MINIMAXNESS PROPERTIES OF EXTENSION FUNCTORS OF LOCAL COHOMOLOGY MODULES

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Abstract. Let $a$ be an ideal of a commutative Noetherian ring $R$ and $M$ an $R$-module. In this paper, it is shown that if $\Ext^i_R(R/a, M)$ is $a$-minimax for all $i \geq 0$, then $M/a^nM$ is $a$-minimax for all $n \geq 0$. Several applications of this result are given. Among other things, we provide a proof of the equivalence of the $a$-minimaxness of the $R$-modules $\Ext^i_R(R/a, M)$, $\Tor^R_i(R/a, M)$ and $H^i(x_1, \ldots, x_t; M)$, for all $i \geq 0$, where $x_1, \ldots, x_t$ are generators for $a$. Using this, we show that, if $b \supseteq a$ is an ideal of $R$ such that $M$ is $b$-minimax and $\text{cd}(b, R) = 1$, then for every finitely generated $R$-module $L$ with $\text{Supp} \ L \subseteq V(b)$, the $R$-modules $\Ext^j_R(L, H^i_a(M))$ are $b$-minimax for all $i$ and $j$. As a consequence, it follows that $H^i_a(M)/b^nH^i_a(M)$ are $b$-minimax $R$-modules for all $i$ and $n$.

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1. Introduction

We continue the study of minimax modules with respect to an ideal $a$ of a commutative Noetherian ring $R$. In [6], H. Zöschinger, introduced the interesting class of minimax modules, and he has in [6] and [7] given many equivalent conditions for a module to be minimax. The $R$-module $M$ is said to be a minimax module, if there is a finitely generated submodule $N$ of $M$, such that $M/N$ is Artinian. The concepts of $a$-minimax and $a$-cominimax modules were introduced in [1] as generalization of minimax and $a$-cofinite modules. We say that an $R$-module $M$ is $a$-minimax if the $a$-relative Goldie dimension of any quotient module of $M$ is finite. Recall that, an $R$-module $M$ is said to have finite Goldie dimension (written $G \dim M < \infty$), if $M$ does not contain an infinite direct sum of non-zero submodules, or equivalently the injective hull $E(M)$ of $M$ decomposes as a finite direct sum of indecomposable (injective) submodules. Also, an $R$-module $M$ is said to have finite $a$-relative Goldie

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dimension if the Goldie dimension of the $\alpha$-torsion submodule $\Gamma_\alpha(M)$ of $M$ is finite. It is known that if $M$ is $\alpha$-torsion, then $M$ is $\alpha$-minimax if and only if $M$ is minimax (see [1, Remark 2.2(ii)]). In addition, we say that an $R$-module $M$ is $\alpha$-cominimax if the support of $M$ is contained in $V(\alpha)$ and $\text{Ext}_R^i(R/\alpha, M)$ is $\alpha$-minimax for all $i \geq 0$.

A brief summary of the contents of this paper will now be given. In Section 2, it is shown that if $M$ is an $\alpha$-cominimax $R$-module, then the $R$-modules $M/\alpha^nM$ are $\alpha$-minimax for all $n \in \mathbb{N}$ (see Theorem 2.3). Several applications of this result are given. Among other things, we provide a proof of the equivalence of the $\alpha$-minimaxness of the $R$-modules $\text{Ext}_R^i(R/\alpha, M)$, $\text{Tor}_R^i(R/\alpha, M)$ and $H^i(x_1, \ldots, x_i; M)$, for all $i \geq 0$, in Theorem 2.7, where $x_1, \ldots, x_i$ are generators for $\alpha$ and $H^i(x_1, \ldots, x_i; M)$ is the $i^{th}$ Koszul cohomology module of $M$ with respect to $x_1, \ldots, x_i$. This theorem is then used to deduce the change of rings principle for $\alpha$-cominimax modules (see Theorem 2.10).

Moreover, in this section by using Theorems 2.3 and 2.7 we show that, if $M$ is an $\alpha$-cominimax $R$-module, then for any finitely generated $R$-module $L$ with support in $V(\alpha)$, the $R$-modules $\text{Ext}_R^i(L, M)$ and $\text{Tor}_R^i(L, M)$ are $\alpha$-minimax, for all $i$. Also, we give a sufficient condition for $\alpha$-cominimax modules, and it is shown that if for an $R$-module $M$ with $\text{Supp} M \subseteq V(\alpha)$, there exists $x \in \sqrt{\alpha}$ such that $0 :_M x$ and $M/xM$ are both $\alpha$-minimax, then $M$ is $\alpha$-cominimax. Finally, we prove that if $b$ is a second ideal of $R$ with $b \supseteq \alpha$, $\text{cd}(b, R) = 1$, and $M$ is a $b$-minimax $R$-module, then for every finitely generated $R$-module $L$ with $\text{Supp} L \subseteq V(b)$, the $R$-modules $\text{Ext}_R^i(L, H^a_\alpha(M))$ are $b$-minimax for all $i$ and $j$, and so the $R$-modules $H^i_\alpha(M)/b^nH^i_\alpha(M)$ are $b$-minimax for all $i$ and $n$.

Throughout this paper, $R$ will always be a commutative Noetherian ring with non-zero identity, and $\alpha$ will be an ideal of $R$. The $i^{th}$ local cohomology module of an $R$-module $M$ with respect to $\alpha$ is defined by

$$H^i_\alpha(M) = \lim_{n \geq 1} \text{Ext}_R^i(R/\alpha^n, M).$$

We refer the reader to [4] or [2] for the basic properties of local cohomology.

2. The results

The purpose of this section is to prove if $\alpha$ is an ideal of a commutative Noetherian ring $R$ and $M$ is an $\alpha$-cominimax module over $R$, then the $R$-modules $M/\alpha^nM$ are $\alpha$-minimax for all $n \in \mathbb{N}$ (see Theorem 2.3). Further, several applications of this result are given.
The following lemmas are needed in the proof of Theorem 2.3

**Lemma 2.1.** Let $M$ be an $R$-module such that $\text{Hom}_R(R/a, M)$ is an $a$-minimax $R$-module. Then $\text{Hom}_R(R/a^n, M)$ is $a$-minimax for all $n \in \mathbb{N}$.

**Proof.** We use induction on $n$. When $n = 1$, there is nothing to prove. Now, let $n > 1$ and suppose that the result has been proved for $n - 1$. Consider the exact sequence

$$0 \longrightarrow 0 : M a \longrightarrow 0 : M a^n \overset{f}{\longrightarrow} a_1(0 : M a^n) \oplus \cdots \oplus a_t(0 : M a^n),$$

where $a = (a_1, \ldots, a_t)$ and $f(x) = (a_1x, \ldots, a_tx)$. As, $a_i(0 : M a^n)$ is a submodule of $0 : M a^{n-1}$, it follows from [1, Proposition 2.3] that $a_i(0 : M a^n)$ is $a$-minimax for all $i = 1, \ldots, t$. Now the result follows from [1, Corollary 2.4 and Proposition 2.3]. □

**Lemma 2.2.** Let $M$ be an $R$-module such that $M/aM$ is $a$-minimax. Then $M/a^nM$ is $a$-minimax for all $n \in \mathbb{N}$.

**Proof.** We use induction on $n$. The case $n = 1$ is true by hypothesis. Now, let $n > 1$ and suppose that the result has been proved for $n - 1$. By [1, Corollary 2.4] and induction hypothesis, $(M/a^{n-1}M)^k$ is $a$-minimax, for all integers $k \geq 0$. Now consider the exact sequence

$$(M/a^{n-1}M)^t \overset{f}{\longrightarrow} M/a^nM \overset{g}{\longrightarrow} M/aM \longrightarrow 0,$$

where $a = (a_1, \ldots, a_t)$ and

$$f(m_1 + a^{n-1}M, \ldots, m_t + a^{n-1}M) = a_1m_1 + \cdots + a_tm_t + a^nM.$$

Therefore, by [1, Proposition 2.3], $M/a^nM$ is $a$-minimax. □

Now, we are prepared to present the following theorem which plays a key role in this paper.

**Theorem 2.3.** Let $M$ be an $R$-module such that $\text{Ext}^i_R(R/a, M)$ is an $a$-minimax $R$-module for all $i \geq 0$. Then $M/a^nM$ is $a$-minimax for all $n \in \mathbb{N}$.

**Proof.** In view of Lemma 2.2, it is enough to prove that $M/aM$ is $a$-minimax. To do this, let $a = (x_1, \ldots, x_n)$. Then

$$M/aM \simeq H^n(x_1, \ldots, x_n; M),$$

where $H^n(x_1, \ldots, x_n; M)$ denotes the $n$th Koszul cohomology module. Consider the co-Koszul complex $K^*(x, M)$ as the following:

$$0 \rightarrow \text{Hom}_R(K_0(x), M) \rightarrow \text{Hom}_R(K_1(x), M) \rightarrow \cdots \rightarrow \text{Hom}_R(K_n(x), M) \rightarrow 0.$$
Then $H^i(x_1, \ldots, x_n; M) = Z^i/B^i$, where $B^i$ and $Z^i$ are the modules of coboundaries and cocycles of the complex $K^•(x, M)$, respectively. Put

$$C = \{ N \mid \text{Ext}_R^i(R/a, N) \text{ is a-minimax for all } i \geq 0 \}.$$ 

By induction we claim that $B^j \in C$ for all $j$. We have $B^0 = 0 \in C$. Now, let $B^j \in C$. Put $C^0 = \text{Hom}_R(K_1(x), M)/B^0$. Since $K_1(x)$ is a finitely generated free $R$-module, it follows from [1, Corollary 2.4] that $\text{Hom}_R(K_1(x), M) \in C$. Now, since $B^j \in C$ and $\text{Hom}_R(K_1(x), M) \in C$, we have $C^j \in C$ by [1, Proposition 2.3]. Hence

$$0 : C^j \simeq \text{Hom}_R(R/a, C^j)$$

is a-minimax. Because of $aH^j(x_1, \ldots, x_n; M) = 0$, it follows that

$$H^j(x_1, \ldots, x_n; M) \subseteq 0 : C^j \ simeq \text{Hom}_R(R/a, C^j),$$

and so $H^j(x_1, \ldots, x_n; M)$ is a-minimax. Consequently, from the short exact sequence

$$0 \rightarrow H^j(x_1, \ldots, x_n; M) \rightarrow C^j \rightarrow B^{j+1} \rightarrow 0$$

and [1, Proposition 2.3] we deduce that $B^{j+1} \in C$. Hence by induction we have proved that $B^j \in C$ for all $j$. Now, since $B^n \in C$ and $\text{Hom}(K_n(x), M) \in C$, we obtain that $C^n \in C$ by [1, Proposition 2.3]. Hence $0 : C^n \simeq \text{Hom}_R(R/a, C^n)$ is a-minimax. Thus $H^n(x_1, \ldots, x_n; M) \subseteq 0 : C^n \simeq \text{Hom}_R(R/a, C^n)$ implies that $H^n(x_1, \ldots, x_n; M)$ is a-minimax. On the other hand, since $M/aM = H^n(x_1, \ldots, x_n; M)$, it follows that $M/aM$ is a-minimax. □

Remark 2.4. We note that if $\dim R = 0$, then each $a$-cominimax $R$-module $M$ is a-minimax. In fact, as $\text{Supp} M \subseteq V(a)$ and $R$ is Artinian, it follows that $M = 0 : M \simeq a^n$, and so $M$ is a-minimax by Lemma 2.1.

In general, we have the following.

Corollary 2.5. Let $M$ be an $a$-cominimax $R$-module. Then $M/a^n M$ is a-minimax for all $n \in \mathbb{N}$.

Proof. The assertion follows from the definition and Theorem 2.3. □

Corollary 2.6. Let $a$ be an ideal of $R$, and let $M$ be an $R$-module such that $H^i_a(M)$ is a-cominimax for all $i$. Then $M/a^n M$ is a-minimax for all $n \in \mathbb{N}$.

Proof. Since $H^i_a(M)$ is a-cominimax for all $i$, in view of [1, Proposition 3.7] the $R$-module $\text{Ext}_R^i(R/a, M)$ is a-minimax for all $i$. Now the result follows from Theorem 2.3. □
The next theorem provides a proof of the equivalence of the $\alpha$-minimaxness of the $R$-modules $\text{Ext}^i_R(R/a, M)$, $\text{Tor}^R_i(R/a, M)$ and $H^i(x_1, \ldots, x_t; M)$, for all $i \geq 0$.

**Theorem 2.7.** Let $a = (x_1, \ldots, x_t)$ be an ideal of $R$, and let $M$ be an $R$-module. Then the following statements are equivalent:

(i) $\text{Ext}^i_R(R/a, M)$ is an $\alpha$-minimax $R$-module for all $i$.
(ii) $\text{Tor}^R_i(R/a, M)$ is an $\alpha$-minimax $R$-module for all $i$.
(iii) The Koszul cohomology modules $H^i(x_1, \ldots, x_t; M)$ are $\alpha$-minimax $R$-modules for all $i$.

**Proof.** (i) $\Rightarrow$ (ii) Let

$$F_\bullet : \cdots \to F_2 \to F_1 \to F_0 \to R/a \to 0$$

be a free resolution of finitely generated $R$-modules for $R/a$. Then it follows that

$$\text{Tor}^R_i(R/a, M) = Z_i/B_i,$$

where $B_i$ and $Z_i$ are the modules of boundaries and cycles of the complex $F_\bullet \otimes_R M$, respectively. Put

$$C = \{ N \mid \text{Ext}^i_R(R/a, N) \text{ is } \alpha\text{-minimax for all } i \geq 0 \}.$$ 

By induction we claim that $Z_i \in C$ for all $i$. We have $Z_0 = F_0 \otimes_R M \in C$. Now let $Z_i \in C$. Consider the exact sequence

$$0 \to C_{i+1} \to Z_i \to \text{Tor}^R_i(R/a, M) \to 0, \quad (\dagger)$$

where $C_i = (F_i \otimes_R M)/Z_i$. Hence we obtain the exact sequence

$$Z_i/aZ_i \to \text{Tor}^R_i(R/a, M) \to 0.$$

Therefore, $\text{Tor}^R_i(R/a, M)$ is a homomorphic image of $Z_i/aZ_i$. Now, since $Z_i \in C$, it follows from Theorem 2.3 that $Z_i/aZ_i$ is $\alpha$-minimax, and so $\text{Tor}^R_i(R/a, M)$ is $\alpha$-minimax. Hence, we deduce from (\dagger) that $C_{i+1} \in C$, and so $Z_{i+1} \in C$. Hence by induction we have proved that $Z_j \in C$ for all $j$. It follows from Theorem 2.3 that $Z_i/aZ_i$ is $\alpha$-minimax for all $i$, and so $\text{Tor}^R_i(R/a, M)$ is $\alpha$-minimax for all $i$.

To prove the implication (ii) $\Rightarrow$ (iii), as

$$H^i(x_1, \ldots, x_t; M) \simeq H_{n-i}(x_1, \ldots, x_t; M),$$

it is sufficient to show that $H_i(x_1, \ldots, x_t; M)$ is $\alpha$-minimax for all $i$. Let $x = x_1, \ldots, x_n$. Consider the Koszul complex

$$K_\bullet(x) : 0 \to K_n(x) \to K_{n-1}(x) \to \cdots \to K_1(x) \to K_0(x) \to 0.$$
Then $H_i(x_1, \ldots, x_t; M) = Z_i/B_i$, where $B_i$ and $Z_i$ are the modules of boundaries and cycles of the complex $K_\bullet(x) \otimes_R M$, respectively. Put

$$\mathcal{C} = \{ N \mid \text{Tor}^R_i(R/a, N) \text{ is } \mathfrak{a}\text{-minimax for all } i \geq 0 \}.$$  

Consider the exact sequence

$$0 \to C_{i+1} \to Z_i \to H_i(x_1, \ldots, x_t; M) \to 0,$$

where $C_i = (K_i(x) \otimes_R M)/Z_i$. Hence we obtain the exact sequence

$$Z_i/aZ_i \to H_i(x_1, \ldots, x_t; M) \to 0.$$  

Now, analogous to the proof of the implication (i) $\implies$ (ii), $Z_i \in \mathcal{C}$ for all $i$. It follows that $Z_i/aZ_i = \text{Tor}^R_i(R/a, Z_i)$ is $\mathfrak{a}$-minimax for all $i$, and so $H_i(x_1, \ldots, x_t; M)$ is $\mathfrak{a}$-minimax for all $i$.

Finally, to prove the implication (iii) $\implies$ (i), let

$$\mathbb{F}_\bullet : \cdots \to F_2 \to F_1 \to F_0 \to R/a \to 0$$

be a free resolution of finitely generated $R$-modules for $R/a$. Then it follows that $\text{Ext}^i_R(R/a, M) = Z^i/B^i$, where $B^i$ and $Z^i$ are the modules of coboundaries and cocycles of the complex $\text{Hom}_R(\mathbb{F}_\bullet, M)$, respectively. Put

$$\mathcal{C} = \{ N \mid H^i(x_1, \ldots, x_t; N) \text{ is } \mathfrak{a}\text{-minimax for all } i \geq 0 \}.$$  

Consider the short exact sequence

$$0 \to \text{Ext}^i_R(R/a, M) \to C^i \to B^{i+1} \to 0,$$

where $C^i = \text{Hom}_R(F_i, M)/B^i$. Then in view of the proof of Theorem 2.3, $B^i \in \mathcal{C}$ for all $i$. Thus $C^i \in \mathcal{C}$ for all $i$. Now, since

$$\text{Ext}^i_R(R/a, M) \subseteq 0 : C^i, \quad a \simeq \text{Hom}_R(R/a, C^i) \simeq H^0(x_1, \ldots, x_t; C^i)$$

and $H^0(x_1, \ldots, x_t; C^i)$ is $\mathfrak{a}$-minimax, we see that $\text{Ext}^i_R(R/a, M)$ is $\mathfrak{a}$-minimax for all $i$. \hfill \Box

The following result is an extension of Theorem 2.3.

**Theorem 2.8.** Let $M$ be an $R$-module such that $\text{Ext}^i_R(R/a, M)$ is an $\mathfrak{a}$-minimax $R$-module for all $i \geq 0$. Then for any finitely generated $R$-module $L$ with support in $V(\mathfrak{a})$, the $R$-modules $\text{Ext}^i_R(L, M)$ and $\text{Tor}^i_R(L, M)$ are $\mathfrak{a}$-minimax for all $i$.  

Proof. Since \( V(\text{Ann}_R L) \subseteq V(\mathfrak{a}) \), there exists \( n \in \mathbb{N} \) such that \( \mathfrak{a}^n L = 0 \). Hence \( \mathfrak{a}^n \operatorname{Ext}_R^i(L, M) = 0 \) and \( \mathfrak{a}^n \operatorname{Tor}_R^i(L, M) = 0 \) for all \( i \). Let

\[
\mathcal{F}_\bullet : \cdots \to F_2 \to F_1 \to F_0 \to L \to 0
\]

be a free resolution of finitely generated \( R \)-modules for \( L \). Then \( \operatorname{Ext}_R^i(L, M) = Z^i/B^i \), where \( B^i \) and \( Z^i \) are the modules of coboundaries and cocycles of the complex \( \operatorname{Hom}_R(\mathcal{F}_\bullet, M) \), respectively. Put

\[
C = \{ N \mid \operatorname{Ext}_R^i(R/\mathfrak{a}, N) \text{ is } \mathfrak{a}\text{-minimax for all } i \geq 0 \},
\]

and consider the short exact sequence

\[
0 \to \operatorname{Ext}_R^i(L, M) \to C^i \to B^{i+1} \to 0,
\]

where \( C^i = \operatorname{Hom}_R(F_i, M)/B^i \). Then in view of the proof of Theorem 2.3 and Lemma 2.1, we have that \( B^i \in C \) for all \( i \). (Note that \( \operatorname{Ext}_R^i(L, M) \subseteq 0 :_C \mathfrak{a}^n \).) Thus \( C^i \in C \) for all \( i \). Hence \( 0 :_C \mathfrak{a} \) is a \( \mathfrak{a} \)-minimax for all \( i \), and so it follows from Lemma 2.1 that \( 0 :_C \mathfrak{a}^n \) is a \( \mathfrak{a} \)-minimax for all \( i \). Now, as \( \operatorname{Ext}_R^i(L, M) \subseteq 0 :_C \mathfrak{a}^n \), it follows that \( \operatorname{Ext}_R^i(L, M) \) is a \( \mathfrak{a} \)-minimax for all \( i \).

Also, we have \( \operatorname{Tor}_R^i(L, M) = Z_i/B_i \), where \( B_i \) and \( Z_i \) are the modules of boundaries and cycles of the complex \( \mathcal{F}_\bullet \otimes_R M \), respectively. Put

\[
C' = \{ N \mid \operatorname{Tor}_R^i(R/\mathfrak{a}, N) \text{ is } \mathfrak{a}\text{-minimax for all } i \geq 0 \}.
\]

In view of Theorem 2.7 and assumption, \( M \in C' \). Consider the exact sequence

\[
0 \to C_i+1 \to Z_i \to \operatorname{Tor}_R^i(L, M) \to 0,
\]

where \( C_i = (F_i \otimes_R M/Z_i) \). As \( \mathfrak{a}^n \operatorname{Tor}_R^i(L, M) = 0 \) for all \( i \), we obtain the exact sequence

\[
Z_i/\mathfrak{a}^n Z_i \to \operatorname{Tor}_R^i(L, M) \to 0.
\]

Now, using the proof of Theorem 2.7((i) \implies (ii)) and Lemma 2.2, we see that \( Z_i \in C \) for all \( i \). Therefore, it follows from Lemma 2.2 that \( Z_i/\mathfrak{a}^n Z_i \) is a \( \mathfrak{a} \)-minimax for all \( i \), and so \( \operatorname{Tor}_R^i(L, M) \) is a \( \mathfrak{a} \)-minimax for all \( i \). \( \square \)

To prove the change of rings principle for cominimaxness, we need to the following lemma. Before presenting it, recall that (cf. [3]), for any ideal \( \mathfrak{a} \) of \( R \) and any \( R \)-module \( M \), the \( \mathfrak{a} \)-relative Goldie dimension of \( M \) is defined as

\[
G \dim_{\mathfrak{a}} M := \sum_{p \in V(\mathfrak{a})} \mu^0(p, M),
\]

where \( \mu^0(p, M) \) denotes the 0-th Bass number of \( M \) with respect to prime ideal \( p \).
Lemma 2.9. Let the ring $T$ be a homomorphic image of $R$, and let $M$ be an $T$-module. Then
\[ G\dim_{aT} M = G\dim_a M. \]
In particular, $M$ is an $aT$-minimax $T$-module if and only if $M$ is an $a$-minimax $R$-module.

Proof. Assume that $T = R/I$ for some ideal $I$ of $R$. Then
\[ \text{Ass}_T M \cap V(aT) = \{ p/I \mid p \in \text{Ass}_R M \cap V(a) \}. \]
On the other hand, for any $p \in \text{Ass}_R M \cap V(a)$ we have
\[ \text{Hom}_{T/\bar{p}}(k(\bar{p}), M_{\bar{p}}) \cong \text{Hom}_{R/p}(k(p), M_p) \]
as $k(p)$-vector spaces, where $\bar{p} = p/I$ and $k(p) = R_p/pR_p$. Therefore $\mu^0(p, M) = \mu^0(p/I, M)$ and this completes the proof. \( \square \)

We are now ready to state and prove the change of rings principle for cominimaxness of modules.

Theorem 2.10. Let the ring $T$ be a homomorphic image of $R$, and let $M$ be an $T$-module. Then $M$ is an $aT$-cominimax as a $T$-module if and only if $M$ is an $a$-cominimax as an $R$-module.

Proof. Assume that $T = R/I$ for some ideal $I$ of $R$. Then we have
\[ \text{Supp}_T M \subseteq V(aT) \text{ if and only if } \text{Supp}_R M \subseteq V(a). \]
Let $a = (x_1, \ldots, x_t)$ and let $\varphi : R \to T$ be the natural epimorphism. As $aT = (\varphi(x_1), \ldots, \varphi(x_t))$, it follows from Theorem 2.7 that $\Ext^i_T(T/aT, M)$ is an $aT$-minimax $T$-module for all $i$ if and only if the Koszul cohomology modules $H^i(\varphi(x_1), \ldots, \varphi(x_t); M)$ are $aT$-minimax $T$-modules for all $i$. But, in view of Lemma 2.9, $H^i(\varphi(x_1), \ldots, \varphi(x_t); M)$ is $aT$-minimax if and only if $H^i(\varphi(x_1), \ldots, \varphi(x_t); M)$ is $a$-minimax. Now the result follows from
\[ H^i(\varphi(x_1), \ldots, \varphi(x_t); M) \cong H^i(x_1, \ldots, x_t; M). \]
and Theorem 2.7. \( \square \)

Theorem 2.11. Let $f : M \to N$ be an $R$-homomorphism such that $\Ext^i_R(R/a, \Ker f)$ and $\Ext^i_R(R/a, \Coker f)$ are both $a$-minimax for all $i$. Then $\Ker \Ext^i_R(id_{R/a}, f)$ and $\Coker \Ext^i_R(id_{R/a}, f)$ are also $a$-minimax for all $i$. 
Proof. The exact sequences

\[ 0 \to \text{Ker} f \to M \xrightarrow{g} \text{Im} f \to 0 \quad \text{and} \quad 0 \to \text{Im} f \xrightarrow{\iota} N \to \text{Coker} f \to 0, \]

where \( \iota \circ g = f \), provides the following two exact sequences

\[ \cdots \to \text{Ext}^i_R(R/a, \text{Ker} f) \to \text{Ext}^i_R(R/a, M) \to \text{Ext}^i_R(R/a, \text{Im} f) \to \cdots \quad (\dagger) \]

and

\[ \cdots \to \text{Ext}^i_R(R/a, \text{Im} f) \to \text{Ext}^i_R(R/a, N) \to \text{Ext}^i_R(R/a, \text{Coker} f) \to \cdots. \quad (\ddagger) \]

Now, since \( \text{Ext}^{i+1}_R(R/a, \text{Ker} f) \) is \( \alpha \)-minimax, it follows from the exact sequence (\dagger) that \( \text{Coker} \text{Ext}^i_R(id_{R/a}, g) \) and \( \text{Ker} \text{Ext}^{i+1}_R(id_{R/a}, g) \) are both \( \alpha \)-minimax for all \( i \).

Also, as \( \text{Ext}^i_R(R/a, \text{Coker} f) \) is \( \alpha \)-minimax, the exact sequence (\ddagger) implies that the \( R \)-modules \( \text{Coker} \text{Ext}^i_R(id_{R/a}, t) \) and \( \text{Ker} \text{Ext}^{i+1}_R(id_{R/a}, t) \) are \( \alpha \)-minimax for all \( i \).

Now, the assertion follows from the exact sequences

\[ 0 \to \text{Ker} \text{Ext}^i_R(id_{R/a}, g) \to \text{Ker} \text{Ext}^i_R(id_{R/a}, f) \to \text{Ker} \text{Ext}^i_R(id_{R/a}, t) \]

\[ \text{Coker} \text{Ext}^i_R(id_{R/a}, g) \to \text{Coker} \text{Ext}^i_R(id_{R/a}, f) \to \text{Coker} \text{Ext}^i_R(id_{R/a}, t) \to 0. \quad \square \]

Corollary 2.12. Let \( M \) be an \( R \)-module with \( \text{Supp} M \subseteq V(a) \). Suppose that \( x \in a \) such that \( 0 :_M x \) and \( M/xM \) are both \( \alpha \)-cominimax. Then \( M \) is also \( \alpha \)-cominimax.

Proof. Put \( f = x1_M \). Then \( \text{Ker} f = 0 :_M x \) and \( \text{Coker} f = M/xM \). Hence in view of Theorem 2.11, the \( R \)-module \( \text{Ker} \text{Ext}^i_R(1_{R/a}, f) \) is \( \alpha \)-minimax. Now, it follows from \( \text{Ext}^i_R(1_{R/a}, f) = 0 \) that \( \text{Ker} \text{Ext}^i_R(1_{R/a}, f) = \text{Ext}^i_R(R/a, M) \). This completes the proof. \( \square \)

Corollary 2.13. Let \( M \) be an \( R \)-module. Suppose that \( x \in \sqrt{a} \) such that \( 0 :_M x \) and \( M/xM \) are both \( \alpha \)-minimax. Then \( \text{Ext}^i_R(R/a, \Gamma_{Rx}(M)) \) is also \( \alpha \)-minimax for all \( i \).

Proof. We have \( x^n \in a \) for some \( n \in \mathbb{N} \). Put \( f = x^n1_{\Gamma_{Rx}(M)} \). Then, we have

\[ \text{Ker} f = 0 :_{\Gamma_{Rx}(M)} x^n \to 0 :_M x^n, \]

and \( \text{Coker} f = \Gamma_x(M)/x^n\Gamma_x(M) \). Now, it follows from the exact sequence

\[ 0 \to \text{Coker} f \to M/x^nM, \]

and Lemma 2.2 that \( M/x^nM \) is \( \alpha \)-minimax. Thus \( \text{Coker} f \) is also \( \alpha \)-minimax. Therefore, in view of [1, Corollary 2.5] and Theorem 2.11, \( \text{Ker} \text{Ext}^i_R(1_{R/a}, f) \) is \( \alpha \)-minimax. But \( x \in \sqrt{a} \) implies that \( \text{Ext}^i_R(1_{R/a}, f) = 0 \), and so

\[ \text{Ker} \text{Ext}^i_R(1_{R/a}, f) = \text{Ext}^i_R(R/a, \Gamma_{Rx}(M)). \]
This completes the proof.

Corollary 2.14. Let $M$ be an $R$-module with support in $V(a)$. Suppose that $x \in \sqrt{a}$ such that $0 :_M x$ and $M/xM$ are both $a$-minimax. Then $M$ is $a$-cominimax.

Proof. The result follows from the Corollary 2.13.

Before bringing the next result we recall that, for an $R$-module $M$, the cohomological dimension of $M$ with respect to an ideal $a$ of $R$ is defined as

$$\text{cd}(a, M) = \sup \{ i \in \mathbb{Z} \mid H^i_a(M) \neq 0 \}.$$ 

Lemma 2.15. Let $\text{cd}(a, R) = 1$, and let $M$ be an $a$-minimax $R$-module. Then $H^i_a(M)$ is $a$-cominimax for all $i$.

Proof. Since $H^0_a(M)$ is a submodule of $M$, it follows that $H^1_a(M)$ is $a$-cominimax.

Also, $\text{cd}(a, R) = 1$ implies that $H^i_a(M) = 0$ for all $i > 1$. Therefore, the result follows from [1, Corollary 3.9].

Lemma 2.16. Let $b$ be an ideal of $R$ with $b \supseteq a$, $\text{cd}(b, R) = 1$, and let $M$ be an $R$-module with $\Gamma_a(M) = 0$. Then

$$H^j_b(H^i_a(M)) \cong \begin{cases} H^i_b(M), & \text{if } j = 0, i = 1, \\ 0, & \text{otherwise.} \end{cases}$$

Proof. The assertion follows from the proof of [5, Proposition 3.15].

Corollary 2.17. Let $b$ be an ideal of $R$ with $b \supseteq a$, $\text{cd}(b, R) = 1$, and $M$ a $b$-minimax $R$-module. Then $H^j_b(H^i_a(M))$ is $b$-cominimax for all $i$ and $j$.

Proof. Since $\text{cd}(b, R) = 1$, it follows from Lemma 2.15 that $H^j_b(\Gamma_a(M))$ is $b$-cominimax for all $j$. Now, let $i > 0$. As $H^j_a(M) \cong H^j_a(M/\Gamma_a(M))$, we may therefore assume that $\Gamma_a(M) = 0$. Thus, the result follows from Lemmas 2.15 and 2.16.

Corollary 2.18. Let $b$ be an ideal of $R$ with $b \supseteq a$, $\text{cd}(b, R) = 1$, and $M$ a $b$-minimax $R$-module. Then for every finitely generated $R$-module $L$ with $\text{Supp} L \subseteq V(b)$, the $R$-modules $\text{Ext}^j_R(L, H^i_a(M))$ are $b$-minimax for all $i$ and $j$. In particular, the $R$-modules $H^j_a(M)/b^n H^j_a(M)$ are $b$-minimax for all $i$ and $n$.

Proof. By Corollary 2.17, $H^j_b(H^i_a(M))$ is $b$-cominimax for all $i$ and $j$. Therefore, it follows from [1, Proposition 3.7] that the $R$-modules $\text{Ext}^j_R(R/b, H^i_a(M))$ are $b$-minimax for all $i$ and $j$. Thus, the result follows from Theorems 2.7 and 2.3.
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