EXTENSIONS OF GM-RINGS OVER GENERALIZED POWER SERIES RINGS

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Abstract. Let \( R \) be a reduced ring, \((S, \leq)\) a cancellative torsion-free strictly ordered monoid, it is shown that ring \( \mathbb{R}^{S, \leq} \) is a GM−ring if and only if \( R \) is a GM−ring. We also investigate GM−rings for some special Morita contexts and module extensions over generalized power series rings.

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1. Introduction

All rings considered here are associative with identity and \( R \) denotes such a ring. We use \( U(R) \) to denote the group of units of \( R \). Any concept and notation not defined here can be found in [6, 7].

A ring \( R \) is said to be a GM−ring provided that for any \( x, y \in R \), there exist idempotents \( e, f \in R \) and \( u \in U(R) \) such that \( x = eu, y = fu^{-1} \in U(R) \). A ring \( R \) is called a clean ring if for any \( x \in R \), there exists \( e^2 = e \in R \) such that \( x - e \in U(R) \). Clearly, all clean rings are GM−rings. Many examples and results of GM−rings are given in [1, 2].

Let \((S, \leq)\) be an ordered set. Recall that \((S, \leq)\) is artinian if every strictly decreasing sequence of elements of \( S \) is finite, and that \((S, \leq)\) is narrow if every subset of pairwise order-incomparable elements of \( S \) is finite. Let \((S, \leq)\) be a strictly ordered monoid and \( R \) a ring. Let \( \mathbb{R}^{S, \leq} \) be the set of all maps \( f : S \to R \) such that \( supp(f) = \{ s \in S | f(s) \neq 0 \} \) is artinian and narrow. With pointwise addition and the operation of convolution

\[
(fg)(s) = \sum_{(u,v) \in X_s(f,g)} f(u)g(v)
\]

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Let $X_s(f, g) = \{(u, v) \in S \times S | s = u + v, f(u) \neq 0, g(v) \neq 0\}$ be a finite set by [8, Theorem 4.1] for every $s \in S$ and $f, g \in [[[R^{S, \leq}]]], [[[R^{S, \leq}]]]$ becomes a ring, with unit element $e^*$, namely

\[ e^*(0) = 1, \quad e^*(s) = 0 \text{ for every } s \in S, \quad s \neq 0. \]

The elements of $[[[R^{S, \leq}]]]$ are called generalized power series with coefficients in $R$ and exponents in $S$. For any $a \in R$, $C_a \in [[[R^{S, \leq}]]]$ is given by $C_a(0) = a, C_a(s) = 0$ for all $0 \neq s \in S$. Ordered monoid $(S, \leq)$ is said to satisfy condition (S0) in case $s \geq 0$ for all $s \in S$. Henceforth, unless otherwise mentioned, in this paper, $(S, \leq)$ will always denote a strictly ordered monoid satisfying condition (S0).

In this paper, we show that if $R$ is a reduced ring, then ring $[[[R^{S, \leq}]]]$ is a $GM$-ring if and only if $R$ is a $GM$-ring. We also investigate $GM$-rings for some special Morita Contexts and module extensions rings over generalized power series rings. These given generalizations of [3, Theorem], [2, Theorem 6] and [2, Theorem 11].

2. Main results

**Lemma 2.1.** [6] Let $R$ be a ring, $M_{n \times n}(R)$ the ring of all $n \times n$ matrices with entries in $R$. Then $[[[M_{n \times n}(R)^{S, \leq}]]] \cong M_{n \times n}([[R^{S, \leq}]]]).$

**Lemma 2.2.** [8] Let $(S, \leq)$ be a cancellative torsion-free strictly ordered monoid and satisfy condition (S0), and let $f \in [[[R^{S, \leq}]]]$. Then $f \in U([[R^{S, \leq}]])$ if and only if $f(0) \in U(R)$.

**Lemma 2.3.** Let $R$ be a ring, and $e_1^2 = e_1, e_2^2 = e_2 \in R$. Then $[[[(e_1 Re_2)^{S, \leq}]]] = C_{e_1}[[[R^{S, \leq}]]]C_{e_2}$.

**Proof.** For any $f \in C_{e_1}[[[R^{S, \leq}]]]C_{e_2}$, there exists $g \in [[[R^{S, \leq}]]]$ such that $f = C_{e_1}gC_{e_2}$. Thus for any $s \in S$, we have $f(s) = (C_{e_1}gC_{e_2})(s) = C_{e_1}(0)(gC_{e_2})(s) = C_{e_1}(0)g(s)e_2(0) = e_1g(s)e_2 \in e_1Re_2$. So $f \in [[[e_2 Re_2]^{S, \leq}]]$. Hence $C_{e_1}[[[R^{S, \leq}]]]C_{e_2} \subseteq [[[e_1 Re_2]^{S, \leq}]]$. Conversely, for any $f \in [[[e_1 Re_2]^{S, \leq}]]$ and any $s \in supp(f)$, there exists $r_s \in R$ such that $0 \neq f(s) = e_1r_s e_2 \in e_1Re_2$. Define a map $g : S \rightarrow R$ via

\[ g(s) = \begin{cases} r_s, & s \in supp(f) \\ 0, & s \in S \setminus supp(f) \end{cases} \]

Clearly, $supp(g) = supp(f)$. Thus $g \in [[[R^{S, \leq}]]]$. For any $s \in supp(f), (C_{e_1}gC_{e_2})(s) = e_1g(s)e_2 = e_1r_se_2 = f(s)$, for any $s \in S \setminus supp(f), (C_{e_1}gC_{e_2})(s) = 0 = f(s)$. Thus $f = C_{e_1}gC_{e_2} \in C_{e_1}[[[R^{S, \leq}]]]C_{e_2}$. This implies that $[[[(e_1 Re_2)^{S, \leq}]]] \subseteq C_{e_1}[[[R^{S, \leq}]]]C_{e_2}$. Therefore we have $[[[(e_1 Re_2)^{S, \leq}]]] = C_{e_1}[[[R^{S, \leq}]]]C_{e_2}$. \qed
Lemma 2.4. If $R$ is a GM–ring. Then $[[R^{S\leq}]]$ is a GM–ring.

Proof. Let $f, g \in [[R^{S\leq}]]$, there exist $e^2 = e, f^2 = f \in R$ and $u \in U(R)$ such that $f(0) = eu, g(0) = fu^{-1} \in U(R)$ by $R$ is a GM–ring. Since $C_0C_{u^{-1}} = e^2$, and $(f - C_0C_u, u)(0) \in U(R)$, it is easy to see that $f - C_0C_u - C_fC_u^{-1} \in U(R)$, and $C_0^2 = C_e, C_f^2 = C_f, C_u \in U(R)$. Thus $[[R^{S\leq}]]$ is a GM–ring.

Example 1 Let $\mathbb{N} \cup \{0\}$ denote the monoid which consists of natural numbers and zero. If $S = \mathbb{N} \cup \{0\}$ with the usual order. Then $[[R^{S\leq}]] \cong R[[X]]$ (rings of formal power series in one indeterminate and coefficients in $R$). So if $R$ is a GM–ring, then $R[[X]]$ is also a GM–ring. [2, Theorem 14]

Example 2 Let $S = \mathbb{N}^n \cup \{0\}$, with the usual order($\Pi \leq_i$), or the lexicographic ($\text{lex} \leq_i$) order, or the reverse lexicographic($\text{revlex} \leq_i$) order. If $R$ is a GM–ring, then $[[R^{\mathbb{N}^n \cup \{0\}, \Pi \leq_i}]]$, $[[R^{\mathbb{N}^n \cup \{0\}, \text{lex} \leq_i}]]$, $[[R^{\mathbb{N}^n \cup \{0\}, \text{revlex} \leq_i}]]$ are also GM–rings. Since rational number field $\mathbb{Q}$ and real number field $\mathbb{R}$ are GM–rings, then $[[\mathbb{Q}^{\mathbb{N}^n \cup \{0\}, \Pi \leq_i}]]$, $[[\mathbb{Q}^{\mathbb{N}^n \cup \{0\}, \text{lex} \leq_i}]]$ and $[[\mathbb{Q}^{\mathbb{N}^n \cup \{0\}, \text{revlex} \leq_i}]]$ are also GM–rings.

Let $(S_1, \leq_1), (S_2, \leq_2), \ldots, (S_n, \leq_n)$ be cancellative torsion-free strictly ordered monoids satisfying the condition (S0). If $R$ is a GM–ring, then $[[R^{S_1 \times S_2 \times \cdots \times S_n, \Pi \leq_i}]]$, $[[R^{S_1 \times S_2 \times \cdots \times S_n, \text{lex} \leq_i}]]$ are also GM–rings.

A ring $R$ is called reduced if it has no nonzero nilpotent element. It was proved in [5, Lemma 3.4] that if $R$ is a reduced ring, and $(S, \leq)$ is a cancellative torsion-free strictly ordered monoid. Then for every idempotent $f^2 = f \in [[R^{S\leq}]]$, there exists an idempotent $e \in R$ such that $f = C_e$.

Lemma 2.5. Let $R$ be a reduced ring, $(S, \leq)$ a cancellative torsion-free strictly ordered monoid. If $[[R^{S\leq}]]$ is a GM–ring, then $R$ is a GM–ring.

Proof. Let $a, b \in R$, then $C_a, C_b \in [[R^{S\leq}]]$. Since $[[R^{S\leq}]]$ is a GM–ring, there exist $C_e^2 = C_e, C_f^2 = C_f \in [[R^{S\leq}]]$ where $e^2 = e \in R, f^2 = f \in R$, and $\tau \in U([[R^{S\leq}]]$ such that $C_a - C_e\tau, C_b - C_f\tau^{-1} \in U([[R^{S\leq}]]$. Thus $(C_a - C_e\tau)(0) = a - e\tau(0) \in U(R)$ and $(C_b - C_f\tau^{-1})(0) = b - f\tau^{-1}(0) \in U(R)$. This implies that $R$ is a GM–ring.

Example 3 Let $R$ be a reduced ring. If the formal power series ring $R[[X]]$ is a GM–ring, then so is $R$ by Lemma 2.5. This can be proved in a directly simple manner. Given any $x, y \in R$, we have $x, y \in R[[X]]$ as well. Thus we can find
idempotents \( e(x), f(x) \in R[[X]] \) and a unit \( u(x) \in R[[X]] \) such that \( x - e(x)u(x), y - f(x)u(x)^{-1} \in U(R[[X]]) \). It is well known that \( h(x) \in R[[X]] \) is a unit if and only if \( h(0) \in R \) is a unit, and if \( R \) is a reduced ring, then the set of all idempotents in \( R[[X]] \) equal to the set of all idempotents in \( R \). Thus we know \( x - e(0)u(0), y - f(0)u(0)^{-1} \in U(R) \), One easily checks that \( e(0) = e, f(0) = f \) are idempotents and \( u(0) \in R \) is a unit. Thus \( R \) is a GM-ring.

Let \( e_1, e_2, \ldots, e_n \in R \) be idempotents. Clearly,

\[
\begin{pmatrix}
C_{e_1}[[R^{S, \leq}]]C_{e_1} & \ldots & C_{e_1}[[R^{S, \leq}]]C_{e_n} \\
\vdots & \ddots & \vdots \\
C_{e_n}[[R^{S, \leq}]]C_{e_1} & \ldots & C_{e_n}[[R^{S, \leq}]]C_{e_n}
\end{pmatrix}
= \begin{pmatrix}
C_{e_1}r_{11}C_{e_1} & \ldots & C_{e_1}r_{1n}C_{e_n} \\
\vdots & \ddots & \vdots \\
C_{e_n}r_{n1}C_{e_1} & \ldots & C_{e_n}r_{nn}C_{e_n}
\end{pmatrix}_{r_{ij} \in [[R^{S, \leq}]](0 \leq i, j \leq n)}
\]

form a ring with the identity \( \text{diag}(C_{e_1}, \ldots, C_{e_n}) \).

**Theorem 2.6.** Let \( e_1, e_2, \ldots, e_n \) be idempotents of a ring \( R \). If all \( e_iRe_i \) are GM-rings, then so is the ring

\[
\begin{pmatrix}
C_{e_1}[[R^{S, \leq}]]C_{e_1} & \ldots & C_{e_1}[[R^{S, \leq}]]C_{e_n} \\
\vdots & \ddots & \vdots \\
C_{e_n}[[R^{S, \leq}]]C_{e_1} & \ldots & C_{e_n}[[R^{S, \leq}]]C_{e_n}
\end{pmatrix}
\]

**Proof.** Clearly, the ring \( \begin{pmatrix}
e_1Re_1 & \ldots & e_1Re_n \\
\vdots & \ddots & \vdots \\
e_nRe_1 & \ldots & e_nRe_n
\end{pmatrix} \) is a GM-ring by virtue of [2, Lemma 1]. Since

\[
\begin{pmatrix}
e_1Re_1 & \ldots & e_1Re_n \\
\vdots & \ddots & \vdots \\
e_nRe_1 & \ldots & e_nRe_n
\end{pmatrix}^{S, \leq}
\]

\[
\begin{pmatrix}
\text{diag}(e_1, \ldots, e_n)M_n(R)\text{diag}(e_1, \ldots, e_n)^{S, \leq}
\end{pmatrix}
\]

\[
\begin{pmatrix}
\text{diag}(e_1, \ldots, e_n)^{S, \leq}\text{diag}(M_n(R))^{S, \leq}\text{diag}(e_1, \ldots, e_n)^{S, \leq}
\end{pmatrix}
\]

\[
\text{diag}(C_{e_1}, \ldots, C_{e_n})M_n([[R^{S, \leq}]]\text{diag}(C_{e_1}, \ldots, C_{e_n}))
\]
\[
\begin{pmatrix}
C_1[R^{S,\leq}]C_1 & \cdots & C_1[R^{S,\leq}]C_n \\
\vdots & \ddots & \vdots \\
C_n[R^{S,\leq}]C_1 & \cdots & C_n[R^{S,\leq}]C_n
\end{pmatrix}
\]

Apply Lemma 2.4, we get the result.

Let \( M \) be an \( R^- \) module. \([M^{S,\leq}]\) denotes the set of all maps \( \phi : S \to M \) such that \( \text{supp}(\phi) = \{ s \in S | \phi(s) \neq 0 \} \) is artinian and narrow. From [9], it is immediate that \([M^{S,\leq}]\) is an \([R^{S,\leq}]\) module. For any \( f \in [R^{S,\leq}], \phi \in [M^{S,\leq}] \) and \( s \in S \), the scalar multiplication is defined as follow:

\[
(f\phi)(s) = \sum_{(u,v) \in X_s(f,\phi)} f(u)\phi(v).
\]

Let \( A_1, A_2, A_3 \) be associative rings with identity. Let \( M_{21}, M_{31}, M_{32} \) be \((A_2, A_1)-, (A_3, A_1)-, (A_3, A_2)-\)bimodule, respectively. Let \( \psi : M_{32} \otimes_{A_2} M_{21} \to M_{31} \) be an \((A_3, A_1)-\)homomorphism, and let

\[
T = \begin{pmatrix} A_1 & 0 & 0 \\ M_{21} & A_2 & 0 \\ M_{31} & M_{32} & A_3 \end{pmatrix},
T^S = \begin{pmatrix}
[[A_1^{S,\leq}]] & 0 & 0 \\
[[M_{21}^{S,\leq}]] & [[A_2^{S,\leq}]] & 0 \\
[[M_{31}^{S,\leq}]] & [[M_{32}^{S,\leq}]] & [[A_3^{S,\leq}]]
\end{pmatrix},
\]

with the usual matrix operations (see[4]), \( T \) is a ring. Now we show that \( T^S \) is also a ring.

**Theorem 2.7.** There exists a \(((A_3^{S,\leq}],[A_1^{S,\leq}]))-\)homomorphism

\[
\psi^S : [[M_{32}^{S,\leq}]] \otimes [[M_{21}^{S,\leq}]] \to [[M_{31}^{S,\leq}]]
\]
such that with the usual matrix operations , \( T^S \) is a ring.

**Proof.** Since \( M_{32}, M_{21} \) is \((A_3, A_2)-,(A_3, A_1)-\)bimodule, respectively, according to [9], it is easy to see that \([M_{32}^{S,\leq}]\) is a \(((A_3^{S,\leq}],[A_2^{S,\leq}]))-\)bimodule, and \([M_{21}^{S,\leq}]\) is a \(((A_2^{S,\leq}],[A_1^{S,\leq}]))-\)bimodule. Consider following diagram:

\[
\begin{array}{ccc}
[[M_{32}^{S,\leq}]] \times [[M_{21}^{S,\leq}]] & \xrightarrow{\pi} & [[M_{32}^{S,\leq}]] \\
\downarrow & & \uparrow \psi^S \\
[[M_{31}^{S,\leq}]] & \xrightarrow{\theta} & [[M_{21}^{S,\leq}]]
\end{array}
\]
Let $n \in [[M_{32}^{S,\leq}]]$ and $m \in [[M_{21}^{S,\leq}]]$. Define a map

$$\alpha_{[n, m]} : S \rightarrow M_{31}, \quad \alpha_{[n, m]}(s) = \sum_{(u, v) \in X_s(n, m)} \psi(n(u) \otimes m(v))$$

for any $s \in S$. It is clearly that $\text{supp}(\alpha_{[n, m]}) \subseteq \text{supp}(n) + \text{supp}(m)$, thus $\alpha_{[n, m]} \in [[M_{31}^{S,\leq}]]$.

Define a map $f : [[M_{32}^{S,\leq}]] \times [[M_{21}^{S,\leq}]] \rightarrow [[M_{31}^{S,\leq}]]$, where $f((n, m)) = \alpha_{[n, m]}$ for any $(n, m) \in [[M_{32}^{S,\leq}]] \times [[M_{21}^{S,\leq}]]$. Let $n_1, n_2 \in [[M_{32}^{S,\leq}]], m \in [[M_{21}^{S,\leq}]]$. By the preceding discussions, there exist $\alpha_{[n_1, m]}, \alpha_{[n_2, m]}, \alpha_{[n_1 + n_2, m]} \in [[M_{31}^{S,\leq}]]$. For all $s \in S$,

$$\alpha_{[n_1 + n_2, m]}(s) = \sum_{(u, v) \in X_s(n_1 + n_2, m)} \psi((n_1 + n_2)(u) \otimes m(v))$$

$$= \sum_{(u, v) \in X_s(n_1 + n_2, m)} \psi(n_1(u) \otimes m(v)) + \sum_{(u, v) \in X_s(n_1 + n_2, m)} \psi(n_2(u) \otimes m(v)).$$

If $(u', v') \in X_s(n_1, m)$, but $(u', v') \in X_s(n_1 + n_2, m)$, then we have $(n_1 + n_2)(u') = 0$. So $n_2(u') \neq 0$, thus $(u', v') \in X_s(n_2, m)$ and $\psi(n_1(u') \otimes m(v')) + \psi(n_2(u') \otimes m(v')) = \psi((n_1(u') + n_2(u')) \otimes m(v')) \neq 0$. Likewise, if $(u', v') \in X_s(n_2, m)$, but $(u', v') \in X_s(n_1 + n_2, m)$, we also have $(u', v') \in X_s(n_1, m)$ and $\psi(n_1(u') \otimes m(v')) + \psi(n_2(u') \otimes m(v')) = \psi((n_1(u') + n_2(u')) \otimes m(v')) = 0$. So

$$\alpha_{[n_1 + n_2, m]}(s) = \sum_{(u, v) \in X_s(n_1 + n_2, m)} \psi(n_1(u) \otimes m(v))$$

$$+ \sum_{(u, v) \in X_s(n_1 + n_2, m)} \psi(n_2(u) \otimes m(v))$$

$$= \alpha_{[n_1, m]}(s) + \alpha_{[n_2, m]}(s) = (\alpha_{[n_1, m]} + \alpha_{[n_2, m]})(s).$$

Thus $\alpha_{[n_1 + n_2, m]} = \alpha_{[n_1, m]} + \alpha_{[n_2, m]}$, hence $f((n_1 + n_2, m)) = f((n_1, m)) + f((n_2, m))$.

Analogously, we see that $f((m, n_1 + n_2)) = f((m, n_1)) + f((m, n_2))$ for all $n \in [[M_{32}^{S,\leq}]], m_1, m_2 \in [[M_{21}^{S,\leq}]]$. 

EXTENSIONS OF GM-RINGS
For any \( n \in [[M_{32}^{S \leq}]], \tau \in [[A_2^{S \leq}]], m \in [[M_{21}^{S \leq}]] \) and any \( s \in S \), we have

\[
f((n \tau, m))(s) = \alpha_{[n \tau, m]}(s) = \sum_{(u', u) \in X_s(n \tau, m)} \psi((n \tau)(u') \otimes m(u)) = \sum_{(u', u) \in X_s(n \tau, m)} \psi(\sum_{(v, w) \in X_u(n, \tau)} (n(v) \tau(w) \otimes m(u)) = \sum_{(u', u) \in X_s(n \tau, m)} \sum_{(v, w) \in X_u(n, \tau)} \psi(n(v) \tau(w) \otimes m(u)) = \sum_{(v,w,u) \in X} \psi(n(v) \tau(w) \otimes m(u)) = \sum_{(v,w,u) \in X_s(n, \tau, m)} \psi(n(v) \otimes \tau(w) m(u)) = \sum_{(v,w,u) \in X_s(n, \tau, m)} f(n, \tau m)(s).
\]

Where \( X = \{(v, w, u) \in X_s(n, \tau, m) | n \tau(v + w) = 0\} \). Thus we have \( f(n \tau, m) = f(n, \tau m) \) and hence \( f \) is a bilinear balanced morphism. Then there exists a homomorphism \( \psi^S : [[M_{32}^{S \leq}]] \otimes [[M_{21}^{S \leq}]] \rightarrow [[M_{21}^{S \leq}]] \) such that the preceding diagram commutes.

Next, we check that \( \psi^S \) is a bimodule homomorphism. For any \( a \in [[A_3^{S \leq}]], n \in [[M_{32}^{S \leq}]], m \in [[M_{21}^{S \leq}]] \) and any \( s \in S \).

\[
\psi^S(an, m)(s) = \alpha_{[an, m]}(s) = \sum_{(u', u) \in X_s(an, m)} \psi((an)(u') \otimes m(u)) = \sum_{(u', u) \in X_s(an, m)} \psi(\sum_{v, w) \in X_u(a, n)} (a(v)n(w) \otimes m(u)) = \sum_{(v,w,u) \in X_s(a, n, m)} a(v) \psi(n(w) \otimes m(u)) = a\psi^S(n, m)(s).
\]
Thus \( \psi^S(\alpha n, m) = a \psi^S(\alpha m, n) \). This implies that \( \psi^S \) is a left \([A_1^{S \leq}]\)-module homomorphism. Analogously, it is easy to verify that \( \psi^S \) is a right \([A_1^{S \leq}]\)-module homomorphism. Thus \( \psi^S \) is a bimodule homomorphism. With the usual matrix operations, \( T^S \) is a ring, see [4].

\[ \text{Lemma 2.5.} \]

**Theorem 2.8.** Let \( A_1, A_2, A_3 \) be reduced rings, \((S, \leq)\) a cancellative torsion-free strictly ordered monoid. Then the following conditions are equivalent:

1. \( A_1, A_2, \) and \( A_3 \) are GM-rings.
2. The formal triangular matrix ring over generalized power series

\[
T^S = \begin{pmatrix}
[A_1^{S \leq}] & 0 & 0 \\
[M_{21}^{S \leq}] & [A_2^{S \leq}] & 0 \\
[M_{31}^{S \leq}] & [M_{32}^{S \leq}] & [A_3^{S \leq}]
\end{pmatrix}
\]

is a GM-ring.

**Proof.** (1) \( \Rightarrow \) (2) Since \( A_1, A_2, \) and \( A_3 \) are GM-rings, so are rings \([A_1^{S \leq}], [A_2^{S \leq}]\) and \([A_3^{S \leq}]\) by virtue of Lemma 2.4. According to [2, Theorem 6], the result follows.

(2) \( \Rightarrow \) (1) Applying [2, Theorem 6], we have \([A_1^{S \leq}], [A_2^{S \leq}]\) and \([A_3^{S \leq}]\) are GM-rings. Then according to Lemma 2.5, we get the result.

**Example 4** Let \( A_1, A_2, A_3 \) be reduced rings and \( N \) the semigroup of natural numbers. Let \( S = N \cup \{0\} \), with the usual order. then

\[
T^S = \begin{pmatrix}
[A_1^{S \leq}] & 0 & 0 \\
[M_{21}^{S \leq}] & [A_2^{S \leq}] & 0 \\
[M_{31}^{S \leq}] & [M_{32}^{S \leq}] & [A_3^{S \leq}]
\end{pmatrix}
\]

\[
\cong \begin{pmatrix}
A_1[[X]] & 0 & 0 \\
M_{21}[[X]] & A_2[[X]] & 0 \\
M_{31}[[X]] & M_{32}[[X]] & A_3[[X]]
\end{pmatrix}
\]

where \( A_1[[X]](i = 1, 2, 3) \) is the ring of formal power series, and \( M_{ij}[[X]](i = 2, 3, j = 1, 2) \) is a bimodule of power series rings. If \( A_1, A_2, A_3 \) are GM-rings, then \( T^S \) is also a GM-ring. Actually, let

\[
F = \begin{pmatrix}
f_1 & 0 & 0 \\
m_{21} & f_2 & 0 \\
m_{31} & m_{32} & f_3
\end{pmatrix} \in T^S, \quad G = \begin{pmatrix}
g_1 & 0 & 0 \\
n_{21} & g_2 & 0 \\
n_{31} & n_{32} & g_3
\end{pmatrix} \in T^S.
\]

Since \( A_i(i = 1, 2, 3) \) is a GM-ring, by Lemma 2.4, we have \( A_i[[X]] \) is also a GM-ring. Thus there exist \( e_i = e_i, p_i^2 = p_i \in A_i[[X]], u_i \in U(A_i[[X]]) \) and \( v_i \in U(A_i[[X]]) \), \( v_i' \in U(A_i[[X]]) \) such that \( f_i = e_i + v_i + v_i' \) and \( g_i = p_i u_i + v_i \).
Set
\[ F_1 = \begin{pmatrix} e_1 & 0 & 0 \\ 0 & e_2 & 0 \\ 0 & 0 & e_3 \end{pmatrix}, \quad W = \begin{pmatrix} u_1 & 0 & 0 \\ 0 & u_2 & 0 \\ 0 & 0 & u_3 \end{pmatrix}, \quad K_1 = \begin{pmatrix} v_1 & 0 & 0 \\ m_{21} & v_2 & 0 \\ m_{31} & m_{32} & v_3 \end{pmatrix}. \]

It is easy to verify that \( F_1^2 = F_1 \in T^S \), and
\[ K_1 = \begin{pmatrix} v_1^{-1} & 0 & 0 \\ -v_2^{-1}m_{21}v_1^{-1} & v_2^{-1} & 0 \\ v_3^{-1}m_{32}v_2^{-1} - v_3^{-1}m_{31}v_1^{-1} & -v_3^{-1}m_{32}v_2^{-1} & v_3^{-1} \end{pmatrix} \]
\[ = \begin{pmatrix} v_1^{-1} & 0 & 0 \\ -v_2^{-1}m_{21}v_1^{-1} & v_2^{-1} & 0 \\ v_3^{-1}m_{32}v_2^{-1} - v_3^{-1}m_{31}v_1^{-1} & -v_3^{-1}m_{32}v_2^{-1} & v_3^{-1} \end{pmatrix} K_1 \]
\[ = \text{diag}(1,1,\ldots,1), \]

This means that \( F_1 \) is an idempotent and \( K_1 \) is a unit. Moreover, \( F = F_1W + K_1 \) and \( W \) is a unit. Analogously, we have an idempotent \( F_2 = \begin{pmatrix} p_1 & 0 & 0 \\ 0 & p_2 & 0 \\ 0 & 0 & p_3 \end{pmatrix} \), and a unit \( K_2 = \begin{pmatrix} v'_1 & 0 & 0 \\ n_{21} & v'_2 & 0 \\ n_{31} & n_{32} & v'_3 \end{pmatrix} \) such that \( G = F_2W^{-1} + K_2 \). Therefore we conclude that \( T^S \) is a \( GM \)-ring. Conversely, if \( T^S \) is a \( GM \)-ring, similar to the proof of Theorem 6 in [2], we obtain that \( A_i[[X]] \) is a \( GM \)-ring. Then by Lemma 2.5, we have \( A_i(i = 1, 2, 3) \) is a \( GM \)-ring.

**Corollary 2.9.** Let \( R \) be a reduced ring, \((S, \leq)\) a cancellative torsion-free strictly ordered monoid. A ring \( R \) is a \( GM \)-ring if and only if the ring of all \( n \times n \) lower triangular matrices over \([[[R^{S,\leq}]])\) is a \( GM \)-ring.

**Proof.** According to Theorem 2.8, the result follows. \( \square \)

Analogously, let \( R \) be a reduced ring, \((S, \leq)\) a cancellative torsion-free strictly ordered monoid. we deduce that a ring \( R \) is a \( GM \)-ring if and only if the ring of all \( n \times n \) upper triangular matrices over \([[[R^{S,\leq}]])\) is a \( GM \)-ring.

Let \( M \) be a \((R, R)\)-bimodule, then the module extension of \( R \) by \( M \) is the ring \( R \bowtie M \) with the usual addition and multiplication defined by \( (r_1, m_1)(r_2, m_2) = (r_1r_2, r_1m_2 + m_1r_2) \) for \( r_1, r_2 \in R \) and \( m_1, m_2 \in M \). Now we investigate \( GM \)-rings for module extension of \([[[R^{S,\leq}]])\) by \([[[M^{S,\leq}]])\) and introduce a large class of such rings.
Lemma 2.10. Let ring $R \bowtie M$ be the module extension of $R$ by $M$. Let $[[R^{S} \leq]] \bowtie [[M^{S} \leq]]$ be the module extension of $[[R^{S} \leq]]$ by $[[M^{S} \leq]]$. Then $[[R^{S} \leq]] \bowtie [[M^{S} \leq]] \cong [[[R \bowtie M]^{S} \leq]]$.

Proof. Let
\[
T(R, M) = \left\{ \begin{pmatrix} r & m \\ 0 & r \end{pmatrix} \mid r \in R, m \in M \right\},
\]
\[
T^{*}(R, M) = \left\{ \begin{pmatrix} f & m \\ 0 & f \end{pmatrix} \mid f \in [[R^{S} \leq]], m \in [[M^{S} \leq]] \right\}.
\]
With the usual matrix operations, $T(R, M)$ and $T^{*}(R, M)$ are rings. As in the proof of [7, Proposition 4.3], it is easy to show that $T^{*}(R, M) \cong [[T(R, M)^{S} \leq]]$. Moreover, $R \bowtie M \cong T(R, M)$ and $[[R^{S} \leq]] \bowtie [[M^{S} \leq]] \cong T^{*}(R, M)$. So $[[R^{S} \leq]] \bowtie [[M^{S} \leq]] \cong [[[R \bowtie M]^{S} \leq]]$, as asserted. \hfill \Box

Theorem 2.11. Let $R$ be a ring, $M$ a $(R, R)$ bimodule. If $R$ is a GM ring, then $[[R^{S} \leq]] \bowtie [[M^{S} \leq]]$ is a GM ring.

Proof. Since $R$ is a GM ring, so is ring $R \bowtie M$ by [2, Theorem 11]. Use the fact that $[[R^{S} \leq]] \bowtie [[M^{S} \leq]] \cong [[[R \bowtie M]^{S} \leq]]$, then the result follows by Lemma 2.4. \hfill \Box

Corollary 2.12. Let $R$ be a ring. If $R$ is a GM ring, then $[[R^{S} \leq]] \bowtie [[R^{S} \leq]]$ is a GM ring.

Proof. It is an immediate consequence of Theorem 2.11. \hfill \Box

Corollary 2.13. Let $R$ be an exchange ring with artinian primitive factors. Then $[[R^{S} \leq]] \bowtie [[R^{S} \leq]]$ is a GM ring.

Proof. Since $R$ is an exchange ring with artinian primitive factors, it is a GM ring. Thus we get the result by Corollary 2.12. \hfill \Box

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