International Electronic Journal of Algebra
Volume 2 (2007) 189-207

DIVISION ALGEBRAS THAT RAMIFY ONLY ON THE ZEROS OF AN ELEMENTARY SYMMETRIC POLYNOMIAL

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Received: 2 March 2007; Revised: 30 May 2007
Communicated by Derya Keskin Tütüncü

Abstract. Let k be an algebraically closed field of characteristic zero. The elementary symmetric polynomial of degree $n-1$ in n variables is a homogeneous polynomial, hence defines both an affine variety in $\mathbb{A}^n_k$ which we denote by C_{n-1} and a projective variety in $\mathbb{P}^{n-1}_k$ denoted V_{n-1}. We describe, up to Brauer equivalence, the central division algebras over the rational function field of $\mathbb{A}^n$ which ramify only on C_{n-1} as well as the central division algebras over the rational function field of $\mathbb{P}^{n-1}$ that ramify only on V_{n-1}. The Brauer group of the cubic surface V_3 in $\mathbb{P}^3$ is computed and is shown to consist solely of Azumaya algebras that are locally trivial in the Zariski topology.

Mathematics Subject Classification (2000): Primary 14F22; Secondary 16K50, 14M99
Key words: Brauer group, division algebra, elementary symmetric polynomial

1. Introduction

Throughout, all cohomology is for the étale topology and all unexplained terminology and notation is as in [10]. Let k be an algebraically closed field of characteristic zero. Fix $n \geq 2$ and $k[x_1, x_2, \dots, x_n]$ be the ring of polynomials in the indeterminates $x_1, x_2, \dots, x_n$ over k. We write $\mathbb{A}^n$ for $\operatorname{Spec} k[x_1, \dots, x_n]$ and $\mathbb{P}^{n-1}$ for $\operatorname{Proj} k[x_1, \dots, x_n]$. Since k is algebraically closed, the Brauer groups $\mathrm{B}(\mathbb{P}^{n-1})$ and $\mathrm{B}(\mathbb{A}^n)$ are trivial.

For each d and m such that $1 \leq d \leq m \leq n$, let $\sigma_d(x_1, x_2, \dots, x_m)$ be the elementary symmetric polynomial of degree d in the m variables $x_1, x_2, \dots, x_m$. So $\sigma_d(x_1, \dots, x_m)$ is equal to the coefficient of λ^{m-d} in the polynomial $(\lambda + x_1)(\lambda + x_2) \dots (\lambda + x_m)$. The $d = 1$ and $d = m$ cases are

$$\begin{aligned} \sigma_1(x_1, \dots, x_m) &= x_1 + \dots + x_m \\ \sigma_m(x_1, \dots, x_m) &= x_1 \cdots x_m \end{aligned} \tag{1}$$

and for $1 < d < m$ the recurrence relation

$$\sigma_d(x_1, \dots, x_m) = \sigma_d(x_1, \dots, x_{m-1}) + x_m \sigma_{d-1}(x_1, \dots, x_{m-1}) \tag{2}$$

is satisfied. Denote by C_d the subvariety of $\mathbb{A}^n$ defined by the zeros of $\sigma_d(x_1, x_2, \dots, x_m)$. We denote by V_d the projective variety in $\mathbb{P}^{n-1}$ defined by the homogeneous polynomial $\sigma_d(x_1, x_2, \dots, x_m)$. Our objective is to study the Brauer group functor on the varieties C_d, $\mathbb{A}^n - C_d$, V_d, and $\mathbb{P}^{n-1} - V_d$.

If X is a variety, with field of rational functions $K = K(X)$, the Brauer group of X classifies the Azumaya algebras defined over the sheaf of regular functions $\mathcal{O}_X$. The group $\mathrm{B}(K/X)$ is defined by the exact sequence

$$0 \to \mathrm{B}(K/X) \to \mathrm{B}(X) \xrightarrow{\eta} \mathrm{B}(K)$$

where η is the natural map. If X is nonsingular, then $\mathrm{B}(K/X) = 0$. In a rough sense, we divide the theory into the study of the image and kernel of η. To understand the image of η we try to study those Azumaya algebras that have nontrivial generic stalk. These algebras classes are represented over K by division algebras. The kernel of η however, consists of Azumaya algebras that are split on some Zariski open subset of X. Different methods tend to be employed when approaching these two problems.

In the context of this article, the image of η is made up of classes of division algebras Λ that are central over the rational function field $k(x_1, \dots, x_n)$. Given any prime divisor $D \subseteq \mathbb{A}^n$, we can measure the ramification of the algebra Λ at D (see [11, Chapter 10]). There are at most finitely many prime divisors where the ramification of Λ is nontrivial. In Section 2 we study the affine varieties C_{n-1} of degree $n-1$. The Brauer group $\mathbb{A}^n - C_{n-1}$ parametrizes those central division algebras over $k(x_1, \dots, x_n)$ that ramify only on the divisor C_{n-1}. It follows from [3, Theorem 3.1] that the Brauer group of $\mathbb{A}^n - C_{n-1}$ is isomorphic to the subgroup of torsion elements in the class group $\mathrm{Cl}(C_{n-1})$. We prove in Theorem 2.1 that this group is cyclic of order two.

The projective varieties V_{n-1} of degree $n-1$ are studied in Section 3. The Brauer group $B(\mathbb{P}^{n-1} - V_{n-1})$ parametrizes the central division algebras Λ over the rational function field of $\mathbb{P}^{n-1}$ that ramify only on V_{n-1}. This group is described in Theorem 3.1. Though the Brauer group and Picard group of the affine cone C_{n-1} are trivial by Lemma 1.3, for the projective variety V_{n-1} the theory is much richer. For $n = 4$, the groups $\mathrm{B}(V_3)$ and $\mathrm{Pic}(V_3)$ are computed in Theorem 3.3. It turns out that the Brauer group of V_3 consists entirely of Azumaya algebras that are split by a finite Zariski open cover of V_3.

The bulk of this article is contained in Section 2 and Section 3 where we are concerned with the varieties of degree $n-1$. In Section 4 we compile some results on the varieties of degree different from $n-1$. As quadric hypersurfaces are fairly well known, Section 4.1 is included mostly for completeness' sake. It seems that the varieties defined by elementary symmetric polynomials have not been mentioned too much in the literature. Hopefully the few results presented here will show that this interesting class of varieties is nontrivial but accessible and deserving of more attention.

Before leaving Section 1 we compute some of the geometric properties of the varieties C_d and V_d.

Lemma 1.1. *Assume $3 \leq m \leq n$ and $1 \leq d \leq m-1$. Then*

1. *C_d is an irreducible rational hypersurface of degree d in $\mathbb{A}^n$.*
2. *V_d is an irreducible rational hypersurface of degree d in $\mathbb{P}^{n-1}$.*

Proof. We see in (2) that $\sigma_d(x_1, \dots, x_m)$ is linear in the variable x_m. Since the linear polynomial $\sigma_1(x_1, \dots, x_{m-d+1})$ is irreducible, a finite induction argument proves that $\sigma_d(x_1, \dots, x_m)$ is irreducible. Therefore both C_d and V_d are irreducible hypersurfaces. The affine coordinate ring of C_d is the integral domain

$$\mathcal{O}(C_d) = \frac{k[x_1, \dots, x_n]}{(\sigma_d(x_1, \dots, x_m))} = \frac{k[x_1, \dots, x_n]}{(\sigma_d(x_2, \dots, x_m) + x_1 \sigma_{d-1}(x_2, \dots, x_m))}$$

where the second equation follows from (2) and symmetry. If we invert $\sigma_{d-1}(x_2, \dots, x_m)$, then there is an isomorphism

$$\mathcal{O}(C_d)[\sigma_{d-1}(x_2, \dots, x_m)^{-1}] \cong k[x_2, \dots, x_n][\sigma_{d-1}(x_2, \dots, x_m)^{-1}] \tag{3}$$

defined by the map

$$x_1 \mapsto -\sigma_d(x_2, \dots, x_m)\sigma_{d-1}(x_2, \dots, x_m)^{-1}.$$

This shows C_d is birational to $\mathbb{A}^{n-1}$. In a similar way one can localize to see that V_d is birational to $\mathbb{A}^{n-2}$. □

The next lemma shows that the varieties V_d in general have singularities but are nonsingular in codimension one. For the $d = n-1$ case, a sharper description of the singular locus is provided by Theorem 2.1.

Lemma 1.2. *Keep the same notation as above but assume $m = n$.*

1. *If $d = 2$ and $n \geq 3$, then V_d is a nonsingular hypersurface in $\mathbb{P}^{n-1}$.*
2. *V_d is nonsingular in codimension one.*
3. *If $d > 2$ and $n \geq 5$, then V_d is a singular hypersurface in $\mathbb{P}^{n-1}$.*

Proof. Let Y_i denote the closed subset of V_d where $\sigma_{d-1}(x_1,\dots,\hat{x_i},\dots,x_n)$ vanishes. We see from (3) that V_d is nonsingular on the open complement of V_i. Then

$$\operatorname{Sing} V_d \subseteq \bigcap_{i=1}^{n} Y_i. \tag{4}$$

If $d=2$, then Y_i is defined by the linear equation $x_1+\cdots+x_n = x_i$. The intersection in (4) is contained in the intersection of n hyperplanes in $\mathbb{P}^{n-1}$ in general position, hence is empty. This proves part 1.

For the remainder of the proof assume $d \geq 3$ and $n \geq d+2$. The case $d=2$ follows from part 1 and we prove the $d = n-1$ case in Theorem 2.1. Let

$$f_1 = \sigma_{d-1}(x_2,\dots,x_n) = \sigma_{d-1}(x_3,\dots,x_n) + x_2\sigma_{d-2}(x_3,\dots,x_n)$$

and

$$f_2 = \sigma_{d-1}(x_1,x_3,\dots,x_n) = \sigma_{d-1}(x_3,\dots,x_n) + x_1\sigma_{d-2}(x_3,\dots,x_n).$$

For each i, $W_i = Z(f_i)$ is an irreducible hypersurface in $\mathbb{P}^{n-1}$. Let $I_i = (f_i)$ be the ideal for W_i in $k[x_1,\dots,x_n]$. Since neither W_1 nor W_2 contains the other, $W_1 \cap W_2$ has codimension at least two in $\mathbb{P}^{n-1}$. Now let $f = \sigma_d(x_1,\dots,x_n)$. We prove that f is not in the radical of $I_1 + I_2$. Maybe this is not clear, so a proof is given. Suppose for sake of contradiction, that $r > 0$ and $f^r = af_1 + bf_2$ for some a, b in $k[x_1,\dots,x_n]$. Using (2) twice,

$$\begin{aligned} f = \sigma_d(x_1,\dots,x_n) &= \sigma_d(x_2,\dots,x_n) + x_1 f_1 \\ &= \sigma_d(x_3,\dots,x_n) + x_2\sigma_{d-1}(x_3,\dots,x_n) + x_1 f_1. \end{aligned}$$

For any polynomial g in $k[x_1,\dots,x_n]$, write g' for the polynomial in $k[x_3,\dots,x_n]$ obtained by substituting $x_1 = x_2 = 0$. Then $(f')^r = a'f_1' + b'f_2'$, hence

$$\begin{aligned} \sigma_d(x_3,\dots,x_n)^r &= a'\sigma_{d-1}(x_3,\dots,x_n) + b'\sigma_{d-1}(x_3,\dots,x_n) \\ &= (a'+b')\sigma_{d-1}(x_3,\dots,x_n). \end{aligned} \tag{5}$$

Since $d \geq 3$, $\sigma_{d-1}(x_3,\dots,x_n)$ is irreducible of degree $d-1 \geq 2$. If $n > d+2$, then $\sigma_d(x_3,\dots,x_n)$ is irreducible of degree $d \geq 3$. If $n = d+2$, the left hand side of (5) is a monomial. In either case (5) is a contradiction and we know that $W_1 \cap W_2$ is not contained in the irreducible hypersurface V_d. It follows that $W_1 \cap W_2 \cap V_d$ has codimension at least two in V_d. This together with (4) proves part 2.

Now let $\{\alpha,\beta,\gamma,\delta\}$ be any 4-set in $\{1,\dots,n\}$ and let $I_{\alpha,\beta,\gamma,\delta}$ denote the ideal generated by $x_\alpha, x_\beta, x_\gamma, x_\delta$ in $k[x_1,\dots,x_n]$. Using (2) and symmetry one sees that $\sigma_d(x_1,\dots,x_n)$ is in $I_{\alpha,\beta,\gamma,\delta}$. Let $Z_{\alpha,\beta,\gamma,\delta}$ denote the set of zeros of $I_{\alpha,\beta,\gamma,\delta}$ in $\mathbb{P}^{n-1}$. This shows $V_d \supseteq Z_{\alpha,\beta,\gamma,\delta}$. For each 3-set $\{\alpha,\beta,\gamma\}$ in $\{1,\dots,n\}$, consider the ideal

$I_{\alpha,\beta,\gamma}$ in $k[x_1,\dots,x_n]$ generated by $x_\alpha, x_\beta, x_\gamma$. Using (2) for $d-1$ and $n-1$ one sees that $\sigma_{d-1}(x_1,\dots,\hat{x_i},\dots,x_n)$ is in $I_{\alpha,\beta,\gamma}$ provided $i \notin \{\alpha,\beta,\gamma\}$. Since the intersection of any $n-1$-set and any 4-set contains at least a 3-set, it follows that $\sigma_{d-1}(x_1,\dots,\hat{x_i},\dots,x_n)$ is always in $I_{\alpha,\beta,\gamma,\delta}$. Then $I_{\alpha,\beta,\gamma,\delta}$ contains each first partial derivative of $\sigma_d(x_1,\dots,x_n)$. This proves

$$\operatorname{Sing} V_d \supseteq \bigcup_{\binom{n}{4}} Z_{\alpha,\beta,\gamma,\delta}. \tag{6}$$

The right-hand side of (6) is a subset of $\mathbb{P}^{n-1}$ of dimension $(n-1)-4$, hence is non-empty. This proves part 3. □

For the record, we state

Lemma 1.3. *In the context of Lemma 1.1,* $\mathrm{B}(C_d)=0$, $\operatorname{Pic}(C_d)=0$ *and* $\operatorname{Pic}(\mathbb{A}^n - C_d)=0$.

Proof. Follows directly from [3, Proposition 1.2]. □

2. The Affine Variety of Degree $n-1$ in $\mathbb{A}^n$

Fix $n \geq 3$ and let $\{x_1,x_2,\dots,x_n\}$ be a set of indeterminates. For each m such that $1 < m \leq n$ let $\sigma_{m-1}(x_1,x_2,\dots,x_m)$ denote the elementary symmetric polynomial of degree $m-1$ in the variables $x_1,x_2,\dots,x_m$. Then

$$\begin{aligned}\sigma_1(x_1,x_2) &= x_1+x_2\\ \sigma_2(x_1,x_2,x_3) &= x_1x_2 + x_3\sigma_1(x_1,x_2)\end{aligned} \tag{7}$$

and recursively

$$\sigma_{m-1}(x_1,x_2,\dots,x_m) = x_1\cdots x_{m-1} + x_m\sigma_{m-2}(x_1,\dots,x_{m-1}) \quad \text{for } 1 < m \leq n. \tag{8}$$

A non-recursive formula for $\sigma_{m-1}(x_1,x_2,\dots,x_m)$ is

$$\sigma_{m-1}(x_1,x_2,\dots,x_m) = \frac{x_1\cdots x_m}{x_1} + \cdots + \frac{x_1\cdots x_m}{x_m}. \tag{9}$$

As in Lemma 1.1, let $V_{m-1} = Z(\sigma_{m-1}(x_1,x_2,\dots,x_m))$ be the projective variety in $\mathbb{P}^{n-1}$ defined by $\sigma_{m-1}(x_1,x_2,\dots,x_m)=0$ and $C_{m-1} = Z\bigl(\sigma_{m-1}(x_1,x_2,\dots,x_m)\bigr)$ the affine variety in $\mathbb{A}^n$ defined by $\sigma_{m-1}(x_1,x_2,\dots,x_m)=0$. The affine coordinate ring of $\mathbb{A}^n - C_{m-1}$ is

$$\mathcal{O}(\mathbb{A}^n - C_{m-1}) = k[x_1,x_2,\dots,x_n][\sigma_{m-1}(x_1,x_2,\dots,x_m)^{-1}].$$

Theorem 2.1. *Let $3 \leq m \leq n$. In the above context,*

1. *The singular locus of C_{m-1} has pure codimension two.*
2. $\operatorname{Cl}(C_{m-1}) \cong \mathbb{Z}/2 \oplus \mathbb{Z}^{(r)}$ *where* $r = \binom{m-1}{2} - 1$.

3. $\mathrm{B}(\mathbb{A}^n - C_{m-1})$ *is cyclic of order two.*
4. $\mathrm{B}(\mathbb{A}^n - C_{m-1})$ *is generated by the algebra class of the following product of symbol algebras*

$$\big(\sigma_{m-1}(x_1, x_2, \dots, x_m), \sigma_{m-2}(x_1, x_2, \dots, x_{m-1})\big)_2 \\ \big(\sigma_{m-2}(x_1, x_2, \dots, x_{m-1}), \sigma_{m-3}(x_1, x_2, \dots, x_{m-2})\big)_2 \cdots \\ \big(\sigma_2(x_1, x_2, x_3), \sigma_1(x_1, x_2)\big)_2.$$

Proof of Part 1: Let $3 \le m \le n$. Let (α, β, γ) be a triple such that $1 \le \alpha < \beta < \gamma \le m$. For each such triple, consider the ideal $I_{\alpha,\beta,\gamma} = (x_\alpha, x_\beta, x_\gamma)$ in $k[x_1, \dots, x_n]$. Then

$$\sigma_{m-1}(x_1, \dots, x_m) = x_2 \cdots x_m + x_1 \sigma_{m-2}(x_2, \dots, x_m) \tag{10}$$

shows that $\sigma_{m-1}(x_1, \dots, x_m)$ is in $I_{1,2,3}$. By symmetry, $\sigma_{m-1}(x_1, \dots, x_m)$ is in $I_{\alpha,\beta,\gamma}$ for all such (α, β, γ). Therefore $C_{m-1} \supseteq Z(I_{\alpha,\beta,\gamma})$. Likewise

$$\sigma_{m-2}(x_2, \dots, x_m) = x_3 \cdots x_m + x_2 \sigma_{m-3}(x_3, \dots, x_m) \tag{11}$$

shows that $\sigma_{m-2}(x_2, \dots, x_m)$ is in $I_{1,2,3}$. By symmetry, $\sigma_{m-2}(x_1, \dots, \hat{x}_i, \dots, x_m)$ is in $I_{\alpha,\beta,\gamma}$ for all such (α, β, γ). Since the partial derivative of $\sigma_{m-1}(x_1, \dots, x_m)$ with respect to x_i is $\sigma_{m-2}(x_1, \dots, \hat{x}_i, \dots, x_m)$, $I_{\alpha,\beta,\gamma}$ contains all the first partial derivatives of $\sigma_{m-1}(x_1, \dots, x_m)$. Therefore $Z(I_{\alpha,\beta,\gamma})$ is a subset of the singular locus, $\operatorname{Sing}(C_{m-1})$. This shows

$$\operatorname{Sing}(C_{m-1}) \supseteq \bigcup_{1 \le \alpha < \beta < \gamma \le m} Z(I_{\alpha,\beta,\gamma}). \tag{12}$$

Next we show the reverse inclusion to (12). To do this, let $P = (a_1, \dots, a_n)$ be a point in $\mathbb{A}^n$ such that $P \in \operatorname{Sing}(C_{m-1})$. We show that at least three of the coordinates $a_1, \dots, a_m$ are zero. By symmetry, it is enough to assume $a_3 \cdots a_m \neq 0$ and argue until a contradiction is reached. Since the first partial derivative with respect to x_1 vanishes at P,

$$0 = \sigma_{m-2}(a_2, \dots, a_m) = a_3 \cdots a_m + a_2 \sigma_{m-3}(a_3, \dots, a_m) \tag{13}$$

implies $a_2 \neq 0$. Since $P \in C_{m-1}$,

$$0 = \sigma_{m-1}(a_1, \dots, a_m) = a_2 \cdots a_m + a_1 \sigma_{m-2}(a_2, \dots, a_m) \tag{14}$$

together with (13) leads to the contradiction $a_2 = 0$. Hence

$$\operatorname{Sing}(C_{m-1}) = \bigcup_{1 \le \alpha < \beta < \gamma \le m} Z(x_\alpha, x_\beta, x_\gamma). \tag{15}$$

If $Z_{\alpha,\beta,\gamma}$ denotes the zeros of $I_{\alpha,\beta,\gamma}$ on C_{m-1}, then

$$\mathcal{O}(Z_{\alpha,\beta,\gamma}) = \frac{\mathcal{O}(C_{m-1})}{I_{\alpha,\beta,\gamma}} \cong \frac{k[x_1,\dots,x_n]}{(x_\alpha, x_\beta, x_\gamma)} \tag{16}$$

shows that $Z_{\alpha,\beta,\gamma}$ is a variety of dimension $n-3$ contained in C_{m-1} which has dimension $n-1$. This proves $\operatorname{Sing}(C_{m-1})$ is a subset of pure codimension two in C_{m-1}. This is part 1. □

Proof of Part 2: Next we compute the class group of C_{m-1}. For each ordered pair (α,β) such that $1 \le \alpha < \beta \le m-1$, consider the ideal $I_{\alpha,\beta}$ generated by (x_α, x_β) in $\mathcal{O}(C_{m-1})$. We see by (9) that $\sigma_{m-1}(x_1,\dots,x_m) \in I_{\alpha,\beta}$. Therefore

$$\frac{\mathcal{O}(C_{m-1})}{I_{\alpha,\beta}} \cong \frac{k[x_1,\dots,x_n]}{(x_\alpha, x_\beta)} \tag{17}$$

shows $I_{\alpha,\beta}$ is a prime ideal. The rings in (17) have dimension $n-2$. Since $\mathcal{O}(C_{m-1})$ has dimension $n-1$, it follows that each ideal $I_{\alpha,\beta}$ is a prime ideal of height one. By the recurrence relation (8), any prime ideal that contains $\sigma_{m-2}(x_1,\dots,x_{m-1})$ must contain one of $x_1, x_2, \dots, x_{m-1}$. Likewise, applying (9) to $\sigma_{m-2}(x_1,\dots,x_{m-1})$, we see that any prime ideal that contains both $\sigma_{m-2}(x_1,\dots,x_{m-1})$ and x_α must also contain x_β for some $\beta \ne \alpha$. The minimal primes in $\mathcal{O}(C_{m-1})$ containing the element $\sigma_{m-2}(x_1,\dots,x_{m-1})$ are $\{I_{\alpha,\beta} \mid 1 \le \alpha < \beta \le m-1\}$ and each of these primes has height one.

Write $\mathcal{O}_{I_{\alpha,\beta}}$ for the localization of $\mathcal{O}(C_{m-1})$ at the prime ideal $I_{\alpha,\beta}$. Rewrite (9) as

$$\sigma_{m-1}(x_1,\dots,x_m) = x_1 \left(\frac{x_2 \dots x_m}{x_2} + \dots + \frac{x_2 \dots x_m}{x_m} \right) + x_2 x_3 \cdots x_m. \tag{18}$$

Notice that for $3 \le j \le m$, $x_j \notin I_{12}$. In the local ring $\mathcal{O}_{I_{1,2}}$ obtained by localizing $\mathcal{O}(C_{m-1})$ at I_{12}, $x_1 = x_2 u$ and $\sigma_{m-2}(x_1,\dots,x_{m-1}) = x_1 x_2 v$ where

$$u = -(x_3 \cdots x_m) \left(\frac{x_2 \dots x_m}{x_2} + \dots + \frac{x_2 \dots x_m}{x_m} \right)^{-1}$$

and

$$v = -(x_3 \cdots x_{m-1}) x_m^{-1}$$

are units in $\mathcal{O}_{I_{1,2}}$. So the maximal ideal in the local ring $\mathcal{O}_{I_{1,2}}$ is principal and x_2 is a local parameter. By symmetry x_1 is a local parameter as well. Then $\mathcal{O}_{I_{1,2}}$ is a discrete valuation ring and we have shown that $\sigma_{m-2}(x_1,\dots,x_{m-1})$ has valuation 2. By symmetry, $\sigma_{m-2}(x_1,\dots,x_{m-1})$ has order 2 along $I_{\alpha,\beta}$ for each such pair (α,β).

The right-hand side of (3) is regular so the singular locus of C_{m-1} is contained in the support of the divisor of the function $\sigma_{m-2}(x_1,\dots,x_{m-1})$. By the computations above, each local ring $\mathcal{O}_{I_{\alpha,\beta}}$ is regular. We remark for curiosity's sake that this gives another proof that C_{m-1} is nonsingular in codimension one. (We already proved this in Part 1.)

The right-hand side of (3) is a unique factorization domain. Therefore the same is true for $\mathcal{O}(C_{m-1})[\sigma_{m-2}(x_1,\dots,x_{m-1})^{-1}]$. By [5, Theorem 7.1, p.35], the divisor class group $\mathrm{Cl}(C_{m-1})$ is generated by the prime divisors in $\mathrm{div}(\sigma_{m-2}(x_1,\dots,x_{m-1}))$. It follows from the above calculations that

$$\mathrm{div}\bigl(\sigma_{m-2}(x_1,\dots,x_{m-1})\bigr) = \sum_{1\le\alpha<\beta\le m-1} 2I_{\alpha,\beta}. \tag{19}$$

By Nagata's Theorem (e.g. [3, Theorem 1.1]) there is an exact sequence

$$\mathbb{Z} \to \sum_{\binom{m-1}{2}} \mathbb{Z}\cdot I_{\alpha,\beta} \to \mathrm{Cl}(C_{m-1}) \to 0 \tag{20}$$

where the first arrow is defined by $1 \mapsto \mathrm{div}\bigl(\sigma_{m-2}(x_1,\dots,x_{m-1})\bigr)$. This proves part 2, namely that $\mathrm{Cl}(C_{m-1}) \cong \mathbb{Z}/2 \oplus \mathbb{Z}^{(r)}$. □

Proof of Part 3. It follows from (20) that the subgroup of $\mathrm{Cl}(C_{m-1})$ consisting of torsion elements is generated by the divisor $\frac{1}{2}\,\mathrm{div}\bigl(\sigma_{m-2}(x_1,\dots,x_{m-1})\bigr) = \sum I_{\alpha,\beta}$ and has order 2. According to [3, Theorem 3.1(c)] the Brauer group $\mathrm{B}(X - C_{m-1})$ is isomorphic to the subgroup of torsion elements in the divisor class group $\mathrm{Cl}(C_{m-1})$. It follows that $\mathrm{B}(X - C_{m-1})$ is a group of order 2 which proves part 3. □

Proof of Part 4. We saw above that x_1 is a local parameter for each $\mathcal{O}_{I_{1,\beta}}$. The divisor of x_1 on C_{m-1} is

$$\mathrm{div}(x_1) = I_{1,2} + \cdots + I_{1,m}.$$

Thus, on C_{m-1} the divisor of the function $x_1\cdots x_m$ is

$$\begin{aligned}\mathrm{div}(x_1\cdots x_m) &= 2\sum_{1\le\alpha<\beta\le m} I_{\alpha,\beta}\\ &= 2\sum_{1\le\alpha<\beta\le m-1} I_{\alpha,\beta} + 2\sum_{1\le\alpha\le m-1} I_{\alpha,m} \quad .\\ &= \mathrm{div}\bigl(\sigma_{m-2}(x_1,\dots,x_{m-1})\bigr) + 2\,\mathrm{div}(x_m)\end{aligned} \tag{21}$$

This shows that the divisors $\mathrm{div}(x_1\cdots x_m)$ and $\mathrm{div}\bigl(\sigma_{m-2}(x_1,\dots,x_{m-1})\bigr)$ differ by the principal divisor $2\,\mathrm{div}(x_m)$, hence the subgroup of torsion elements of $\mathrm{Cl}(C_{m-1})$ is generated by the divisor $\frac{1}{2}\,\mathrm{div}(x_1\cdots x_m)$. Define the symbol algebra

$$\Lambda_{m-1} = \bigl(\sigma_{m-1}(x_1,\dots,x_m), \sigma_{m-2}(x_1,\dots,x_{m-1})\bigr)_2$$

over the function field of $\mathbb{A}^n$, $k(x_1, \dots, x_n)$. Compute the ramification divisor of Λ_{m-1} following [4, Section 1]. The ramification is given by the tame symbol. Then Λ_{m-1} ramifies on the divisors $C_{m-1} = Z(\sigma_{m-1}(x_1, \dots, x_m)$ and $C_{m-2} = Z(\sigma_{m-2}(x_1, \dots, x_{m-1}))$ in $\mathbb{A}^n$. The ramification of Λ_{m-1} on C_{m-1} corresponds to the square root of the divisor $\operatorname{div}\big(\sigma_{m-2}(x_1, \dots, x_{m-1})\big)$ which is given by (19). So this is an unramified quadratic extension of $K(C_{m-1})$. The ramification of Λ_{m-1} on C_{m-2} corresponds to the square root of the divisor $\operatorname{div}(x_1 \cdots x_{m-1})$ on C_{m-2}. Since

$$\sigma_{m-1}(x_1, \dots, x_m) = x_1 \cdots x_{m-1} + \sigma_{m-2}(x_1, \dots, x_{m-1}) x_m,$$

upon restriction to C_{m-2}, the function $\sigma_{m-1}(x_1, \dots, x_m)$ is equal to $x_1 \cdots x_{m-1}$. By (21) this gives rise to the element of order two in $\operatorname{Cl}(C_{m-2})$. So the ramification of Λ_{m-1} on C_{m-2} is an unramified quadratic extension of $K(C_{m-2})$. Therefore Λ_{m-1} factors in the Brauer group of $K(\mathbb{A}^2)$ into the algebra class which represents a generator of $\mathrm{B}(\mathbb{A}^2 - C_{m-1})$ and the class that represents a generator of $\mathrm{B}(\mathbb{A}^2 - C_{m-2})$.

Proceed by induction on m. Consider the case $m = 3$. Look at

$$\Lambda_2 = \big(\sigma_2(x_1, x_2, x_3), \sigma_1(x_1, x_2)\big)_2$$

over $k(x_1, \dots, x_n)$. Upon restriction to C_1 the function $\sigma_2(x_1, x_2, x_3)$ is a square. So Λ_2 ramifies only on C_2 and the ramification corresponds to the unramified quadratic extension $K(C_2)\sqrt{\sigma_1(x_1, x_2)}$. This corresponds to the element of order two in $\operatorname{Cl}(C_2)$. As mentioned above, this corresponds to the generator of $\mathrm{B}(\mathbb{A}^2 - C_2)$. So $\mathrm{B}(\mathbb{A}^2 - C_2)$ is generated by the class of Λ_2.

Now fix m such that $3 < m \le n$ and assume $\mathrm{B}(\mathbb{A}^2 - C_{m-2})$ is generated by the class of the algebra $\Lambda_{m-2} \otimes \cdots \otimes \Lambda_2$ defined by the product of the symbols

$$\big(\sigma_{m-2}(x_1, \dots, x_{m-1}, \sigma_{m-3}(x_1, \dots, x_{m-2})\big)_2 \cdots \big(\sigma_2(x_1, x_2, x_3), \sigma_1(x_1, x_2)\big)_2. \quad (22)$$

Multiplying, we get

$$\begin{aligned} \Lambda_{m-1} \otimes \Lambda_{m-2} \otimes \cdots \otimes \Lambda_2 = & \\ & \big(\sigma_{m-1}(x_1, \dots, x_m, \sigma_{m-2}(x_1, \dots, x_{m-1})\big)_2 \\ & \big(\sigma_{m-2}(x_1, \dots, x_{m-1}, \sigma_{m-3}(x_1, \dots, x_{m-2})\big)_2 \\ & \cdots \big(\sigma_2(x_1, x_2, x_3), \sigma_1(x_1, x_2)\big)_2 \end{aligned} \quad (23)$$

is an algebra class that ramifies only on C_{m-1} and therefore represents the generator of $\mathrm{B}(X - C_{m-1})$. This completes part 4. □

3. The Projective Variety of Degree $n-1$ in $\mathbb{P}^{n-1}$

Fix $n \geq 4$ and let $\{x_1, x_2, \dots, x_n\}$ be a set of indeterminates. For each m such that $2 \leq m \leq n$ let $\sigma_{m-1}(x_1, x_2, \dots, x_m)$ denote the elementary symmetric polynomial of degree $(m-1)$ in the variables $x_1, x_2, \dots, x_m$. Then (7) (8) and (9) are satisfied by $\sigma_{m-1}(x_1, x_2, \dots, x_m)$. As in Section 1, the projective variety $\sigma_{m-1}(x_1, x_2, \dots, x_m) = 0$ in $\mathbb{P}^{n-1}$ is

$$V_{m-1} = Z(\sigma_{m-1}(x_1, x_2, \dots, x_m)),$$

the affine cone over V_{m-1} in $\mathbb{A}^n$ is

$$C_{m-1} = Z\big(\sigma_{m-1}(x_1, x_2, \dots, x_m)\big),$$

and the homogeneous coordinate ring of V_{m-1} is

$$\mathcal{O}(C_{m-1}) = k[x_1, \dots, x_n]/\big(\sigma_{m-1}(x_1, \dots, x_m)\big)$$

which is equal to the affine coordinate ring of C_{m-1}.

Theorem 3.1. *Let $4 \leq m \leq n$. In the above context,*

1. *The singular locus of V_{m-1} has pure codimension two.*
2. *If $s = \binom{m-1}{2}$, then*
$$\mathrm{Cl}(V_{m-1}) \cong \begin{cases} \mathbb{Z}^{(s)} & \text{, if } m \text{ is odd} \\ \mathbb{Z}/2 \oplus \mathbb{Z}^{(s)} & \text{, if } m \text{ is even.} \end{cases}$$
3. $$\mathrm{B}(\mathbb{P}^{n-1} - V_{m-1}) \cong \begin{cases} 0 & \text{, if } m \text{ is odd} \\ \mathbb{Z}/2 & \text{, if } m \text{ is even.} \end{cases}$$
4. *If m is even, then $\mathrm{B}(\mathbb{P}^{n-1} - V_{m-1}) \cong \mathrm{B}(\mathbb{A}^n - C_{m-1})$.*

Proof of Part 1: Part 1 follows from Theorem 2.1.1. □

Proof of Part 2: For each ordered pair (α, β) such that $1 \leq \alpha < \beta \leq m$, consider the homogeneous ideal $I_{\alpha,\beta}$ generated by (x_α, x_β) in $\mathcal{O}(C_{m-1})$. We see by (17) that $I_{\alpha,\beta}$ is a prime ideal. Let $Z_{\alpha,\beta} = Z(I_{\alpha,\beta})$ be the prime divisor on V_{m-1} defined by the ideal $I_{\alpha,\beta}$. There is an exact sequence [8]

$$0 \to \mathbb{Z} \to \mathrm{Cl}(V_{m-1}) \to \mathrm{Cl}(C_{m-1}) \to 0 \tag{24}$$

where second arrow maps $1 \in \mathbb{Z}$ to the divisor class of a hyperplane section. From the proof of Theorem 2.1 we know that $\mathrm{Cl}(C_{m-1})$ is generated by the divisors $Z_{\alpha,\beta}$ where $1 \leq \alpha < \beta \leq m-1$. The support of the hyperplane section $x_m = 0$ is contained in $\cup_{\alpha=1}^{m-1} Z_{\alpha,m}$. We have shown that $\mathrm{Cl}(V_{m-1})$ is generated by the $\binom{m}{2}$

divisors $\{Z_{\alpha,\beta} \mid 1 \leq \alpha < \beta \leq m\}$. The only relations are $\operatorname{div}(x_\alpha/x_\beta) \sim 0$. Let F denote the free $\mathbb{Z}$-module with basis $\{Z_{\alpha,\beta} \mid 1 \leq \alpha < \beta \leq m\}$, where we order the basis lexicographically on the (α, β). Let M denote the $\mathbb{Z}$-submodule of F generated by $\{\operatorname{div}(x_\alpha/x_\beta) \mid 1 \leq \alpha < \beta \leq m\}$. Therefore $\operatorname{Cl}(V_{m-1}) \cong F/M$.

Check that the following identities hold for any α, β, γ.

$$\begin{aligned} \operatorname{div}(x_\alpha/x_\beta) &= -\operatorname{div}(x_\beta/x_\alpha) \\ \operatorname{div}(x_\alpha/x_\gamma) &= \operatorname{div}(x_\alpha/x_\beta) + \operatorname{div}(x_\beta/x_\gamma) \end{aligned} \tag{25}$$

Then M is spanned by the $m-1$ divisors

$$\begin{aligned} \operatorname{div}(x_1/x_m) &= Z_{1,2} + Z_{1,3} + \cdots + Z_{1,m-1} - Z_{2,m} - \dots \\ \operatorname{div}(x_2/x_m) &= Z_{1,2} - Z_{1,m} + Z_{2,3} + \dots \\ \operatorname{div}(x_3/x_m) &= Z_{1,3} - Z_{1,m} + Z_{2,3} + \dots \\ \operatorname{div}(x_i/x_m) &= Z_{1,i} - Z_{1,m} + \dots \\ \operatorname{div}(x_{m-1}/x_m) &= Z_{1,m-1} - Z_{1,m} + \dots \end{aligned} \tag{26}$$

If we set

$$\begin{aligned} A &= \operatorname{div}(x_1/x_m) - \operatorname{div}(x_2/x_m) - \operatorname{div}(x_3/x_m) - \cdots - \operatorname{div}(x_{m-1}/x_m) \\ &= (m-2)Z_{1,m} - \sum_{1 \leq \alpha < \beta \leq m-1} 2Z_{\alpha,\beta}. \end{aligned}$$

Then M is spanned by the $m-1$ divisors in this list:

$$L = \{A, \operatorname{div}(x_2/x_m), \dots, \operatorname{div}(x_{m-1}/x_m)\}.$$

Look at the matrix associated to the homomorphism $\mathbb{Z}^{(m-1)} \to F$ which maps the ith generator of $\mathbb{Z}^{(m-1)}$ to the ith element in L. The top m rows are

$$\begin{bmatrix} 0 & 1 & 0 & 0 & \dots & 0 \\ 0 & 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \dots & 1 \\ m-2 & -1 & -1 & -1 & \dots & -1 \\ -2 & 1 & 1 & 0 & \dots & 0 \end{bmatrix} \tag{27}$$

The matrix (27) clearly has $\mathbb{Q}/\mathbb{Z}$-rank $m-1$. If m is odd, then the columns of (27) span a direct summand of F and F/M is free of rank $s = \binom{m-1}{2}$. Otherwise, $F/M \cong \mathbb{Z}/2 \oplus \mathbb{Z}^{(s)}$. This proves part 2. $\square$

Proof of Part 3: Next we prove that part 3 follows from part 2. For this argument, let

$$W_{m-1} = \operatorname{Sing} V_{m-1}$$

which has codimension two in V_{m-1}, and codimension three in $\mathbb{P}^{n-1}$. Consider the exact sequence

$$\begin{aligned}\mathrm{B}(\mathbb{P}^{n-1}) \to \mathrm{B}(\mathbb{P}^{n-1} - W_{m-1}) \to \mathrm{H}^3_{W_{m-1}}(\mathbb{P}^{n-1}, \mu) \to \\ \mathrm{H}^3(\mathbb{P}^{n-1}, \mu) \to \mathrm{H}^3(\mathbb{P}^{n-1} - W_{m-1}, \mu) \to \mathrm{H}^4_{W_{m-1}}(\mathbb{P}^{n-1}, \mu) \quad (28)\end{aligned}$$

of [4, Lemma 0.1]. By [10, p. 245] $\mathrm{H}^3(\mathbb{P}^{n-1}, \mu) = 0$ and by [10, Lemma 9.1, p. 268] $\mathrm{H}^i_{W_{m-1}}(\mathbb{P}^{n-1}, \mu) = 0$ for $i < 6$. It follows that $\mathrm{B}(\mathbb{P}^{n-1}) \cong \mathrm{B}(\mathbb{P}^{n-1} - W_{m-1}) = 0$. Additionally $\mathrm{H}^3(\mathbb{P}^{n-1}, \mu) \cong \mathrm{H}^3(\mathbb{P}^{n-1} - W_{m-1}, \mu) = 0$. It follows from [4, Theorem 1.1] that

$$\mathrm{B}(\mathbb{P}^{n-1} - V_{m-1}) \cong \mathrm{H}^1(V_{m-1} - W_{m-1}, \mu). \quad (29)$$

Since V_{m-1} is a projective variety which is regular in codimension one,

$$\mathrm{H}^0(V_{m-1}, \mathbb{G}_m) = \mathrm{H}^0(V_{m-1} - W_{m-1}, \mathbb{G}_m) = k^*.$$

By Kummer theory, $\mathrm{H}^1(V_{m-1} - W_{m-1}, \mu)$ is isomorphic to the subgroup of torsion elements in

$$\mathrm{H}^1(V_{m-1} - W_{m-1}, \mathbb{G}_m) = \mathrm{Cl}(V_{m-1} - W_{m-1}) = \mathrm{Cl}(V_{m-1}).$$

Now part 3 follows from (29) and part 2. □

Proof of Part 4: Next we prove that part 4 follows from part 3. View $\mathbb{A}^n$ as an open subset of $\mathbb{P}^n$ by introducing a new homogeneous variable x_0. Let $\bar{C}_{m-1}$ denote the completion of C_{m-1} in $\mathbb{P}^n$. We have the projection along the x_0-axis

$$\pi \colon \mathbb{P}^n - \bar{C}_{m-1} \to \mathbb{P}^{n-1} - V_{m-1},$$

the open immersion where $x_0 \neq 0$

$$\psi \colon \mathbb{A}^n - C_{m-1} \to \mathbb{P}^n - \bar{C}_{m-1},$$

and the closed immersion where $x_0 = 0$

$$\phi \colon \mathbb{P}^{n-1} - V_{m-1} \to \mathbb{P}^n - \bar{C}_{m-1}.$$

Since ϕ is a section to ψ, the map on Brauer groups

$$\pi^* \colon \mathrm{B}\big(\mathbb{P}^{n-1} - V_{m-1}\big) \to \mathrm{B}\big(\mathbb{P}^n - \bar{C}_{m-1}\big)$$

is an isomorphism [3, Theorem 3.1(a)]. Since ψ is a localization,

$$\psi^*\colon \mathrm{B}\big(\mathbb{P}^n - \bar{C}_{m-1}\big) \to \mathrm{B}\big(\mathbb{A}^n - C_{m-1}\big)$$

is one-to-one. It follows that

$$\psi^*\pi^*\colon \mathrm{B}\big(\mathbb{P}^{n-1} - V_{m-1}\big) \to \mathrm{B}\big(\mathbb{A}^n - C_{m-1}\big)$$

is one-to-one so part 4 follows from part 3. □

Lemma 3.2. *In the notation of Theorem 3.1, let U_1 denote the open set where $x_1 \neq 0$ and let P_1 be the closed point of U_1 where $x_2 = \cdots = x_m = 0$. Then*

$$\mathrm{Cl}(U_1) \cong Cl(\mathcal{O}_{P_1}) \cong Cl(\mathcal{O}^h_{P_1}) \cong Cl(\hat{\mathcal{O}}_{P_1}) \cong \mathbb{Z}/2 \oplus \mathbb{Z}^{(s)}$$

where $s = \binom{m-1}{2} - \binom{m-1}{1}$.

Proof. By Theorem 3.1.2, $\mathrm{Cl}(U_1)$ is generated by the divisors $Z_{\alpha,\beta}$ passing through P_1, which are precisely the elements of the set $\{Z_{\alpha,\beta} |\ 2 \le \alpha < \beta \le m\}$. Order the set $\{Z_{\alpha,\beta} |\ 2 \le \alpha < \beta \le m\}$ lexicographically on the (α, β) and let F denote the free $\mathbb{Z}$-module with this basis. The ideal corresponding to P_1 is generated by $x_2, \ldots, x_m$. Therefore $\mathrm{Cl}(U_1)$ is the quotient of F modulo the submodule M spanned by the principal divisors

$$\begin{aligned}
\operatorname{div}(x_2) &= Z_{2,3} + Z_{2,4} + Z_{2,5} + \cdots + Z_{2,m} \\
\operatorname{div}(x_3) &= Z_{2,3} + Z_{3,4} + Z_{3,5} + \cdots + Z_{3,m} \\
\operatorname{div}(x_4) &= Z_{2,4} + Z_{3,4} + Z_{4,5} + \cdots + Z_{4,m} \\
&\ \vdots \\
\operatorname{div}(x_m) &= Z_{2,m} + Z_{3,m} + Z_{4,m} + \cdots + Z_{m-1,m}.
\end{aligned} \tag{30}$$

As a generator for M we can replace $\operatorname{div}(x_2)$ with

$$\begin{aligned}
A &= \operatorname{div}(x_2) - \operatorname{div}(x_3) - \operatorname{div}(x_4) - \cdots - \operatorname{div}(x_m) \\
&= \sum_{3 \le \alpha < \beta \le m} 2Z_{\alpha,\beta}.
\end{aligned}$$

Then M is spanned by the $m-1$ divisors in the list

$$L = \{\operatorname{div}(x_3), \operatorname{div}(x_4), \ldots, \operatorname{div}(x_m), A\}.$$

Look at the matrix for the homomorphism $\mathbb{Z}^{(m-1)} \to F$ which maps the ith generator to the ith element of L. The image of this map is M. In block form, the matrix is

$$\left[\begin{array}{c|c} I_{m-2} & z \\ \hline * & t \end{array}\right]$$

where I_{m-2} is the $m-2$-by-$m-2$ identity matrix, z is a column of zeros, and t is a column of twos. Since $\mathrm{Cl}(U_1) \cong F/M$ we have shown that $\mathrm{Cl}(U_1) \cong \mathbb{Z}/2\mathbb{Z}^{(s)}$ as claimed. The same computation applies to the local rings. □

Theorem 3.3. *The context is the same as in Theorem 3.1, but we assume $n = m = 4$. In this notation V_3 denotes the cubic surface in $\mathbb{P}^3$ defined by the elementary symmetric polynomial of degree three in four variables.*

1. *The singular locus of V_3 consists of four isolated rational double points.*
2. $\mathrm{B}(K/V_3) = \mathrm{B}(V_3)$ *is cyclic of order two and* $\mathrm{B}\big(V_3 - \operatorname{Sing} V_3\big) = 0$.
3. $\mathrm{Cl}(V_3)/\operatorname{Pic}(V_3) = (\mathbb{Z}/2)^{(3)}$.
4. $\operatorname{Pic}(V_3) = \mathbb{Z}^{(3)}$.

Proof of Part 1: Follows from the computation of the class groups in Lemma 3.2 and [9, Theorem 17.4]. □

Proof of Parts 2 and 3: There are four triples (α, β, γ) such that $1 \le \alpha < \beta < \gamma \le 4$. For each such triple, let $I_{\alpha,\beta,\gamma}$ denote the homogeneous prime ideal of height two in $\mathcal{O}(C_3)$ generated by $x_\alpha, x_\beta, x_\gamma$. Let $Z_{\alpha,\beta,\gamma} = Z(I_{\alpha,\beta,\gamma})$ be the corresponding closed subset of V_3. By (15), the singular locus of V_3 is equal to the union of the sets $Z_{\alpha,\beta,\gamma}$.

The singular locus of V_3 is equal to the union of the four closed points $P_1 = Z_{2,3,4}$, $P_2 = Z_{1,3,4}$, $P_3 = Z_{1,2,4}$ and $P_4 = Z_{1,2,3}$. Consider the local ring of V_3 at the point $P_1 = Z_{2,3,4}$ which we denote by $\mathcal{O}_{P_1}$. The class group $\mathrm{Cl}(\mathcal{O}_{P_1})$ is cyclic of order two, by Lemma 3.2. To set up our notation, restrict to the open neighborhood of P_1 where $x_1 \neq 0$. The maximal ideal at P_1 is generated by x_2/x_1, x_3/x_1 and x_4/x_1. Therefore, $\mathrm{Cl}(\mathcal{O}_{P_1})$ is the free group on the three prime divisors $Z_{2,3}$, $Z_{2,4}$, $Z_{3,4}$ modulo the subgroup spanned by

$$\begin{aligned} \operatorname{div}(x_2/x_1) &= Z_{2,3} + Z_{2,4} \\ \operatorname{div}(x_3/x_1) &= Z_{2,3} + Z_{3,4} \\ \operatorname{div}(x_4/x_1) &= Z_{2,4} + Z_{3,4} \end{aligned} \tag{31}$$

We can eliminate one of the generators, say $Z_{2,4}$ and simplify the relations down to two. Then $\mathrm{Cl}(\mathcal{O}_{P_1})$ is generated by $Z_{2,3}$ and $Z_{3,4}$ subject to the relations $\operatorname{div}(x_2/x_4) = Z_{2,3} - Z_{3,4} \sim 0$ and $\operatorname{div}\big((x_2x_3/(x_1x_4)\big) = 2Z_{2,3} \sim 0$. By symmetry, the same is true at the other three singular points. We also see that any one of the prime divisors $Z_{\alpha,\beta}$ passing through the point P_γ is a generator for the class group at P_γ.

On V_3 let $D = Z_{1,2} + Z_{1,4} + Z_{2,3} + Z_{3,4}$. Define these five free abelian groups:

$$\begin{aligned} F &= \mathbb{Z} \cdot Z_{1,2} \oplus \mathbb{Z} \cdot Z_{1,4} \oplus \mathbb{Z} \cdot Z_{2,3} \oplus \mathbb{Z} \cdot Z_{3,4} \\ F_1 &= Z \cdot Z_{2,3} \oplus \mathbb{Z} \cdot Z_{3,4} \\ F_2 &= Z \cdot Z_{1,4} \oplus \mathbb{Z} \cdot Z_{3,4} \\ F_3 &= \mathbb{Z} \cdot Z_{1,2} \oplus \mathbb{Z} \cdot Z_{1,4} \\ F_4 &= \mathbb{Z} \cdot Z_{1,2} \oplus \mathbb{Z} \cdot Z_{2,3} \end{aligned} \tag{32}$$

There is a commutative diagram

$$\begin{tikzcd} & \bigoplus_{i=1}^4 \mathbb{Z}^2 \arrow[d, "b"] & & \\ F \arrow[r, "a"] \arrow[d, "\phi"] & \bigoplus_{i=1}^4 F_i \arrow[d, "c"] & & \\ \mathrm{Cl}(V_3)/\operatorname{Pic}(V_3) \arrow[r, "\epsilon"] \arrow[d] & \bigoplus_{i=1}^4 \mathrm{Cl}(\mathcal{O}_{P_i}) \arrow[r, "\delta"] \arrow[d] & \mathrm{B}(K/V_3) \arrow[r] & 0 \\ 0 & 0 & & \end{tikzcd} \tag{33}$$

whose maps are now explained. The bottom row of (33) is exact and is from [1]. There are projection maps $F \to F_i$ and a is the direct sum

$$F \xrightarrow{a} \bigoplus_{i=1}^{4} F_i. \tag{34}$$

Since the singular locus of V_3 is contained in the support of D, $V_3 - |D|$ is locally factorial and there is an exact sequence

$$F \xrightarrow{\phi} \mathrm{Cl}(V_3)/\operatorname{Pic}(V_3) \to 0 \tag{35}$$

which is the map ϕ. For each singular point P_i there is an exact sequence

$$0 \to \mathbb{Z}^2 \xrightarrow{b_i} F_i \xrightarrow{c_i} \mathcal{O}_{P_i} \to 0 \tag{36}$$

where the arrow b_i in (36) maps two generators for $\mathbb{Z}^2$ to the two defining relations for the class group of the local ring. So the matrix for b_i is

$$\begin{pmatrix} 1 & 2 \\ -1 & 0 \end{pmatrix} \tag{37}$$

and b and c are the direct sum maps. From (33) it is routine to derive

$$F \oplus \bigoplus_{i=1}^{4} \mathbb{Z}^2 \xrightarrow{a+b} \bigoplus_{i=1}^{4} F_i \xrightarrow{\psi} \mathrm{B}(K/V_3) \to 0 \tag{38}$$

where $\psi = \delta c$. The matrix for $a+b$ in (38) is

$$\begin{bmatrix}
0 & 0 & 1 & 0 & 1 & 2 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 1 & 2 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 2 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 2 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0
\end{bmatrix}$$

Since the cokernel of ψ is an elementary 2-group, we can compute its rank by tensoring (38) with $\mathbb{Z}/2$. Then all but half of the relations become trivial and (38) simplifies to

$$F \otimes \mathbb{Z}/2 \oplus \bigoplus_{i=1}^{4} \mathbb{Z}/2 \xrightarrow{a+b} \bigoplus_{i=1}^{4} F_i \otimes \mathbb{Z}/2 \xrightarrow{\psi} \mathrm{B}(K/V_3) \to 0. \tag{39}$$

and the matrix simplifies down to

$$\begin{bmatrix}
0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & -1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & -1 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & -1 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & -1
\end{bmatrix}$$

This matrix has rank 7. This proves $\mathrm{B}(K/V_3)$ is a cyclic group of order two. If $\tilde{V_3} \to V_3$ is a desingularization of V_3, then the sequence

$$0 \to \mathrm{B}(K/V_3) \to \mathrm{B}(V_3) \to \mathrm{B}(\tilde{V_3}) \to 0$$

is exact [2, Theorem 1]. The Brauer group is a birational invariant between complete nonsingular surfaces [7, Corollary 7.2], so $\mathrm{B}(\tilde{V_3}) = \mathrm{B}(\mathbb{P}^2) = 0$. The second equation in part 2 follows from [2, Corollary 3]. In (33) the map ϵ is injective, which proves part 3. $\square$

Proof of Part 3: Now we prove part 4 assuming part 3. Combining the sequence (24) with the natural maps from Picard groups to class groups we get the commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & \operatorname{Pic}\mathbb{P}^3 \arrow[r] \arrow[d, "="] & \operatorname{Pic}V_3 \arrow[r] \arrow[d, "\alpha"] & \operatorname{Pic}(V_3)/\langle H\rangle \arrow[r] \arrow[d, "\beta"] & 0 \\
0 \arrow[r] & \operatorname{Cl}\mathbb{P}^3 \arrow[r] & \operatorname{Cl}V_3 \arrow[r] & \operatorname{Cl}C_3 \arrow[r] & 0
\end{tikzcd}$$

with exact rows. The map α is injective. By the Snake Lemma, β is injective and the cokernel of α is isomorphic to the cokernel of β. But Part 3 says the cokernel of α is an elementary 2-group of rank $s = \binom{3}{2}$. By Theorem 2.1.2 $\mathrm{Cl}(C_3) \cong \mathbb{Z}/2 \oplus \mathbb{Z}^{(s-1)}$. We deduce from this that $\operatorname{Pic}V_3/\langle H\rangle$ is necessarily free of rank $s-1$. This proves part 4. □

4. Miscellaneous Results

4.1. The Quadric Hypersurface. Fix $n \geq 3$ and let $\{x_1, x_2, \dots, x_n\}$ be a set of indeterminates. Let $\sigma_2(x_1, \dots, x_n)$ denote the elementary symmetric polynomial of degree two in the variables $x_1, x_2, \dots, x_n$. As in Section 1, let $V_2 \subseteq \mathbb{P}^{n-1}$ be the projective variety defined by $\sigma_2(x_1, \dots, x_n) = 0$ and C_2 the affine variety in $X = \mathbb{A}^n$ defined by $\sigma_2(x_1, \dots, x_n) = 0$. As shown in Lemma 1.2.1, V_2 is a nonsingular rational hypersurface in $\mathbb{P}^{n-1}$ of degree two. The reader is referred to [8, p. 147] where the computations of $\mathrm{Cl}(C_2)$ and $\mathrm{Pic}(V_2)$ are given as exercises.

Theorem 4.1. *In the above context,*

1. *If* $n = 3$*,* $\mathrm{B}(X - C_2) \cong \mathbb{Z}/2$.
2. *If* $n > 3$*,* $\mathrm{B}(X - C_2) = 0$.

Proof. The first case was included in Theorem 2.1. The second case follows directly from [3, Theorem 2.1(c)]. □

Theorem 4.2. *Let $n \geq 3$ and $V_2 \subseteq \mathbb{P}^{n-1}$ the projective variety defined by the elementary symmetric polynomial of degree two in n variables. Then* $\mathrm{B}(\mathbb{P}^{n-1} - V_2) = 0$ *and* $\mathrm{B}(V_2) = 0$.

Proof. After a suitable linear change of variables, the quadratic form $\sigma_2(x_1, \dots, x_n)$ can be transformed into $y_1 y_2 + y_3^2 + \cdots + y_n^2$. Dehomogenize with respect to y_1 and eliminate y_2 to see that V_2 has an open set U that is isomorphic to the affine space $\mathbb{A}^{(n-2)}$. Since V_2 is nonsingular, the natural map $\mathrm{B}(V_2) \to \mathrm{B}(U)$ is injective [10, p. 145]. Hence $\mathrm{B}(V_2) = \mathrm{B}(k) = 0$. By an exercise [8, p. 147], $\mathrm{Pic}(V_2)$ is torsion-free. It follows from Kummer theory that $\mathrm{H}^1(V_2, \mu) = 0$. This implies that $\mathrm{B}(\mathbb{P}^{n-1} - V_2) = 0$. □

4.2. Other Cases. We summarize what is known in the cases not covered above. It follows from the proof of Lemma 1.2 that the class group of C_d is generated by the prime divisors in $\operatorname{div}\big(\sigma_{d-1}(x_1, \dots, x_{n-1})\big)$. Using [6], we can show that if (d, n) is any pair in this list $\{(3,5), (3,6), (4,6), (3,7), (4,7), (5,7)\}$, then $\operatorname{div}\big(\sigma_{d-1}(x_1, \dots, x_{n-1})\big)$ is a principal prime divisor on C_d. Hence $\mathrm{Cl}(C_d) = 0$ and it follows by [3, Theorem 3.1(c)] that $\mathrm{B}(\mathbb{A}^n - C_d) = 0$. Is $C_{d,n}$ factorial for all pairs d, n such that $2 \le d < n - 1$?

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