 5 \), and the proof is quite similar as (i). If \( q \) is odd, then the left-hand side of Eq.( 3) is congruent to 0 (mod 4), while the right-hand side of Eq.( 3) is congruent to 2 (mod 4), a contradiction.

Assume next that \((5, q-1) = 5\). In this case, we have
\[
q^4 + q^3 + q^2 + q + 1 = 5(2^p - 1),
\]
or equivalently
\[
(q - 1)(q^3 + 2q^2 + 3q + 4) = 10(2^{p-1} - 1).
\]
In the case when \( q \) is even, one can easily deduce that \( q = 2 \), and so \( 13 = 5(2^{p-1} - 1) \), a contradiction. Moreover, if \( q \) is odd, then from the equality \( q(q + 1)(q^2 + 1) = 5 \cdot 2^p - 6 \) it is easily seen that the left-hand side of this equation is congruent to 0 (mod 4), while the right-hand side is congruent to 2 (mod 4), a contradiction.

(iii) \( r = 3 \). In this case, we have \((q^3 - 1)/(q - 1)(3, q - 1) = 2^p - 1\). First of all, if \( q \) is even, then we obtain \( p = 3 \), which is not the case. Thus, we can assume that \( q \) is odd.
If \((3, q - 1) = 1\), then
\[ q(q + 1) = 2\left(2^{(p-1)/2} - 1\right)\left(2^{(p-1)/2} + 1\right). \tag{4} \]

If \(q\) divides \(2^{(p-1)/2} - 1\), then
\[ q \leq 2^{(p-1)/2} - 1, \quad q + 1 < 2^{(p-1)/2} + 1. \]
Hence, we obtain
\[ q(q + 1) < 2\left(2^{(p-1)/2} - 1\right)\left(2^{(p-1)/2} + 1\right), \]
a contradiction. If \(q\) divides \(2^{(p-1)/2} + 1\), then \(2^{(p-1)/2} + 1 = kq\). Now, from Eq. (4), it follows that
\[ 2k\left(2^{(p-1)/2} - 1\right) = q + 1. \]
When \(k = 1\), we conclude that \(p = 5\) and \(q = 5\). But then, we have
\[ |K/H| = |A_2(5)| = 2^5 \cdot 3 \cdot 5^3 \cdot 31, \]
while \(|G| = |C_5(2)| = 2^{25} \cdot 3^6 \cdot 5^2 \cdot 7 \cdot 11 \cdot 17 \cdot 31; this is a contradiction because |K/H|_5 > |G|_5. If \(k \geq 2\), then \(q \geq 2\left(2^{(p+1)/2} - 2\right) - 1\). Therefore, we have
\[ 2\left(2^{(p+1)/2} - 2\right) - 1 \leq q < q + 1 \leq 2^{(p-1)/2} + 1, \]
a contradiction.
If \((3, q - 1) = 3\), then \(q(q + 1) = 2^2\left(3 \cdot 2^{p-2} - 1\right)\), which implies that \((q + 1)_2 = 4\) and so \((q - 1)_2 = 2\). Moreover, under these conditions, one can easily deduce that \(f\) is odd, otherwise \(8|q - 1\) where
\[ q - 1 = t^f - 1 = (t^{f/2} - 1)(t^{f/2} + 1), \]
which is a contradiction. Thus, we have \(|A_2(q)|_2 = 2^4\), while
\[ |G/K|_2 \cdot |H|_2 = \frac{|G|_2}{|A_2(q)|_2} = 2^{p^2 - 4}. \]
Since \(|G/K|\) divides \(2f(3, q - 1)\) and \(f\) is odd, \(|G/K|_2 \leq 2\). Therefore a Sylow 2-subgroup of \(H\) has order either \(2^{p^2 - 4}\) or \(2^{p^2 - 5}\). Applying Lemma 2.5 we deduce that \(2^p - 1|2^{p^2 - 4} - 1\) or \(2^p - 1|2^{p^2 - 5} - 1\). Now, one can easily check that the second divisibility is possible only for \(p = 5\). But then, we get \(q(q + 1) = 2^2 \cdot 23\), which is a contradiction.

**Step 3.30.** \(K/H \not\cong 2A_r(q), where (q + 1)|r + 1 and (r, q) \neq (3, 3), (5, 2).\)
In this case, we have
\[ |K/H| = |2A_r(q)| = \frac{1}{(r+1)q} q^{r(r+1)/2} \prod_{i=2}^{r+1} (q^i - (-1)^i), \]
and the only odd order component of \(2A_r(q)\) is \((q^r + 1)/(q + 1)\). Therefore, we get
\[(q^r + 1)/(q + 1) = 2^p - 1.\]

An argument similar to that in the previous cases shows that
\[ q^r - 1 > (q^r + 1)/(q + 1) = 2^p - 1, \]
and so \(q^r > 2^p\). Let \(q = t^f\), where \(t\) is a prime and \(f\) is a natural number. We now consider three cases separately.

(i) If \(r \geq 7\). Then \(q^{r(r+1)/2} > q^{3(r+1)} \geq 2^3q^{3r} > 2^{3(p+1)}\), which forces by Lemma 2.9 that \(t = 2\). Thus \((2^{fr} + 1)/(2^f + 1) = 2^p - 1\), and, consequently, \(f = 1\), \(r = 3\) and \(p = 2\), which is a contradiction.

(ii) If \(r = 5\), then \((q^5 + 1)/(q + 1) = 2^p - 1\). Arguing as in the case (i), we conclude that \(t = 2\) and \(f = 1\), whence \(12 = 2^p\), a contradiction.

(iii) If \(r = 3\), then \((q^3 + 1)/(q + 1) = 2^p - 1\). It follows that \(q(q - 1) = 2(2^{p-1} - 1)\), and so \(q = p = 2\), which is impossible.

**Step 3.31.** \(K/H \not\cong 2A_{r-1}(q)\).

In this case, we have
\[ |K/H| = |2A_{r-1}(q)| = \frac{1}{(r,q+1)} q^{r(r-1)/2} \prod_{i=2}^{r} (q^i - (-1)^i), \]
and the only odd order component of \(2A_{r-1}(q)\) is \((q^r + 1)/(q + 1)(r, q + 1)\). Thus
\[ \frac{q^r + 1}{(q + 1)(r, q + 1)} = 2^p - 1, \]
As before, we deduce that \(q^r \geq 2^p\). Let \(q = t^f\), where \(t\) is a prime and \(f\) is a natural number. We now consider three cases separately.

(i) If \(r \geq 7\). It follows that \(q^{r(r-1)/2} \geq q^{3r} > 2^{3p}\), which implies that \(t = 2\) by Lemma 2.9. Now, we obtain
\[ \frac{2^{fr} + 1}{(2^f + 1)(r, 2^f + 1)} = 2^p - 1, \]
which contradicts Lemma 2.1 because \(2^p - 1\) is the largest prime in \(\pi(G)\).
(ii) $r = 5$. In this case we have $q^5 + 1 = (q + 1)(2^p - 1)(5, q + 1)$. Assume first that $q$ is even, that is $q = 2^f$. If $(5, q + 1) = 1$, then we obtain $2^{5f} = 2^{f+p} + 2^p - 2^f - 2$, which is impossible. If $(5, q + 1) = 5$, then $2^{5f} = 5(2^{f+p} + 2^p - 2^f) - 6$, which is again a contradiction. Assume next that $q$ is odd. Noting that $q(q + 1)(2^p - 1)(5, q + 1) - 1$, it is easily seen that the left hand side is congruent to 0 (mod 4), while the right hand side is congruent to 2 (mod 4), a contradiction.

(iii) $r = 3$. In this case, we have $(q^3 + 1)/(q+1)(3, q+1) = 2^p - 1$. If $(3, q+1) = 1$, then we obtain

$$q(q-1) = 2^p - 2 = 2(2^{(p-1)/2} - 1)(2^{(p-1)/2} + 1).$$

If $q$ divides 2, than $p = 2$, a contradiction. If $q$ divides $2^{(p-1)/2} - 1$ or $2^{(p-1)/2} + 1$, then

$$q(q-1) < 2^p - 2 = 2(2^{(p-1)/2} - 1)(2^{(p-1)/2} + 1),$$

a contradiction. Therefore we may assume that $(3, q+1) = 3$. If $q$ is even, then we conclude that $q = 4$, which is a contradiction. We now suppose that $q$ is odd. Since $q(q-1) = 2^2(3 \cdot 2^{p-2} - 1)$, it follows that $(q-1)_2 = 4$, and so $(q+1)_2 = 2$. Moreover, under these hypotheses, one can easily deduce that $f$ is odd, otherwise $8|q - 1 = t^f - 1 = (t^{f/2} - 1)(t^{f/2} + 1)$, which is a contradiction. On the other hand, $|G/K|$ divides $f(3, q + 1)$ and since $f$ is odd, $|G/K|_2 = 1$. Therefore a Sylow 2-subgroup of $H$ has order $2^{p^2-4}$. Again, using Lemma 2.5, we see that $2^p - 1|2^{p^2-4} - 1$, which implies that $p = 2$. This is a contradiction.

**Step 3.32.** $K/H \not\cong C_r(2)$.

The only odd order component of $C_r(2)$ is $2^r - 1$. Thus $2^r - 1 = 2^p - 1$. It follows that $r = p$, $G/K = 1$ and $H = 1$, which means $G \cong C$. This completes the proof of the theorem.

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