1. Introduction and notation

All rings $R$ considered are commutative, nonzero and unital; all morphisms of rings are unital. Let $R \subseteq S$ be a (ring) extension. The set of all $R$-subalgebras of $S$ is denoted by $[R, S]$. The extension $R \subseteq S$ is said to have FIP (for the “finitely many intermediate algebras property”) if $[R, S]$ is finite. A chain of $R$-subalgebras of $S$ is a set of elements of $[R, S]$ that are pairwise comparable with respect to inclusion. We say that the extension $R \subseteq S$ has FCP (for the “finite chain property”) if each chain of $R$-subalgebras of $S$ is finite. It is clear that each extension that satisfies FIP must also satisfy FCP. If the extension $R \subseteq S$ has FIP (FCP), we will sometimes say that $R \subseteq S$ is an FIP (FCP) extension. Our main tool are the minimal (ring) extensions, a concept introduced by Ferrand-Olivier [10]. Recall that an extension $R \subseteq S$ is called minimal if $[R, S] = \{R, S\}$. The key connection between the above ideas is that if $R \subseteq S$ has FCP, then any maximal (necessarily finite) chain $R = R_0 \subset R_1 \subset \cdots \subset R_n = S$, of $R$-subalgebras of $S$, with length $n < \infty$, results from juxtaposing $n$ minimal extensions $R_i \subset R_{i+1}$, $0 \leq i \leq n-1$. The length of $[R, S]$, denoted by $\ell[R, S]$, is the supremum of the lengths of chains of $R$-subalgebras of $S$. In particular, if $\ell[R, S] = r$, for some integer $r$, there exists a maximal chain $R = R_0 \subset R_1 \subset \cdots \subset R_{r-1} \subset R_r = S$ of $R$-subalgebras of $S$ with length $r$. Against the general trend, we characterized arbitrary FCP and FIP extensions in [8], a joint paper by D. E. Dobbs and ourselves whereas most of papers on the subject are concerned with extensions of integral domains. Note that
other papers by D. E. Dobbs [6], and D. E. Dobbs with P.-J. Cahen, T. G. Lucas [5], J. Shapiro [9], B. Mullins and ourselves [7] also went against the same trend. It is worth noticing here that FCP extensions of integral domains are (ignoring fields) extensions of overrings as a quick look at [5, Theorems 4.1,4.4] shows because FCP extensions are composites of finitely many minimal extensions.

The seminal work on FIP and FCP by R. Gilmer is settled for \( R \)-subalgebras of \( K \) (also called overrings of \( R \)), where \( R \) is a domain and \( K \) its quotient field. In particular, [12, Theorem 2.14] shows that \( R \subseteq S \) has FCP for each overring \( S \) of \( R \) only if \( R/C \) is an Artinian ring, where \( C = (R : R) \) is the conductor of \( R \) in its integral closure. This necessary Artinian condition is not surprisingly present in all our results.

This paper is concerned with \( R \)-modules \( M \) over a ring \( R \) and ring extensions \( R \subseteq R(+)M \), where \( R(+)M \) is the idealization of \( M \). The main results are as follows. Proposition 2.2 shows that \( R \subseteq R(+)M \) has FCP if and only if the length of the \( R \)-module \( M \) is finite, while Proposition 2.4 says that \( R \subseteq R(+)M \) has FIP if and only if \( M \) has finitely many \( R \)-submodules. This leads us to characterize \( R \)-modules having finitely many \( R \)-submodules in Corollary 2.7. An \( R \)-module \( M \), with \( C := (0 : M) \), has finitely many submodules if and only if the three following conditions are satisfied: \( M \) is finitely generated, \( R/C \) has finitely many ideals and \( M_P \) is cyclic for any prime ideal \( P \) of \( R \) containing \( C \) such that \( R/P \) is infinite.

Then Theorem 2.13 gives a structure theorem for these modules that are faithful.

Let \( R \) be a ring. As usual, Spec(\( R \)) (resp Max(\( R \))) denotes the set of all prime ideals (resp. maximal ideals) of \( R \). If \( I \) is an ideal of \( R \), we set \( V_R(I) := \{ P \in \text{Spec}(R) \mid I \subseteq P \} \). If \( R \subseteq S \) is a ring extension and \( P \in \text{Spec}(R) \), then \( S_P \) is the localization \( S_{R \setminus P} \) and \( (R : S) \) is the conductor of \( R \subseteq S \). If \( E \) is an \( R \)-module, \( L_R(E) \) is its length. We will shorten finitely generated module to f.g. module. Recall that a special principal ideal ring (SPIR) is a principal ideal ring \( R \) with a unique nonzero prime ideal \( M = Rt \), such that \( M \) is nilpotent of index \( p > 0 \). Hence a SPIR is not a field. Each nonzero element of a SPIR is of the form \( utk \) for some unit \( u \) and some unique integer \( k < p \). Finally, as usual, \( \subset \) denotes proper inclusion and \( |X| \) denotes the cardinality of a set \( X \).

There are four types of minimal extension, but we only need ramified minimal extensions.

**Theorem 1.1.** \([10, \text{Théorème 2.2}], [16, \text{Theorem 3.3}]\) Let \( R \subseteq T \) be a ring extension and \( M := (R : T) \). Then \( R \subseteq T \) is a ramified minimal extension if and only if \( M \in \text{Max}(R) \) and there exists \( M' \in \text{Max}(T) \) such that \( M'^2 \subseteq M \subseteq M' \). [\( T/M :
$R/M = 2$ (resp. $L_R(M'/M) = 1$), and the natural map $R/M \to T/M'$ is an isomorphism.

**Definition 1.2.** An integral extension $f : R \to S$ is termed *subintegral* if all its residual extensions are isomorphisms and $af$ is bijective [18].

A minimal morphism is ramified if and only if it is subintegral.

According to J. A. Huckaba and I. J. Papick [14], an extension $R \subseteq S$ is termed a $\Delta_0$-extension provided each $R$-submodule of $S$ containing $R$ is an element of $[R,S]$. We recall here for later use an unpublished result of the Gilbert’s dissertation.

**Proposition 1.3.** [11, Proposition 4.12] Let $R \subseteq S$ be a ring extension with conductor $I$ and such that $S = R + Rt$ for some $t \in S$. Then the $R$-modules $R/I$ and $S/R$ are isomorphic. Moreover, each of the $R$-modules between $R$ and $S$ is a ring (and so there is a bijection from $[R,S]$ to the set of ideals of $R/I$).

We will use the following result. If $R_1, \ldots, R_n$ are finitely many rings, the ring $R_1 \times \cdots \times R_n$ localized at the prime ideal $P_1 \times \cdots \times P_n$ is isomorphic to $(R_1)_{P_1}$ for $P_1 \in \text{Spec}(R_1)$. This rule works for any prime ideal of the product.

Rings which have finitely many ideals are characterized by D. D. Anderson and S. Chun [1], a result that will be often used.

**Proposition 1.4.** [1, Corollary 2.4] A commutative ring $R$ has only finitely many ideals if and only if $R$ is a finite direct product of finite local rings, SPIRs, and fields, and these are the localizations of $R$ at its maximal ideals.

Note that if $(R,M)$ is a local Artinian ring, then $R$ is finite if and only if $R/M$ is finite, since $M^n = 0$ for some integer $n$. If $(R,M)$ is an Artinian local ring, we denote by $v(R)$ the nilpotency index of $M$.

From now on, a ring $R$ with finitely many ideals is termed an FMIR.

### 2. Idealizations which are FCP or FIP extensions

Let $M$ be an $R$-module. We consider the ring extension $R \subseteq R(+)M$, where $R(+)M$ is the idealization of $M$ in $R$.

Recall that $R(+)M := \{(r,m) \mid (r,m) \in R \times M\}$ is a commutative ring whose operations are defined as follows:

$$(r,m) + (s,n) = (r + s, m + n) \quad \text{and} \quad (r,m)(s,n) = (rs, rn + sm)$$

Then $(1,0)$ is the unit of $R(+)M$, and $R \subseteq R(+)M$ is a ring morphism defining $R(+)M$ as an $R$-module, so that we can identify any $r \in R$ with $(r,0)$. The following lemma will be useful for all this section.
Lemma 2.1. Let $M$ be an $R$-module, then $R \subseteq R(+)M$ is a subintegral extension with conductor $(0 : M)$.

Proof. If $(r, m) \in R(+)M$, then $(r, m)^2 = 2r(r, m) - r^2(1, 0)$ shows that $R(+)M$ is integral over $R$. Moreover, by [13, Theorem 25.1(3)], $\text{Spec}(R(+)M) = \{ P(+)M \mid P \in \text{Spec}(R) \}$ implies that $R \subseteq R(+)M$ is subintegral.

Let $\nu := R(+)M$ and set $x \in (R : S)$. Then, we have $(x, 0)(0, m) = (0, xm) \in R$ for any $m \in M$, so that $x \in (0 : M)$. Conversely, any $x \in (0 : M)$ gives $x(r, m) = (xr, 0) \in R$ for any $(r, m) \in R(+)M$, which implies $x \in (R : S)$. So, we get $(R : S) = (0 : M)$.

Proposition 2.2. Let $M$ be an $R$-module, then $R \subseteq R(+)M$ has FCP if and only if $L_R(M) < \infty$ and, if and only if $R/(0 : M)$ is Artinian and $M$ is f.g. over $R$.

Proof. Set $S := R(+).M$. Since $R \subseteq S$ is integral, $R \subseteq S$ has FCP if and only if $L_R(S/R) < \infty$ by [8, Theorem 4.2]. By the same reference, this condition is equivalent to $R/(0 : M) \cong R/(R : S)$ is Artinian and $R \subseteq S$ is module finite. Finally, note that $S/R \cong M$; and that $S$ is f.g over $R$ if (and only if) $S/R$ is f.g. over $R$.

For a submodule $N$ of an $R$-module $M$, we denote by $[N, M]$ the set of all submodules of $M$ containing $N$ and set $[M] := [0, M]$. Recall that $M$ is called uniserial if $[M]$ is linearly ordered.

Proposition 2.3. (Dobbs) Let $M$ be an $R$-module, then $R \subseteq R(+)M$ is a $\Delta_0$-extension because $[R, R(+)M] = \{ R(+)N \mid N \in [M] \}$.

Proof. The equality $[R, R(+)M] = \{ R(+)N \mid N \in [M] \}$ was proved by D. E. Dobbs in [6, Remark 2.9] using the bijection $[M] \to [R, R(+)M], N \mapsto R(+)N$.

We say that an $R$-module $M$ is an FMS module if $M$ has finitely many $R$-submodules. An FMS $R$-module $M$ is Noetherian and Artinian and $R/(0 : M)$ is a Noetherian and Artinian ring. We denote by $\nu_R(M)$ (or $\nu(M)$) the number of submodules of an FMS $R$-module $M$. Hence, $\nu(R)$ is the number of ideals of an FMIR $R$.

Proposition 2.4. Let $M$ be an $R$-module, then $R \subseteq R(+)M$ has FIP if and only if $M$ is an FMS module. In this case, $|[R, R(+)M]| = \nu(M)$.

Proof. Set $S := R(+).M$. By Proposition 2.3, it follows that $R \subseteq S$ has FIP if and only if $M$ is an FMS module. In this case, $|[R, R(+)M]| = \nu(M)$.

We now intend to characterize FMS modules by using the previous proposition.
Theorem 2.5. An $R$-module $M$ over a quasi-local ring $(R,P)$ is an FMS module if and only if the next conditions (1) and (2) hold with $C := (0 : M)$:

(1) $M$ is finitely generated, and cyclic when $|R/P| = \infty$.

(2) $R/C$ is an FMIR.

If $M$ is an FMS $R$-module, $(R,P)$ is quasi-local, $|R/P| = \infty$, and $M = Re$ for some $e \in M$, then $M$ is uniserial, $[M] = \{ P^j e \mid j = 0, \ldots , m \}$, with $m := n(R/C) = \nu(R/C) - 1$ and $|[R, R(+)M]| = m + 1$.

Assume in addition that $P = (0 : M)$ and $|R/P| = \infty$. Then $R \subseteq R(+)M$ has FIP if and only if $M$ is simple, if and only if $R \subseteq R(+)M$ is minimal ramified.

Proof. Note that $R$-submodules and $R/C$-submodules of $M$ coincide.

Assume that $M$ is an FMS module. We first prove (1). Then Proposition 2.4 shows that $R \subseteq R(+)M$ has FIP, whence has FCP. We deduce from Proposition 2.2 that $M$ is f.g. and $(R/C, P/C)$ is local Artinian. Assume that $|R/P| = \infty$. Denote by $Re_1, \ldots , Re_n$, with $e_i \in M$, the finitely many cyclic submodules of $M$. Then for any $m \in M$, there is some $i$ such that $Rm = Re_i$, so that $M = \cup_{i=1}^n Re_i$. We can then suppose that $M = \cup_{i=1}^n Rf_i$, where $f_i \in \{ e_1, \ldots , e_n \}$ and the $Rf_i$ are incomparable. If $p = 1$, then $M$ is cyclic. The case $p = 2$ cannot happen because a group cannot be the union of two proper incomparable subgroups. We now show that $p > 2$ leads to a contradiction. Let $\mathcal{F}$ be an (infinite) set of representatives of the non-zero elements of $R/P$. Then, each $\alpha \in \mathcal{F}$ is a unit of $R$. For each $\alpha \in \mathcal{F}$, set $m_\alpha := f_1 + \alpha f_2$. Obviously $m_\alpha \notin Rf_1 \cup Rf_2$. It follows that $m_\alpha \in Rf_i$, for some $i \neq 1, 2$. Let $\alpha, \beta \in \mathcal{F}$, $\alpha \neq \beta$. We claim that $m_\alpha$ and $m_\beta$ are not in the same $Rf_i$. Deny, then $m_\alpha - m_\beta = (\alpha - \beta)f_2 \in Rf_i$ and $\alpha - \beta$ is a unit implies $f_2 \in Rf_i$, a contradiction. Therefore, $M$ is cyclic and (1) is proved.

To prove (2), we consider two cases. If $|R/P| < \infty$, then $|R/C| < \infty$ (see the remark after Proposition 1.4), so that $R/C$ is an FMIR.

Assume that $|R/P| = \infty$. It follows from (1) that $M = Re$ for some $e \in M$, so that $C = (0 : e)$. Set $R' := R/C$, $P' := P/C$ and $I_N := (N : R e)$ for $N \in [M]$. Then, $I_N \in [C, R]$ and is such that $N = I_N e$. Conversely, $I \in [C, R]$ is such that $I = I_{Ie}$ with $Ie \in [M]$, since $C \subseteq I$. We define a preserving order bijective map $\psi : [C, R] \to [M]$ by $I \mapsto Ie$. It follows that $R'$ is an FMIR (either a field or a SPIR) and $\nu(M) = \nu(R/C)$. Then, (2) is proved.

Now, assume that (1) and (2) hold. There is no harm to suppose that $C = 0$ and that $R$ is an FMIR, so that $(R, P)$ is local Artinian. If $|R/P| < \infty$, we get that $|M| < \infty$ and then $M$ is an FMS module. Assume that $|R/P| = \infty$, and that $M = Re$ is cyclic. The assertion is clear if $M = 0$. Assume $M \neq 0$. If $P = 0$, then
\( M \) is a one-dimensional vector space over the field \( R \), so that \( \nu(M) = 2 = \nu(R) \). If \( P \neq 0 \), consider \( S := R(+)M = R + Rf \), where \( f = (0, e) \). From Proposition 1.3 we deduce that \( \| [R, S] \| < \infty \), since \( R \) is an FMIR and also that there are bijective maps \([R] \rightarrow [R, S] \) and \([R, S] \rightarrow [M] \). In fact \( \| [R, S] \| = \{ R(+)N \mid N \in \{ M \} \} \). By Proposition 2.3, \( M \) is an FM module.

Assume that \( M \) is an FMS \( R \)-module, \((R, P)\) is quasi-local, \( |R/P| = \infty \), and \( M = Re \) for some \( e \in M \). If \( R' \) is a SPIR, there is some \( x \in P \), whose class \( \bar{x} \in R' \) is such that \( P' = R\bar{x}, \bar{x}^m = 0 \) and \( \bar{x}^{m-1} \neq 0 \), for \( m := n(R') > 1 \). It follows that \([C, R]\) = \\{ \( P^j + C \mid j \in \{ 0, \ldots, m \} \\} \) and \([M]\) = \\{ \( P^j e \mid j \in \{ 0, \ldots, m \} \\} \) (to see this, use the above bijection \( \psi \)). If \( R' \) is a field, then \( P = C \) gives \( m = 1 \). In both cases, \( M \) is uniserial, \( m := n(R/C) = \nu(R/C) - 1 \) and \( \| [R, R(+)M] \| = m + 1 \).

To end, assume that \((R, P)\) is quasi-local with \( |R/P| = \infty \). Let \( M \) be a simple \( R \)-module, with \( P = (0 : M) \). Then \([R, R(+)M] = \{ R, R(+)M \} \) by Proposition 2.3. It follows that \( R \subseteq R(+)M \) has FIP and is a minimal ramified extension since minimal subintegral. The converse is obvious.

**Example 2.6.** We give this example due to the referee showing that the condition \( |R/P| = \infty \) in Theorem 2.5 is necessary in order to have \( M \) a simple module when \( M \) is an FMS module. Let \( R \) be a finite field, and let \( M := R \oplus R \). Then, \( R \subseteq R(+)M \) has FIP since \( M \) has only finitely many submodules and \( (0 : M) = \{ 0 \} = P \), but \( M \) is not a simple \( R \)-module.

**Corollary 2.7.** Let \( M \) be an \( R \)-module and \( C := (0 : M) \). Then \( M \) is an FMS module if and only if the two following conditions hold:

1. \( M \) is f.g. and \( M_P \) is cyclic over \( R_P \) for all \( P \in V(C) \) such that \( |R/P| = \infty \).
2. \( R/C \) is an FMIR.

In case (1), (2) both hold, set \( \{ P_1, \ldots, P_n \} = V(C) \) and suppose that each \( |R/P_i| = \infty \). Then, for each \( i \), there exist some \( e_i \in M \), such that \( M_{P_i} = R_{P_i} e_i / 1 \) and, \( M \) is generated by the \( e_1, \ldots, e_n \).

**Proof.** If \( M \) is an FMS module, Proposition 2.4 shows that \( R \subseteq R(+)M \) has FIP, and then has FCP. Hence, \( M \) is f.g. and \( R/C \) is Artinian by Proposition 2.2. Let \( P \in V(C) \), then \( M_P \) is an FMS \( R_P \)-module, so that we can use Theorem 2.5. It follows that \( R_P/C_P \cong (R/C)_P \) is an FMIR, and so is \( R/C \), since \( |V(C)| < \infty \), which gives (2). Moreover, for \( P \in V(C) \) with \( |R/P| = \infty \), Theorem 2.5 gives that \( M_P \) is cyclic and (1) holds.

Conversely, if (1) and (2) hold, they also hold for each \( M_P \), where \( P \in V(C) \). Theorem 2.5 gives that \( M_P \) is an FMS module for any \( P \in V(C) \). To show that \( M \)
is an FMS module, there is no harm to suppose that $C = 0$, so that $R$ is Artinian, with $\text{Max}(R) = \{P_1, \ldots, P_n\}$. Now if $N$ is a submodule of $M$, it is well known that $N = \cap_{i=1}^{n} \varphi_i^{-1}(N_{P_i})$, where $\varphi_i : M \to M_{P_i}$ is the natural map and thus $M$ is an FMS module.

Now, assume that (1) and (2) hold and that $|R/P| = \infty$ for any $P \in V(C) = \{P_1, \ldots, P_n\}$. For each $j = 1, \ldots, n$, there is some $e_j \in M$ such that $M_{P_j} = R_{P_j}(e_j/1)$. Set $M' := Re_1 + \cdots + Re_n$. It is easy to show that $M'_{P_j} = M_{P_j}$ for $j = 1, \ldots, n$. Observe that $V(C) = \text{Supp}(M)$, because $M$ is f.g. ([2, Proposition 17, ch. II, p.133]). Now let $P \in \text{Max}(R) \setminus V(C)$. We get that $M'_{P} \subseteq M_P = 0$ and then $M' = M$. \hfill $\Box$

Let $N$ be a submodule of an $R$-module $M$. By Proposition 2.3, $R(+)^N$ is an $R$-subalgebra of $R(+)^M$ and then $R(+)^M$ is an $(R(+)^N)$-algebra. Even if $R \subseteq R(+)^M$ does not have FCP (resp. FIP), it may be that $R(+)^N \subseteq R(+)^M$ has FCP (resp. FIP).

Any $(R(+)^N)$-subalgebra of $R(+)^M$ is an $R$-subalgebra of $R(+)^M$, and then is of the form $R(+)^N'$, for some $N' \in [N, M]$ since $R(+)^N \subseteq R(+)^N'$. Conversely, for any $R$-subalgebra $N'$ of $M$ containing $N$, $R(+)^N'$ is an $(R(+)^N)$-subalgebra of $R(+)^M$. In particular, $R(+)^N \subseteq R(+)^M$ is a minimal extension if and only if $M/N$ is a simple module.

**Proposition 2.8.** Let $N$ be a submodule of an $R$-module $M$. Then:

1. $R(+)^N \subseteq R(+)^M$ is a $\Delta_0$-extension.
2. $R(+)^N \subseteq R(+)^M$ has FCP if and only if $\text{L}_R(M/N) < \infty$. In this case, $\ell([R(+)^N, R(+)^M]) = \text{L}_R(M/N)$.
3. $R(+)^N \subseteq R(+)^M$ has FIP if and only if $M/N$ is an FMS module. In this case, $[[R(+)^N, R(+)^M]] = \nu(M/N)$.

**Proof.** (1) By Proposition 2.3, $R \subseteq R(+)^M$ is a $\Delta_0$-extension. Since an $(R(+)^N)$-submodule $S$ of $R(+)^M$ containing $R$ is also an $R$-submodule of $R(+)^M$, we get that $S$ is a ring, so that $R(+)^N \subseteq R(+)^M$ is a $\Delta_0$-extension.

(2) By Lemma 2.1, $R \subseteq R(+)^M$ is integral and so is $R(+)^N \subseteq R(+)^M$. Therefore, the following conditions are equivalent:
- $R(+)^N \subseteq R(+)^M$ has FCP
- there exists a finite chain of minimal finite extensions going from $R(+)^N$ to $R(+)^M$ ([8, Theorem 4.2(2)])
- there is a finite maximal chain of $R$-submodules of $M$ going from $N$ to $M$
- $\text{L}_R(M/N) < \infty$.  

\vspace{1cm}
In this case, \( \ell[R(+)^N, R(+)^M] = L_R(M/N) \), the supremum of the lengths of chains of submodules of \( M \) containing \( N \).

(3) The following conditions are equivalent:
- \( R(+)N \subseteq R(+)M \) has FIP
- there are finitely many \((R(+)N)\)-subalgebras of \( R(+)M \)
- there are finitely many \( R \)-subalgebras of \( R(+)M \) containing \( R(+)N \)
- there are finitely many \( R \)-submodules of \( M \) containing \( N \)
- \( M/N \) is an FMS module.

In this case, \( ||R(+)N, R(+)M|| \) is also the number of \( R \)-submodules of \( M \) containing \( N \), which is also \( \nu(M/N) \). □

We consider now the special case where \( M \) is an ideal \( I \) of \( R \).

**Proposition 2.9.** Let \( I \) be an ideal of a ring \( R \), \( S := R(+)R \) and \( T := R(+)I \).

Then:
(1) \( R \subseteq S \) has FCP if and only if \( L_R(R) < \infty \) if and only if \( R \) is Artinian. In this case, \( \ell[R, R(+)R] = L_R(R) \).
(2) \( R \subseteq T \) has FCP if and only if \( L_R(I) < \infty \) if and only if \( I \) is finitely generated and \( R/(0 : I) \) is Artinian. In this case, \( \ell[R, R(+)I] = L_R(I) \).
(3) \( R \subseteq S \) has FIP if and only if \( R \) is an FMIR. In this case, \( ||R, R(+)R|| = \nu(R) \).
(4) \( R \subseteq T \) has FIP if and only if \( [I] \) is finite. In this case, \( ||R, R(+)I|| = \nu(I) \).

**Proof.** Propositions 2.2 and 2.8 with \( M \) equal to \( R \) or \( I \) give most of the results because taking \( N = 0 \) gives \( \ell(R(+)^0) \cong R \). □

**Proposition 2.10.** Any f.g. module over a ring \( R \) is an FMS module if and only if \( R \) is a finite ring.

**Proof.** If \( R \) is finite, then \([M]\) is finite for any f.g. \( R \)-module \( M \). Conversely, let \( R \) be a ring such that any f.g. \( R \)-module is an FMS module. Set \( S := R[X,Y]/(X^2, XY, Y^2) = R[x,y] \), where \( x \) and \( y \) are respectively the classes of \( X \) and \( Y \) in \( S \). Then \( S \) is an \( R \)-module with basis \( \{1, x, y\} \). For each \( \alpha \in R \), set \( S_\alpha := R(x + \alpha y) \), which is an \( R \)-submodule of \( S \). If \( \alpha, \beta \in R \) and \( \alpha \neq \beta \), then \( S_\alpha \neq S_\beta \). Therefore, \( |R| = \infty \) gives a contradiction and \( R \) is a finite ring. □

**Remark 2.11.** If \( N \) is a submodule of an \( R \)-module \( M \), Proposition 2.2 shows that \( R \subseteq R(+)M \) has FCP if and only if \( R \subseteq R(+)N \) and \( R \subseteq R(+)M/N \) have FCP. This property does not hold for FIP. It is enough to consider a 2-dimensional vector space \( M \) over an infinite field, and a 1-dimensional subspace \( N \) because \( N \) and \( M/N \) are FMS modules, while \( M \) is not.
Example 2.12. In the following examples, we mix properties of this section and [17, Section 3].

(1) Let $k$ be a field, $n > 1$ an integer, $E$ an $n$-dimensional $k$-vector space with basis $\{e_1, \ldots, e_n\}$ and set $R := k^n$. We can equip $E$ with the structure of an $R$-module by the following law: for $(a_1, \ldots, a_n) \in R$ and $x = \sum_{i=1}^n x_i e_i$, $x_i \in k$, we set $(a_1, \ldots, a_n)x := \sum_{i=1}^n a_i x_i e_i$. Then $E$ is generated over $R$ by $\{e_1, \ldots, e_n\}$ and is faithful, while $R$ is an FMIR. Finally, the prime (maximal) ideals of $R$ are the ideals $P_i := \{(a_1, \ldots, a_n) \in R \mid a_i = 0\}$ for $i = 1, \ldots, n$, so that $RP_i \cong k$. The canonical base $\{e_1, \ldots, e_n\}$ of $R$ over $k$ is such that each $e_i \notin P_i$. We have $e_i e_j = 0$ for each $i, j \in \{1, \ldots, n\}$ such that $i \neq j$, so that $e_j/1 = 0$ in $R_{P_i}$ for $j \neq i$. It follows that $E_{P_i} = \sum_{j=1}^n R_{P_i}(e_j/1) = R_{P_i}(e_i/1)$ is cyclic over $R_{P_i} \cong k$. Then, whatever $|k|$ may be, Corollary 2.7 gives that $E$ is an FMS $R$-module. But, as soon as $|k| = \infty$ and $n \geq 2$, $E$ is infinite. Since $E_{P_i} \cong k(e_i/1)$ is one-dimensional over $k$, $E_{P_i}$ has only two $R_{P_i}$-submodules. Set $F := \prod_{i=1}^n E_{P_i}$ and consider the canonical injective morphism of $R$-modules $\varphi : E \to F$ and the projections $\varphi_i : F \to E_{P_i}$. Any $R$-submodule $N$ of $F$ is of the form $N' := \prod_{i=1}^n N_i$, where $N_i = \varphi_i(N)$, because $N \subseteq N' \subseteq \sum_{i=1}^n e_i N$. Now $\varphi$ is a $k$-isomorphism because $\dim_k(E) = \dim_k(F)$, whence an $R$-isomorphism. It follows that $\nu_R(E) = 2^n$.

By Proposition 2.4, $k^n \subseteq k^n(+E)$ has FIP, and $k \subseteq k^n$ has FIP by [4, Proposition 3, p. 29] (another proof follows from [7, Theorem III.5]). But, always in view of Proposition 2.4, if $|k| = \infty$ and $n \geq 2$, then $k \subseteq k(+E)$ has not FIP, so that $k \subseteq k^n(+E)$ has not FIP.

(1') We keep the context of (1). Set $\mathcal{R} := \prod_{i=1}^n (k/(0 : e_i))$. Since $(0 : e_i) = 0$ for each $i$, we get $\mathcal{R} \cong k^n$. Then $k \subseteq \mathcal{R}$ has FIP while $k \subseteq k(+E)$ has not FIP.

(2) Let $k$ be an infinite field, $n > 1$ an integer and $E$ an $n$-dimensional vector space over $k$. Let $u \in \text{End}(E)$ with minimal polynomial $X^n$. Then, $u^n = 0$ and $u^{n-1}(e_1) \neq 0$ for some $e_1 \in E$. If $e_i := u^{i-1}(e_1)$ for any $i \in \{1, \ldots, n\}$, an easy induction shows that $\{e_1, \ldots, e_n\}$ is a basis of $E$ over $k$. Set $R := k[u]$, then $E$ is a faithful $R$-module with scalar multiplication defined by $P(u) \cdot x := P(u(x))$, for $P(X) \in k[X]$ and $x \in E$. Since $R \cong k[X]/(X^n)$ is a SPIR and $E = R \cdot e_1$ because $e_i = u^{i-1} \cdot e_1$ for each $i$, then by Theorem 2.5, $E$ is an FMS $R$-module and $R \subseteq R(+E)$ has FIP by Proposition 2.4.

(2') Let $R$ be a ring, $n > 1$ an integer and $I_1, \ldots, I_n$ ideals of $R$ distinct from $R$, but not necessarily distinct, such that $\cap_{j=1}^n I_j = 0$. Such a family $\{I_1, \ldots, I_n\}$ of ideals of $R$ is called a separating family, a reference to Algebraic Geometry where a finite family of morphisms $\{f_j : M \to M_j \mid j = 1, \ldots, n\}$ of $R$-modules is
called separating if $\cap_{j=1}^{n} \ker f_j = 0$. In [17, Section 3], we study the ring extension $R \subseteq \prod_{j=1}^{n} (R/I_j) =: \mathcal{R}$ associated to a separating family.

We keep the context of (2). Since $u^n = 0$, $u^{n-1}(e_1) \neq 0$ and $e_j = u^{j-1}(e_1)$ for any $j \in \{1, \ldots, n\}$, a short calculation gives $I_j := (0 :_R e_j) = Ru^{n-j+1}$. Then, $\cap_{j=1}^{n} I_j = 0$ because $I_1 = Ru^n = 0$ and $\{I_1, \ldots, I_n\}$ is a separating family such that $I_j \subseteq I_{j+1}$ for each $j \in \{1, \ldots, n-1\}$. Moreover, $R/I_j = R/Ru^{n-j+1} \cong k[X]/(X^{n-j+1})$. Set $M := Ru$, $\mathcal{R} := \prod_{i=1}^{n} (R/(0 :_R e_i))$ and $J_j := \cap_{k=1,k\neq j}^{n} I_k$. Then, $J_1 = I_2 \cong (X^{n-1})/(X^n)$ and $J_j = 0$ for each $j > 1$. Apply [17, Corollary 3.10]. We have $\sum_{j=1}^{n} J_j = I_2$, giving that $R/ \sum_{j=1}^{n} J_j = R/I_2 \cong k[X]/(X^{n-1})$ is a SPIR and $|R/M| = \infty$, because $R/M \cong k$. Since $I_1 + J_1 = I_2 \cong (X^{n-1})/(X^n)$ and $I_j + J_j = I_j \cong (X^{n-j+1})/(X^n)$ for each $j > 1$, it is enough to take $n > 3$ to get that $R \subseteq \mathcal{R}$ has not FIP.

(3) Let $M = \sum_{i=1}^{n} Re_i$ be a faithful Artinian $R$-module and set $\mathcal{R} := \prod_{i=1}^{n} (R/(0 :_R e_i))$. Since $M$ is faithful, we have $(0 : M) = 0$. Then, $R$ is an Artinian ring in view of [15, Theorem 2, page 180] because $M$ is a finitely generated Artinian module, and $R \subseteq R(+)$ has FCP by Proposition 2.2. Since $(0 : M) = \cap_{i=1}^{n} (0 :_R e_i) = 0$, the family $\{(0 : e_i)\}_{i=1,\ldots,n}$ is separating and $R \subseteq \mathcal{R}$ has FCP by [17, Proposition 3.1].

Examples (1’) and (2’) show that for a finitely generated $R$-module $M = \sum_{i=1}^{n} Re_i$ such that $\{(0 : e_1), \ldots, (0 : e_n)\}$ is a separating family, we may have only one of the two extensions $R \subseteq R(+)$ and $R \subseteq \prod_{i=1}^{n} (R/(0 :_R e_i))$ which has FIP, and not the other one.

(4) Let $k$ be an infinite field, $n > 1$ an integer and $E$ an $n$-dimensional vector space over $k$. Let $u \in \text{End}(E)$ with minimal polynomial $\pi_u(X) := \prod_{i=1}^{n} P_i(X)$, with each $P_i(X) \in k[X]$ of degree 1, $P_i(X) \neq P_j(X)$ for $i \neq j$, and such that $n = \sum_{i=1}^{n} \alpha_i$. For each $i$, set $E_i := \ker(P_i\alpha_i(u))$. The “Lemme des noyaux” [4, Proposition 3, ch. VII, p. 30] gives that $E = \bigoplus_{i=1}^{n} E_i$ (∗), with $\alpha_i = \text{dim}_k(E_i)$. If $R := k[u]$, then $E$ is a faithful $R$-module for the scalar multiplication defined by $P(u) \cdot x := P(u)(x)$, for $P(X) \in k[X]$ and $x \in E$. Since $R \cong k[X]/(\pi_u(X))$ is an Artinian FMIR, we conclude that $E$ is an FMS module over $R$ by applying Corollary 2.7, we need only to show that $E_M$ is cyclic for each $M \in \text{Max}(R) = \{M_1, \ldots, M_s\}$ where $M_i := P_i(u)R$. We next prove that $E_{M_i} \cong (E_i)_{M_i}$ as $R_{M_i}$-modules. Let $x \in E_j$ for some $j \neq i$, then $P_{j\alpha_i}(u)(x) = 0$ and $P_{j\alpha_i}(u)$ is a unit in $R_{M_i}$, since $P_j(X) \notin (P_i(X))$. It follows that $x/1 = 0$ in $E_{M_i}$, so that $E_{M_i} \cong (E_i)_{M_i}$ by (∗). Now, we are reduced to (2) with $P_{j\alpha_i}(u) = 0$ in $(E_i)_{M_i}$, so that each $(E_i)_{M_i}$ is cyclic over $R_{M_i}$ and Corollary 2.7 holds.
Theorem 2.13. A faithful $R$-module $M$ is an FMS module if and only if the two following conditions are satisfied:

1. $R$ is an FMIR which is a direct product of two rings $R' \times R''$, where $|R'| < \infty$ and $|R''/P| = \infty$ for any $P \in \text{Spec}(R'')$.

2. $M$ is the direct product of a finite $R'$-module and a rank one projective $R''$-module.

Proof. If $M$ is an FMS module, $R$ is an FMIR and $M$ is f.g. over $R$ by Corollary 2.7. Then by Proposition 1.4, $R = \prod_{i=1}^{n} R_{i}$, a product of local rings that are either finite, or a SPIR, or a field. Let $R'$ be the ring product of the $R_{i}$ that are finite and $R''$ the product of the others. Then $|R'| < \infty$ and a SPIR factor $(R_{i}, P_{i})$ of $R''$ is such that $|R_{i}/P_{i}| = \infty$ because $R_{i}$ is local Artinian. When $R_{i}$ is an infinite field, take $P_{i} = 0$. So, (1) holds with $R = R' \times R''$.

Set $M' := R'M = \{(r', 0) : r' \in R', \ m \in M\}$ and $M'' := R''M = \{(0, r'') : m \in M\}$. By [3, Remarque 3, ch.II, p.32], we get $M = M' \bigoplus M'' \cong M' \times M''$, $R'M'' = R''M' = 0$ and $(0 :_{R''} M'') = 0$. Clearly, $|M'\}| < \infty$ since $M'$ is f.g. over the finite ring $R'$. In the same way, $M''$ is f.g. over $R''$. Now an $R''$-submodule $N$ of $M''$ gives an $R$-submodule of $M$ by the one-to-one function $N \mapsto M' \times N$. It follows that $M''$ is an FMS $R''$-module. Therefore, we can assume that $R$ is an FMIR with $|R/P| = \infty$ for each $P \in \text{Spec}(R) = \{P_{1}, \ldots, P_{n}\}$. By Corollary 2.7, $M$ is generated over $R$ by some $e_{1}, \ldots, e_{n} \in M$ such that $M_{P_{i}} = R_{P_{i}}(e_{i}/1)$ for each $i$. Actually, $e_{i}/1$ is free over $R_{P_{i}}$: suppose that $(a/t)(e_{i}/1) = 0$ for $a \in R$ and $t \in R \setminus P_{i}$. There is some $s_{i} \in R \setminus P_{i}$ such that $s_{i}ae_{i} = 0$. Moreover, $e_{j}/1 \in M_{P_{i}} = R_{P_{i}}(e_{i}/1)$ for $j \neq i$ gives that $e_{j}/1 = (b_{j}/t_{j})(e_{i}/1)$, for some $b_{j} \in R$, $t_{j} \in R \setminus P_{i}$ for each $j \neq i$. This allows us to pick up some $s_{j} \in R \setminus P_{i}$ such that $s_{j}ae_{j} = 0$. Setting $s := s_{1} \cdots s_{n}$, we get $sa_{e_{k}} = 0$ for each $k \in \{1, \ldots, n\}$. Since $M$ is faithful, $sa = 0$, so that $a/t = 0$. By [2, Théorème 2, ch.II, p.141], $M$ is a rank one projective $R$-module and (2) follows.

Conversely, assume that (1) and (2) hold and keep the above notation with $R = R' \times R''$, $|R'| < \infty$, $|R''/P| = \infty$ for any $P \in \text{Spec}(R'')$ and $M = M' \times M''$, where $M'$ is a finite $R'$-module and $M''$ is a rank one projective $R''$-module. Then, from [2, Théorème 2, ch. II, p. 141], we deduce that $M''$ is f.g. over $R''$, with $M''_{P_{i}}$ cyclic for each maximal ideal $P$ of $R''$. Since $M'$ is also f.g. over $R'$ because finite, $M$ is f.g. over $R$. For each $N \in \text{Max}(R)$ such that $|R/N| = \infty$, there exists $P \in \text{Max}(R'')$ such that $N = R' \times P$ and in this case $M_{N} \cong M''_{P}$ as $R_{N}$-modules. Indeed, consider the $R_{N}$-linear isomorphism $u : M_{N} \cong (M' \times M'')_{R' \times P} \rightarrow M''_{P}$ defined by $u((m', m'')/(s, t)) = m''/t$, using the ring isomorphism $R_{N} \cong R''_{P}$. It
follows that $M_N$ is cyclic over $R_N$. By Corollary 2.7, we can conclude that $M$ is an FMS module.

**Remark 2.14.** (1) For the proof of Theorem 2.13, it was convenient to suppose that $M$ is a faithful $R$-module. However, one should note that Theorem 2.13 can be used to characterize when an arbitrary (not necessarily faithful) module is FMS. In fact, an $R$-module $M$ is FMS (as an $R$-module) if and only if $M$ is an FMS module over the ring $R/(0 : M)$.

(2) The rings $R'$ and $R''$ in the statement of Theorem 2.13 are necessarily each FMIRs. In fact, if $A$ and $B$ are rings, then $A \times B$ is an FMIR if and only if both $A$ and $B$ are FMIRs.

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