WEAKLY REGULAR SEMINEARRINGS

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Abstract. Weakly regular seminearrings are defined and characterized. The space of irreducible ideals is topologized and a sheaf representation is given for a class of distributive left weakly regular seminearrings.

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1. Introduction and Preliminaries

A right seminearring is a set $R$ together with two binary operations “+” and “.” such that $(R, +)$ and $(R, .)$ are semigroups and for all $a, b, c \in R: (a + b)c = ac + bc$ ([5]). A right seminearring $R$ is said to have an absorbing zero 0 if $a + 0 = 0 + a = a$ and $a.0 = 0.a = 0$ hold for all $a \in R$. A non-empty subset $I$ of a seminearring $R$ is called a right (left) ideal if

(i) for all $x, y \in I$, $x + y \in I$ and

(ii) for all $x \in I$ and $r \in R$, $xr(rx) \in I$.

The word ideal will always mean a subset of $R$ which is both a left and a right ideal of $R$. An element $a$ of a seminearring $R$ is called distributive if for all $x, y \in R$, $a(x + y) = ax + ay$; $R$ will be called distributive if each of its element is distributive. A seminearring $R$ is called distributively generated, or d.g. for short, if $R$ contains a multiplicative subsemigroup $D$ of distributive elements which generates $(R, +)$. If $A, B$ are the non-empty subsets of a seminearring $R$, then $AB$ will denote the set of all finite sums of the form $\sum a_kb_k$ with $a_k \in A$ and $b_k \in B$.

In particular, for each $a \in R$, $aR (Ra)$ will denote the set of all finite sums of the form $\sum ar_k (\sum r_k a)$ with $r_k \in R$. Since $R$ is right distributive, $Ra = \{ra : r \in R\}$. Clearly $aR(Ra)$ is a right (left) ideal of $R$.

For any subset $A$ of $R$, $< A >$ will denote the ideal of $R$ generated by $A$. If $A$ and $B$ are ideals of $R$ then the product $AB = \{\sum_{k=1}^{n} a_kb_k : a_k \in A$ and $b_k \in B\}$ is not an ideal. However, if $R$ is distributively generated seminearring then $AB$ is an
ideal of $R$. For ideals $A, B$ of $R$, the sum $A + B$ is defined as the set of all finite sums $\sum (a_k + b_k)$ with $a_k \in A$ and $b_k \in B$.

If $R$ is d.g. seminearring then $A + B$ is the smallest ideal of $R$ containing both $A$ and $B$. More generally, if $\{A_i : i \in I\}$ is an arbitrary family of ideals of a seminearring $R$ with an absorbing zero, then the sum $\sum A_i$ is the set of all finite sums $\sum x_j$ where $x_j = \sum_{i \in I} a_{ij}$ such that $a_{ij} \in A_i$ and $a_{ij} = 0$ for all except finitely many $i \in I$. If $R$ is d.g. seminearring, then $\bigcap_{i \in I} A_i$ is the greatest ideal of $R$ contained in all ideals $\{A_i : i \in I\}$.

Let $R$ be a seminearring with multiplicative identity 1 (i.e. $1 \cdot x = x \cdot 1 = x$ for all $x \in R$). An additive semigroup $(M, +)$ with neutral element zero is called a left $R$-seminearmodule if there exist a function $\alpha : R \times M \to M$ such that if $(r; m)$ is denoted by $rm$, then

(i) $(r_1 + r_2)m = r_1m + r_2m$
(ii) $(r_1r_2)m = r_1(r_2m)$
(iii) $1 \cdot m = m$
(iv) $r0 = 0m = 0$, for all $r_1, r_2, r \in R$ and $m \in M$.

A mapping $\alpha : A \to B$ between left $R$-seminearmodules $A$ and $B$ is a left $R$-homomorphism if

(i) $\alpha (a + a_1) = \alpha (a) + \alpha (a_1)$
(ii) $\alpha (ra) = r\alpha (a)$ for all $a, a_1 \in A$ and $r \in R$.

Generally, the sum of two $R$-homomorphisms is not an $R$-homomorphism. Let $R$ and $L$ be two seminearrings. We shall say that $L$ is an $R$-seminearring if $L$ has the structure of $R$-seminearmodule and $r(xy) = (rx)y$, for all $x, y \in L$ and $r \in R$.

For two $R$-seminearrings $L_1$ and $L_2$, a seminearring homomorphism $f : L_1 \to L_2$ is called a homomorphism of $R$-seminearrings if $f$ is an $R$-homomorphism.

In [3], Brown and McCoy considered the notion of weakly regular rings. These rings were later studied by Ramamurthi [6], [4] and others. In this paper we initiate the study of weakly regular seminearrings. In Section 2, we define and characterize these seminearrings. In Section 3, we construct irreducible spectrum of a d.g. weakly regular seminearring. In Section 4, we prove a representation theorem for distributive left weakly regular seminearrings by sections in a presheaf.
2. Weakly Regular Seminearrings

A ring \( R \) is called left weakly regular if \( x \in (Rx)^2 \), for each \( x \in R \). Adopting this definition, a seminearring \( R \) will be called left weakly regular if for each \( x \in R \), \( x \in (Rx)^2 \). Every seminearring in this paper contains multiplicative identity.

2.1. Theorem. The following assertions for a seminearring \( R \) are equivalent.

(i) \( R \) is left weakly regular.

(ii) Every left ideal of \( R \) is idempotent (i.e. \( A^2 = A \) for every left ideal \( A \) of \( R \)).

Proof. (i) \( \Rightarrow \) (ii) Let \( A \) be a left ideal of \( R \). Clearly \( A^2 \subseteq A \). For the reverse inclusion, let \( x \in A \), then \( x \in (Rx)^2 \). But \( Rx \subseteq A \), so \( (Rx)^2 \subseteq A^2 \Rightarrow x \in A^2 \).

(ii) \( \Rightarrow \) (i) Suppose that every left ideal of \( R \) is idempotent. Let \( x \in R \) then \( x \in Rx \) implies that \( x \in (Rx)^2 \). Thus \( R \) is left weakly regular.

2.2. Proposition. Suppose \( R \) is distributively generated seminearring. Let \( x \in R \) then \( RxR \) is a two sided ideal of \( R \) generated by \( x \).

Proof. Let \( a, b \in RxR \) then \( a = \sum_{finite} s_i xt_i \) and \( b = \sum_{finite} s_i' xt_i' \) where \( s_i, s_i', t_i \) and \( t_i' \in R \).

\[
a + b = \left( \sum_{finite} s_i xt_i + \sum_{finite} s_i' xt_i' \right) \in RxR.
\]

If \( r \in R \) then

\[
ar = ( \sum_{finite} s_i xt_i ) r = \sum_{finite} s_i x(t, r) \in RxR.
\]

Since \( R \) is d.g. so there exist distributive elements \( d_1, d_2, \ldots, d_n \) such that \( r = d_1 + d_2 + \ldots + d_n \). Thus

\[
ra = (d_1 + d_2 + \ldots + d_n)(\sum_{finite} s_i xt_i)
\]

\[
= d_1(\sum_{finite} s_i xt_i) + d_2(\sum_{finite} s_i xt_i) + \ldots + d_n(\sum_{finite} s_i xt_i)
\]

\[
= \sum_{finite} (d_1 s_i) xt_i + \sum_{finite} (d_2 s_i) xt_i + \ldots + \sum_{finite} (d_n s_i) xt_i \in RxR.
\]

Thus \( RxR \) is a two sided ideal of \( R \). As \( R \) contains multiplicative identity, so \( x \in RxR \). If \( A \) is any ideal of \( R \) containing \( x \), then \( sxt \in A \) for all \( s, t \in R \). Also \( \sum_{finite} s_i xt_i \in A \) implies that \( RxR \subseteq A \). Hence \( RxR \) is the two sided ideal of \( R \) generated by \( x \).
2.3. Theorem. The following assertions for a distributively generated seminearring \( R \) are equivalent.

(i) \( R \) is left weakly regular.

(ii) Every left ideal of \( R \) is idempotent.

(iii) For each (two sided) ideal \( I \) of \( R \), \( J \cap I = IJ \), for any left ideal \( J \) of \( R \).

Proof. (i) \( \Leftrightarrow \) (ii) From Theorem 2.1.

(ii) \( \Rightarrow \) (iii) Let \( I \) be an ideal and \( J \) be a left ideal of \( R \). Since \( IJ \subseteq J \) and \( IJ \subseteq I \implies IJ \subseteq J \cap I \). Let \( x \in J \cap I \implies x \in J \) and \( x \in I \). By (ii) \( x \in Rx = (Rx)^2 \) which implies \( x = \sum_{\text{finite}} (r_i)(t_i) = (\sum_{\text{finite}} (r_ixt_i))x \in IJ \). Since \( x \in I \), so \( \sum_{\text{finite}} r_ixt_i \in I \). Thus \( J \cap I \subseteq IJ \). Hence \( J \cap I = IJ \).

(iii) \( \Rightarrow \) (i) Let \( x \in R \) then \( x \in Rx \) and \( x \in RxR \). As \( Rx \) is a left ideal and \( RxR \) is an ideal of \( R \), so by (iii), \( RxR = RxR(Rx) \). As \( x \in Rx \cap RxR = (Rx)(Rx) \), therefore \( x \in (Rx)^2 \). Hence \( R \) is a left weakly regular.

2.4. Proposition. Each ideal of a left weakly regular seminearring is left weakly regular (as a seminearring).

Proof. Let \( J \) be an ideal of a left weakly regular seminearring \( R \). Let \( x \in J \) then \( Jx \) is a left ideal of \( R \) (since \( Jx = \{ jx : j \in J \} \) if \( j_1x \text{ and } j_2x \in Jx \) then \( j_1x + j_2x = (j_1 + j_2)x \in Jx \), if \( r \in R \) then \( r(jx) = (rj)x \in Jx \). By Theorem 2.1, \( (Jx)^2 = Jx \). As \( x \in R \), so \( x \in (Rx)^2 \) that is

\[
x = r_1xt_1 + r_2xt_2 + \ldots + r_nxt_n
\]

\[
= (r_1xt_1 + r_2xt_2 + \ldots + r_nxt_n)x \in Jx, \text{ since } x \in J.
\]

As \( Jx = (Jx)^2 \), so \( x \in (Jx)^2 \). Thus \( J \) is a left weakly regular (as a seminearring).

2.5. Definition. A two sided ideal \( I \) of a seminearring \( R \) is called left pure if for each \( x \in I \) there exists \( y \in I \) such that \( x = yx \). In other words, \( I \) is left pure if and only if for each \( a \in I \) the equation \( a = xa \) has a solution in \( I \).

2.6. Proposition. A distributively generated seminearring \( R \) is left weakly regular if and only if every two sided ideal \( I \) of \( R \) is left pure.

Proof. Suppose \( R \) is a left weakly regular seminearring and \( I \) a two sided ideal of \( R \). Let \( a \in I \). Then \( a \in (Ra)^2 \), that is

\[
a = \sum_{\text{finite}} (r_i)(t_i) = (\sum_{\text{finite}} r_ia_1)a = ya, \text{ where } y = \sum_{\text{finite}} r_ia_1 \in I.
\]

Thus \( I \) is left pure.
Conversely, assume that every ideal of $R$ is left pure. Let $a \in R$, then $RaR$ is a two sided ideal of $R$ generated by $a$. By hypothesis $a \in (RaR)a = (Ra)(Ra) = (Ra)^2$. Thus $R$ is left weakly regular.

2.7. Proposition. For a distributively generated left weakly regular seminearring $R$, the set of all ideals of $R$ (ordered by inclusion) form a complete lattice $\mathcal{L}_R$ under the sum and intersection of ideals with $I \cap J = IJ$ for ideals $I$, $J$ of $R$.

A lattice $\mathcal{L}$ is called Brouwerian if for any $a, b \in \mathcal{L}$; the set of all $x \in \mathcal{L}$ satisfying $a \land x \leq b$ contains a greatest element $c$, the pseudo-complement of $a$ relative to $b$.

A (complete) Brouwerian lattice is distributive.

2.8. Proposition. If $R$ is a distributively generated left weakly regular seminearring, then the lattice $\mathcal{L}_R$ of all ideals of $R$ (ordered by inclusion) is distributive.

Proof. Follows from [2, Proposition 3.3].

2.9. Proposition. Let $R$ be a distributively generated left weakly regular seminearring. For an ideal $P$ of $R$ the following assertions are equivalent.

(i) For ideals $I, J$ of $R$, $I \cap J = P$ implies $I = P$ or $J = P$, 
(ii) $I \cap J \subseteq P \implies I \subseteq P$ or $J \subseteq P$, 
(iii) $< a > \cap < b > \subseteq P \implies a \in P$ or $b \in P$, for any $a, b \in R$.

Proof. (i) $\Rightarrow$ (ii) Suppose $I \cap J \subseteq P$ for ideals $I, J$ of $R$. Then $P = (I \cap J) + P = (I + P) \cap (J + P)$ by Proposition 2.8. Hence by hypothesis $I + P = P$ or $J + P = P$ that is $I \subseteq P$ or $J \subseteq P$.

(ii) $\Rightarrow$ (iii) It is obvious.

(iii) $\Rightarrow$ (i) Suppose $I, J$, are ideals of $R$ containing $P$ properly. Then there exist $a \in I \setminus P$ and $b \in J \setminus P$. By the contrapositivity of (iii), we have $< a > \cap < b > \not\subseteq P$. Hence $I \cap J \neq P$. □

2.10. Definition. An ideal $P$ of $R$ is called irreducible if it is proper (i.e. $P \neq R$) and satisfies one of the equivalent conditions of the above Proposition.

2.11. Proposition. Let $R$ be a distributively generated left weakly regular seminearring. If $I$ is a proper ideal of $R$ and $a \notin I$, then there exist an irreducible ideal $J$ of $R$ such that $I \subseteq J$ and $a \notin J$.

Proof. By Zorn’s Lemma, there exists an ideal $J$ of $R$ which is maximal with respect to the property that $J$ is proper, $I \subseteq J$ and $a \notin J$. Then $J$ is irreducible. For if $J = P \cap L$ but both $P$ and $L$ are properly contain $J$, then $P$ and $L$ are both
contain a. Hence \( a \in P \cap L \). Since \( a \notin J \), this contradicts the assumption that \( J = P \cap L \).

The following is an immediate consequence of the above Proposition.

**2.12. Proposition.** Let \( R \) be a distributively generated left weakly regular seminearring. Then each proper ideal of \( R \) is the intersection of all irreducible ideals which contain it.

**2.13. Definition.** An ideal \( J \) of \( R \) is called a direct summand of \( R \) if there exists an ideal \( J' \), called Cosummand of \( J \), such that \( J + J' = R \) and \( J \cap J' = \{0\} \).

**2.14. Proposition.** Let \( R \) be a distributively generated left weakly regular seminearring. Then the set of direct summands of \( R \) is a Boolean sublattice of \( \mathcal{L}_R \).

### 3. Irreducible spectrum of a distributively generated left weakly regular seminearring

**3.1. Definition.** We denote by \( \mathcal{L}_R \) the lattice of ideals of \( R \) and by \( H(R) \) the set of irreducible ideals of \( R \). For any ideal \( I \) of \( R \), we define

\[
\Theta_I = \{ J \in H(R) : I \nsubseteq J \},
\]

\[
\Im H(R) = \{ \Theta_I : I \in \mathcal{L}_R \}.
\]

In the rest of this section, \( R \) will denote a d.g. left weakly regular seminearring.

**3.2. Theorem.** The set \( \Im(H(R)) \) forms a topology on the set \( H(R) \). Moreover, the assignment \( I \mapsto \Theta_I \) is an isomorphism between the lattice \( \mathcal{L}_R \) of ideals of \( R \) and the lattice of open subsets of \( H(R) \).

**Proof.** First we show that \( \Im(H(R)) \) forms a topology on the set \( H(R) \). Note that \( \Theta_0 = \{ J \in H(R) : (0) \nsubseteq J \} = \emptyset \), since \( (0) \) is contained in every (irreducible) ideal. Thus \( \Theta_0 \) is the empty subset of \( \Im(H(R)) \). On the other hand \( \Theta_R = \{ J \in H(R) : R \nsubseteq J \} = H(R) \). This is true since irreducible ideals are proper. So \( \Theta_R = H(R) \) is an element of \( \Im(H(R)) \). Now let \( \Theta_{I_1}, \Theta_{I_2} \in \Im(H(R)) \) with \( I_1, I_2 \in \mathcal{L}_R \). Then

\[
\Theta_{I_1} \cap \Theta_{I_2} = \{ J \in H(R) : I_1 \nsubseteq J \text{ and } I_2 \nsubseteq J \}
\]

\[
= \{ J \in H(R) : I_1 \cap I_2 \nsubseteq J \}
\]

\[
= \Theta_{I_1 \cap I_2}.
\]

This follows from Proposition 2.9.
Now consider an arbitrary family \( \{ I_\lambda \}_{\lambda \in \Lambda} \) of ideals of \( R \). Then
\[
\bigcup_{\lambda \in \Lambda} \Theta I_\lambda = \bigcup_{\lambda \in \Lambda} \{ J \in H(R) : I_\lambda \nsubseteq J \}
= \{ J \in H(R) : \exists \lambda \in \Lambda \text{ such that } I_\lambda \nsubseteq J \}
= \{ J \in H(R) : \sum_{\lambda} I_\lambda \nsubseteq J \}
= \Theta \sum_{\lambda} I_\lambda.
\]
Since \( \sum_{\lambda} I_\lambda \) is an ideal of \( R \) it follows that \( \bigcup_{\lambda \in \Lambda} \Theta I_\lambda \in \Im(H(R)) \). This shows that \( \Im(H(R)) \) is a topology on \( H(R) \). Define
\[
\Phi : \mathcal{L}_R \rightarrow \Im(H(R))
\]
by setting \( \Phi(I) = \Theta I \).

It is easily verified that \( \Phi \) preserves finite intersection and arbitrary union. Hence \( \Phi \) is a lattice homomorphism. Finally we show that \( \Phi \) is an isomorphism. For this purpose we show that \( \Phi \) is a lattice homomorphism. Finally we show that \( \Phi \) is an isomorphism. For this purpose we show that \( I_1 = I_2 \iff \Theta I_1 = \Theta I_2 \) for \( I_1, I_2 \) in \( \mathcal{L}_R \). Suppose \( \Theta I_1 = \Theta I_2 \). If \( I_1 \neq I_2 \), then \( \exists x \in I_1 \) such that \( x \notin I_2 \). Then by Proposition 2.11, there exists an irreducible ideal \( J \) such that \( I_2 \subseteq J \) and \( x \notin J \). Hence \( I_1 \nsubseteq J \) and so \( J \in \Theta I_1 \). By the assumption \( \Theta I_1 = \Theta I_2 \) so \( J \in \Theta I_2 \). Hence \( I_2 \nsubseteq J \). But this is a contradiction. Hence \( I_1 = I_2 \).

3.3. Definition. The set \( H(R) \) of irreducible ideals of \( R \) will be called irreducible spectrum of \( R \). The topology \( \Im(H(R)) \) in the above Theorem will be called the irreducible spectral topology on \( H(R) \). We shall denote by \( H(R) \) the corresponding topological space. \( H(R) \) will be called irreducible spectral space.

3.4. Proposition. (i) \( H(R) \) is a compact space (but not, in general, Hausdorff).
(ii) For \( I \in \mathcal{L}_R \), \( \Theta I \) is open and closed in \( H(R) \) iff \( I \) is a direct summand of \( R \).

Proof. (i) Suppose \( \bigcup_{\lambda \in \Lambda} \Theta I = H(R) \) be an open covering of \( H(R) \). Then \( \sum_{\lambda} I_\lambda = R \). Since \( 1 \in R \implies 1 = \sum_{\text{finite}} x_i \) where \( x_i = \sum_{\lambda \in \Lambda} a_{\lambda i} \) such that \( a_{\lambda i} \in I_\lambda \) and \( a_{\lambda_i} = 0 \) for all except finitely many \( \lambda \in \Lambda \).

Suppose \( 1 = x_1 + x_2 + \ldots + x_n \) and each \( x_i \) is a sum of \( m_i \) non-zero \( a_{\lambda_i} \) then \( 1 = \sum_{\text{finite}} I_\lambda \) where number of \( I_\lambda \) is not more than \( m_1 + m_2 + \ldots + m_n \). Thus the open cover \( \{ \Theta I : \lambda \in \Lambda \} \) is reducible to a finite subcover. Thus \( H(R) \) is compact.

(ii) Suppose \( \Theta I (I \in \mathcal{L}_R) \in \Im(H(R)) \) is both open and closed. Then there exist \( \Theta J \) with \( J \in \mathcal{L}_R \) so that \( \Theta I \cup \Theta J = H(R) \) and \( \Theta I \cap \Theta J = \emptyset \). This implies that \( I + J = R \) and and \( I \cap J = \{0\} \). Therefore, \( I \) is a direct summand of \( R \).

The following example shows that \( H(R) \) need not be a Hausdorff space.
3.5. Example. Let $R = \{0, x, 1\}$ with the following operations

$$
\begin{array}{c|ccc}
+ & 0 & x & 1 \\
\hline
0 & 0 & x & 1 \\
x & x & x & x \\
1 & 1 & x & 1 \\
\end{array}
\quad
\begin{array}{c|ccc}
\cdot & 0 & x & 1 \\
\hline
0 & 0 & 0 & 0 \\
x & 0 & x & x \\
1 & 1 & 0 & x \\
\end{array}
$$

$R$ is of course a weakly regular seminearring, all of whose ideals are linearly ordered.

$\mathcal{L}_R = \{\{0\}, \{0, x\}, \{0, x, 1\}\}$ and $H(R) = \{\{0\}, \{0, x\}\}$. The spectral space $H(R)$ is clearly not Hausdorff. Note that $H(R) = \{\emptyset, \{0\}, H(R)\}$.

4. Representation of distributive left weakly regular Seminearrings

In this section $R$ will denote a distributive left weakly regular seminearring with multiplicative identity $1$.

4.1. Proposition. Let $I$ and $J$ be ideals of $R$ with $J \subseteq I$. Then any $R$-homomorphism from $J$ to $I$ factors through $J$.

Proof. Let $f : J \to I$ be an $R$-homomorphism. If $a \in J$, then by Proposition 2.6, there exist $x \in J$ such that $a = xa$. Hence $f(a) = f(xa) = xf(a) \in J$, since $x \in J$. \hfill $\square$

4.2. Proposition. For each ideal $I$ of $R$, $I^* = \{\sum_{finite} f_i : f_i \in \text{End}_R(I)\}$ is an $R$-seminearring.

Proof. Clearly $I^*$ is a seminearring with neutral element 0 with respect to pointwise addition and composition of mappings. Define the action of $R$ on $I^*$ by $(r \sum_{finite} f_i)(x) = (\sum_{finite} f_i(x))r$ for all $r \in R$. Now we show that $I^*$ becomes an $R$-seminearmodule. If $f$ is an $R$-endomorphism of $I$ then we show that $rf$ is also an $R$-homomorphism of $I$.

$$(rf)(x + y) = (f(x + y))r = (f(x) + f(y))r = f(x)r + f(y)r = (rf)(x) + (rf)(y), \text{ and}$$

$$(rf)(ax) = (f(ax))r = (af(x))r = a(f(x)r) = a((rf)(x)).$$

Thus $rf$ is an $R$-endomorphism of $I$. Now

$$(r \sum_{finite} f_i)(x) = (\sum_{finite} f_i(x))r = \sum_{finite} f_i(x)r = \sum_{finite} (rf_i)(x)$$
As $r f_i \in \text{End} R(I) \implies r(\sum_{\text{finite}} f_i) \in I^*$. Let $r_1, r_2 \in R$. Then

\[(r_1 + r_2)(\sum_{\text{finite}} f_i)(x) = (\sum_{\text{finite}} f_i(x)(r_1 + r_2)\]

\[= (\sum_{\text{finite}} f_i(x)r_1) + (\sum_{\text{finite}} f_i(x)r_2), \text{ R is distributive}\]

\[= \sum_{\text{finite}} (f_i(x)r_1) + \sum_{\text{finite}} (f_i(x)r_2)\]

\[= \sum_{\text{finite}} (r_1f_i)(x) + \sum_{\text{finite}} (r_2f_i)(x)\]

\[= (\sum_{\text{finite}} (r_1f_i)) + \sum_{\text{finite}} (r_2f_i)(x)\]

Thus

\[(r_1 + r_2)\sum_{\text{finite}} f_i = \sum_{\text{finite}} r_1f_i + \sum_{\text{finite}} r_2f_i\]

\[= r_1(\sum_{\text{finite}} f_i) + r_2(\sum_{\text{finite}} f_i)\]

\[(r_1r_2)(\sum_{\text{finite}} f_i)(x) = (\sum_{\text{finite}} f_i(x))(r_1r_2) = \sum_{\text{finite}} (f_i(x)(r_1r_2)\]

\[= \sum_{\text{finite}} (r_1r_2f_i(x)) = \sum_{\text{finite}} r_1(r_2f_i(x))\]

\[= r_1(\sum_{\text{finite}} r_2f_i(x)) = r_1(\sum_{\text{finite}} r_2 f_i(x))\]

\[= (r_1(r_2(\sum_{\text{finite}} f_i)))(x).\]

Thus

\[(r_1r_2)\sum_{\text{finite}} f_i = r_1(r_2 \sum_{\text{finite}} f_i)\]

\[1.(\sum_{\text{finite}} f_i) = (\sum_{\text{finite}} 1.f_i) = \sum_{\text{finite}} f_i\]

\[r.0 = 0(\sum_{\text{finite}} f_i) = 0\]

So $I^*$ is an $R$-seminearmodule.
Further, let \( r \in R \) and

\[
\sum_{i=1}^{n} f_i, \sum_{j=1}^{m} g_j \in I^*, \text{ then }
\]

\[
r((\sum_{i=1}^{n} f_i)(\sum_{j=1}^{m} g_j)) = r(\sum_{i=1}^{n} \sum_{j=1}^{m} f_i g_j) = \sum_{i=1}^{n} \sum_{j=1}^{m} r(f_i g_j)
\]

\[
= \sum_{i=1}^{n} \sum_{j=1}^{m} (rf_i) g_j = (\sum_{i=1}^{n} rf_i)(\sum_{j=1}^{m} g_j).
\]

\[
\square
\]

4.3. Definition. Let \( X \) be a topological space and \( \mathfrak{S}(X) \) the category of open sets of \( X \) and inclusion maps. A presheaf \( P \) of \( R \)-seminearrings on \( X \) is a contravariant functor from the category \( \mathfrak{S}(X) \) to the category \( M_R \) of \( R \)-seminearrings, that is, it consists of the data:

(a) for every open set \( U \subseteq X \), an \( R \)-seminearring \( P(U) \), and

(b) for every inclusion \( V \subseteq U \) of open sets, an \( R \)-homomorphism

\[
P_{\mu UV}: P(U) \longrightarrow P(V) \text{ subject to the following conditions:}
\]

(i) \( P(\emptyset) = (0) \), where \( \emptyset \) is the empty set,

(ii) \( P_{\mu UV}: P(U) \longrightarrow P(U) \) is the identity map, and

(iii) If \( W \subseteq V \subseteq U \) are three open sets then \( P_{\mu UV} = P_{\nu WV} \circ P_{\mu UV} \): If \( P \) is a presheaf on \( X \), \( P(U) \) is called a section of the presheaf \( P \) on the open set \( U \) and the maps \( P_{\mu UV} \) are called restriction maps, and often the notation \( \alpha|_V \) is used instead of \( P_{\mu UV}(\alpha) \) if \( \alpha \in P(U) \).

4.4. Definition. A presheaf \( P \) on a topological space \( X \) is called a sheaf if the following additional conditions are satisfied.

(iv) If \( U \) is an open set and \( (V_\lambda)_{\lambda \in \Lambda} \) is an open covering of \( U \), and if \( \alpha|_{V_\lambda} = \beta|_{V_\lambda} \) for \( \alpha, \beta \in P(U) \) and for all \( V_\lambda \), then \( \alpha = \beta \).

(v) If \( U \) is an open set and \( (V_\lambda)_{\lambda \in \Lambda} \) is an open covering of \( U \) and if there are elements \( \alpha_\lambda \in P(V_\lambda) \) for \( \lambda \in \Lambda \) with the properties that for each \( \lambda, \mu \in \Lambda \), \( \alpha_\lambda|_{V_\lambda \cap V_\mu} = \alpha_\mu|_{V_\lambda \cap V_\mu} \) then \( \exists \alpha \in P(U) \) such that \( \alpha|_{V_\lambda} = \alpha \) for each \( \lambda \in \Lambda \). If a presheaf satisfies condition (iv) only, it is called separated.

4.5. Theorem. Let \( R \) be a distributive left weakly regular seminearring. For every ideal \( I \) of \( R \) the assignment \( \Theta_I \longrightarrow I^* = P_R(I) \) defines a separated presheaf \( P_R \) of \( R \)-seminearrings on \( H(R) \). The seminearring of the global sections of this presheaf is isomorphic to \( R \).
Proof. First, we prepare the data for the existence of a presheaf. By Proposition 4.2, \( P_R(I) = I^* \) is an \( R \)-seminearring for every ideal \( I \) of \( R \). We need to define a restriction map \( P_{\partial I} : I^* \rightarrow J^* \), \( \Theta_J \subseteq \Theta_I \), that is when \( J \subseteq I \). By Proposition 4.1, if \( f : I \rightarrow I \) is an \( R \)-endomorphisms then \( f|_J : J \rightarrow J \). If \( \sum_{\text{finite}} f_i \in I^* \) then \( P_{\partial I} (\sum_{\text{finite}} f_i) = \sum_{\text{finite}} f_i|_J \). As

\[
P_{\partial I} \left( \sum_{\text{finite}} f_i + \sum_{\text{finite}} g_j \right) = \sum_{\text{finite}} f_i|_J + \sum_{\text{finite}} g_j|_J
\]

\[
= P_{\partial I} \left( \sum_{\text{finite}} f_i \right) + P_{\partial I} \left( \sum_{\text{finite}} g_j \right)
\]

\[
P_{\partial I} \left( \sum_{\text{finite}} f_i \right) \left( \sum_{\text{finite}} g_j \right) = P_{\partial I} \left( \sum_{\text{finite}} f_i g_j \right) = \sum_{\text{finite}} (f_i g_j)|_J
\]

\[
= \sum_{\text{finite}} (f_i|_J)(g_j|_J) = \left( \sum_{\text{finite}} f_i|_J \right) \left( \sum_{\text{finite}} g_j|_J \right)
\]

\[
= P_{\partial I} \left( \sum_{\text{finite}} f_i \right) P_{\partial I} \left( \sum_{\text{finite}} g_j \right).
\]

If \( r \in R \) then

\[
F_{\partial I} \left( r \sum_{\text{finite}} f_i \right) = F_{\partial I} \left( \sum_{\text{finite}} r f_i \right) = \sum_{\text{finite}} (r f_i)|_J
\]

\[
= \sum_{\text{finite}} r (f_i|_J) = r \sum_{\text{finite}} (f_i|_J) = r F_{\partial I} \left( \sum_{\text{finite}} f_i \right).
\]

Thus \( P_{\partial I} \) is a homomorphism of \( R \)-seminearrings. Thus \( P_R \) satisfies the conditions of a presheaf. Thus, we have described the presheaf \( P_R \). In order to show that \( P_R \) is separated, we verify condition \((iv)\) in Definition 4.4. Let \( I = \sum_{\lambda \in \Lambda} I_\lambda \in \mathcal{L}_R \), and suppose \( \sum f_i, \sum g_i \in F_R(I) = I^* \) such that \( (\sum f_i)|I_\lambda = (\sum g_i)|I_\lambda \) for all \( \lambda \in \Lambda \). For each \( x_i \in I \) we have \( x = \sum_{\text{finite}} x_i \), where \( x_i = \sum_{\lambda \in \Lambda} a_{\lambda i} \) such that \( a_{\lambda i} \in I_\lambda \) and \( a_{\lambda i} = 0 \) for all except finitely many \( \lambda \in \Lambda \).

\[
(\sum f_i)(x) = \sum_{\text{finite}} f_i x_i = \sum_{\text{finite}} f_i(\sum x_i) = \sum_{\text{finite}} f_i(x_i)
\]

\[
= \sum_{\text{finite}} \sum_{\lambda \in \Lambda} f_i(\sum a_{\lambda i}) = \sum_{\text{finite}} \sum_{\lambda \in \Lambda} f_i(a_{\lambda i})
\]

\[
= \sum_{\text{finite}} \sum_{\lambda \in \Lambda} \sum g_j(a_{\lambda i}) = (\sum g_j)(x).
\]

Hence \( \sum f_i = \sum g_i \), and so \( P_R \) is separated. Now we show that \( F_R(R) = R^* \cong R \). Define \( h : R^* \rightarrow R \) by \( h(\sum_{\text{finite}} f_i) = \sum_{\text{finite}} f_i(1) \). Then \( h \) is homomorphism of \( R \)-seminearrings. Suppose \( h(\sum_{\text{finite}} f_i) = h(\sum_{\text{finite}} g_i) \). Then \( \sum_{\text{finite}} f_i(1) = \sum_{\text{finite}} g_i(1) \).
Let \( r \in R \),
\[
\sum_{\text{finite}} f_i(r) = \sum_{\text{finite}} f_i(r.1) = \sum_{\text{finite}} rf_i(1) = r \sum_{\text{finite}} f_i(1) = \sum_{\text{finite}} f_i(r.1) = \sum_{\text{finite}} \sum_{\text{finite}} g_i(1) = \sum_{\text{finite}} g_i(r.1) = \sum_{\text{finite}} g_i(r) = \sum_{\text{finite}} g_i.
\]

So \( h \) is 1 - 1. To show that \( h \) is surjective, let \( t \in R \) and define \( t : R \to R \) by \( \alpha_t(r) = rt \). Clearly \( \alpha_t \) is an \( R \)-homomorphism. Hence \( \alpha_t \in R^* \) and \( h(\alpha_t) = \alpha_t(1) = 1.t = t \). Thus \( h \) is also surjective and hence bijective.

\[\Box\]

4.6. Theorem. Let \( R \) be a distributive left weakly regular seminearring all of whose ideals are linearly ordered. For every ideal \( I \) of \( R \), the assignment \( \Theta_I : I^* = PR(I) \) defines a sheaf \( PR \) of \( R \)-seminearrings on \( H(R) \).

The seminearring of the global sections of this sheaf is isomorphic to \( R \).

Proof. We need only to check condition (v) in Definition 4.4. Let \( I = \sum_{\lambda \in \Lambda} I_\lambda \in \mathcal{L}_R \). Consider \( \sum_{\text{finite}} f_\lambda \in I^* \) and \( \sum_{\text{finite}} f_\mu \in I^* \) which coincides on \( I_\lambda \cap I_\mu \).

Since ideals of \( R \) are linearly ordered, \( I_\lambda \subseteq I_\mu \) or \( I_\mu \subseteq I_\lambda \). Hence \( I_\lambda + I_\mu = I_\mu \) or \( I_\lambda + I_\mu = I_\lambda \). Now define \( f : I_\lambda + I_\mu \) by

\[
f(x) = \begin{cases} 
\sum_{\text{finite}} f_\mu(x) & \text{if } I_\lambda + I_\mu = I_\mu \\
\sum_{\text{finite}} f_\lambda(x) & \text{if } I_\lambda + I_\mu = I_\lambda
\end{cases}
\]

Hence, \( f \) is an obvious extension of \( \sum f_\lambda \) and \( \sum f_\mu \). From this it follows that the family \( \{I_\lambda\}_{\lambda \in \Lambda} \) is stable under finite sums. Hence \( f = \sum f_i \) where \( f_i : \sum_{\lambda \in \Lambda} I_\lambda \to \sum_{\lambda \in \Lambda} I_\lambda \) can be defined with no ambiguity. Clearly, \( f \) extends each \( \sum_{\text{finite}} f_k \).

Hence, \( PR \) is a sheaf.

\[\Box\]

References


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