International Electronic Journal of Algebra
Volume 2 (2007) 71-82

MODULES SATISFYING THE ASCENDING CHAIN CONDITION ON SUBMODULES WITH A BOUNDED NUMBER OF GENERATORS

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Received: 18 November 2006; Revised: 26 February 2007
Communicated by Nguyen Viet Dung

Abstract. It is proved that if R is a right and left Noetherian ring then the right R-module R^I satisfies the ascending chain condition on n-generated submodules, for every positive integer n.

Mathematics Subject Classification (2000): 16P70
Keywords: Chain conditions; Bimodule; Noetherian module; Direct product.

All rings have identity elements and all modules are unital right modules, unless stated otherwise. Let R be a ring and let M be an R-module. Given a positive integer n, the module M satisfies *n-acc* provided every ascending chain of n-generated submodules terminates. Moreover, the module M satisfies *pan-acc* in case M satisfies n-acc for every positive integer n. Modules with n-acc have been considered by many authors (see, for example, [1]-[7], [10], [12] and [13]). In particular, Renault [13, Corollaire 3.3] proved that if R is a right and left Noetherian ring then every free right (or left) R-module satisfies pan-acc. He also gave an example of a right Noetherian ring R such that every free right R-module of infinite rank does not satisfy 1-acc (see [13, p. 274]). Renault's paper was the inspiration for this present work. More recently, Frohn [7, Theorem 3.3] proved that if R is a commutative Noetherian ring then every direct product R^I of copies of the R-module R indexed by a set I is an R-module satisfying pan-acc, for every such index set I.

The purpose of this note is to show that if R is a right and left Noetherian ring then the right (or left) R-module R^I satisfies pan-acc, thus generalizing the theorems of both Renault and Frohn. In fact, we shall prove rather more, namely that if S and R are rings and M a left S-, right R-bimodule such that M is Noetherian both as a left S-module and as a right R-module then the right R-module M^I satisfies pan-acc and the left S-module M^I satisfies pan-acc, for every index set I. Note that in [1, Corollary 1.6] Antunes Simões and Smith proved that if R is a ring with finite right uniform dimension then any direct product of nonsingular Noetherian right R-modules satisfies pan-acc. In particular, if R is a right nonsingular right Noetherian ring then the right R-module R^I satisfies pan-acc, for every index set I.

It might be worth reminding ourselves of what happens for Abelian groups. Let $\mathbf{Z}$ denote the ring of integers. Pontrjagin proved that a countably generated torsion-free $\mathbf{Z}$-module A satisfies pan-ac if and only if A is free (see [8, Vol I Theorem 19.1]). More generally, Baumslag and Baumslag [4, Theorem 3] proved that a $\mathbf{Z}$-module A satisfies pan-acc if and only if A satisfies the following conditions:

(i) A is reduced,

(ii) there exists only a finite number of primes p in $\mathbf{Z}$ such that $pa = 0$ for some non-zero element a in A, and

(iii) every countably generated torsion-free submodule of A is free.

In particular, if A is a torsion $\mathbf{Z}$-module then A satisfies pan-acc if and only if A satisfies 1-acc. The situation for torsion-free $\mathbf{Z}$-modules is quite different. For each positive integer n, Fuchs [8, Vol II p. 125] gave an example of a torsion-free $\mathbf{Z}$-module A_n which satisfies n-acc but not $(n+1)$-acc (see also [10, p. 272]).

Let A be a $\mathbf{Z}$-module. Given a prime p in $\mathbf{Z}$, we shall say that A is a *p-module* if for each $a \in A$ there exists a positive integer n such that $p^n a = 0$. Note the following simple fact which may be well known but which we include for convenience. Recall that a $\mathbf{Z}$-module A is called *reduced* provided A does not contain a non-zero divisible submodule.

Lemma 1. *Let p be any prime in $\mathbf{Z}$ and let a $\mathbf{Z}$-module A be a p-module. Then every homomorphic image of A satisfies pan-acc if and only if there exists a positive integer k such that $p^k A = 0$.*

Proof. The sufficiency follows by [4, Theorem 3] because every homomorphic image of A is clearly reduced if $p^k A = 0$ for some k. Conversely, suppose that every homomorphic image of A satisfies pan-acc. Suppose that there does not exist a positive integer k such that $p^k A = 0$. By [8, Vol I Theorem 32.3] there exists a submodule B of A such that B is a direct sum of cyclic submodules and A/B is divisible. Thus $A = B$ and A is a direct sum of cyclic submodules. There exists a submodule C of A such that $C = \bigoplus_{i \geq 1} \mathbf{Z}c_i$, where c_i is an element of A of order p^i for every positive integer i. Let $D = \mathbf{Z}(c_1 - pc_2) \bigoplus \mathbf{Z}(c_2 - pc_3) \bigoplus \dots$. Then $C \neq D$ and C/D is a non-zero divisible submodule of A/D so that A has a non-zero divisible homomorphic image, a contradiction. Thus $p^k A = 0$ for some positive integer k. $\square$

Our first theorem is a consequence of the above theorem of Baumslag and Baumslag.

Theorem 2. *Let* $\mathbf{Z}$ *denote the ring of integers and let* A *be a* $\mathbf{Z}$*-module. Then every homomorphic image of* A *is a* $\mathbf{Z}$*-module satisfying pan-acc if and only if* $A = F \bigoplus T$ *for some finitely generated free submodule* F *of* A *and some submodule* T *of* A *such that* $nT = 0$ *for some positive integer* n*.*

Proof. ($\Rightarrow$) Note first that [4, Theorem 3] gives that A/B is reduced for every submodule B of A. Let T denote the torsion submodule of A. Then A/T is a torsion free $\mathbf{Z}$-module satisfying pan-acc. Suppose that A/T is not finitely generated. Then there exists a submodule C of A, containing T, such that C/T is countably, but not finitely, generated. By [4, Theorem 3] (or see [8, Vol I Theorem 19.1]), C/T is free and hence the $\mathbf{Z}$-module $\mathbf{Q}$ is a homomorphic image of C/T. This implies that $\mathbf{Q}$ is a homomorphic image of A, a contradiction. Thus A/T is finitely generated and hence $A = T \bigoplus F$ for some finitely generated free submodule F of A. Now suppose that T is non-zero. Again using [4, Theorem 3] there exist finitely many distinct primes p_i $(1 \leq i \leq t)$ in $\mathbf{Z}$, for some positive integer t, such that $T = T(p_1) \bigoplus \cdots \bigoplus T(p_t)$, where $T(p_i)$ is the p_i-primary component of T, for each $1 \leq i \leq t$. By Lemma 1, for each $1 \leq i \leq t$ there exists a positive integer k_i such that $p_i^{k_i} T(p_i) = 0$. Let $n = p_1^{k_1} \dots p_t^{k_t}$. Then $nT = 0$.

($\Leftarrow$) Now suppose that $A = F \bigoplus T$ where F is a finitely generated free submodule and T is a torsion submodule such that $nT = 0$ for some non-zero n in $\mathbf{Z}$. Let D be any proper submodule of A. Let E be the submodule of A containing D such that E/D is the torsion submodule of A/D. Note that $D+T \subseteq E$. In particular, $T \subseteq E$ so that A/E is finitely generated torsion-free and hence free. Moreover $E/(D+T)$ is also finitely generated, so that $mE \subseteq D + T$ for some positive integer m. Thus $mnE \subseteq D$. By [4, Theorem 3] it follows that A/D satisfies pan-acc. □

We chose to prove Theorem 2 to point out that if A is a $\mathbf{Z}$-module such that every homomorphic image of A satisfies pan-acc then A is a direct sum of cyclic submodules (see [8, Vol I Theorem 17.2]). In view of this fact and Pontrjagin's Theorem above it would appear that there is some relationship between direct sum decompositions and the property pan-acc. Now in order to prove the above results of Renault and Frohn we shall look at modules satisfying a particular property which can be stated in terms of direct sum decompositions.

Let R be a ring. An R-module M will be said to *satisfy the direct sum condition* provided every countably generated submodule is contained in a direct sum of finitely generated submodules of M. Clearly every free module and every semisimple module satisfies the direct sum condition. More generally, every direct sum of finitely generated R-modules satisfies the direct sum condition. Note also that if M_i is an R-module satisfying the direct sum condition, for all i in some index set I, then the R-module $\bigoplus_{i\in I} M_i$ also satisfies the direct sum condition. For, let N be any countably generated submodule of the module $M = \bigoplus_{i\in I} M_i$. For each $i \in I$, let $\pi_i : M \to M_i$ denote the canonical projection. Because, for each $i \in I$, $\pi_i(N)$ is a countably generated submodule of M_i, there exists a submodule K_i of M_i such that K_i is a direct sum of finitely generated submodules and $\pi_i(N) \subseteq K_i$. Let $K = \bigoplus_{i\in I} K_i$. Then N is contained in the submodule K of M and K is a direct sum of finitely generated submodules.

We want to show that certain direct products satisfy the direct sum condition, in particular modules of the form M^I, the direct product of copies of a module M indexed by a set I. If J is a non-empty subset of I then M^J will be considered a submodule of M^I in the natural way. If R and S are rings and M a left S-, right R-bimodule then M^I is a left S-, right R-bimodule in the natural way.

We first note the following simple fact.

Lemma 3. *Let R be a ring and let L be a countably generated submodule of an R-module M. Then the following statements are equivalent.*

(i) L is contained in a direct sum of finitely generated submodules of M.

(ii) There exists a submodule K of M containing L such that every finitely generated submodule of L is contained in a finitely generated direct summand of K.

Proof. (i) $\Rightarrow$ (ii). Let M_i $(i \in I)$ be a collection of finitely generated submodules of M such that $L \subseteq \bigoplus_{i\in I} M_i$. Then $K = \bigoplus_{i\in I} M_i$ satisfies (ii).

(ii) $\Rightarrow$ (i). Let $L = x_1R + x_2R + \dots$. By hypothesis there exist submodules E_1 and F_1 of K such that $K = E_1 \bigoplus F_1$, E_1 is finitely generated and $x_1R \subseteq E_1$. Again, by hypothesis, there exist submodules E_2 and F_2 of K such that $K = E_2 \bigoplus F_2$, E_2 is finitely generated and $E_1 + x_2R \subseteq E_2$. Note that $x_1R + x_2R \subseteq E_2 = E_1 \bigoplus (E_2 \bigcap F_1)$. Repeat this argument. For each positive integer $n \geq 2$, there exist submodules E_n and F_n of K such that $K = E_n \bigoplus F_n$, E_n is finitely generated and contains $x_1R + \cdots + x_nR$. Note that $E_n = E_1 \bigoplus (E_2 \bigcap F_1) \bigoplus \dots (E_n \bigcap F_{n-1})$. It follows that L is contained in the direct sum $E_1 \bigoplus (E_2 \bigcap F_1) \bigoplus (E_3 \bigcap F_2) \dots$, which is a direct sum of finitely generated submodules because E_n is a finitely generated submodule for each positive integer n. $\square$

The proof of the next result is adapted from the proof of [14, Splitting Lemma 6].

Lemma 4. *Let R and S be rings and let M be a left S-, right R-bimodule such that the left S-module M is Noetherian. Let I denote an index set and X the left S-, right R-bimodule M^I. Then, for each finitely generated submodule F of the right R-module X, there exist a finite subset J of I and an R-isomorphism $\varphi : X \to X$ such that $\varphi(F) \subseteq M^J$.*

Proof. Let F be any finitely generated submodule of the right R-module X. Then there exist a positive integer k and elements $x_i \in F$ $(1 \leq i \leq k)$ such that $F = x_1R + \cdots + x_kR$. Let $x = x_1$. There exist elements $m_i \in M$ $(i \in I)$ such that $x = (m_i)$. The S-submodule $\sum_{i \in I} Sm_i$ is finitely generated and hence there exists a finite subset J_1 of I such that $\sum_{i \in I} Sm_i = \sum_{j \in J_1} Sm_j$. For each i in I there exist elements $s_{ij} \in S$ $(j \in J_1)$ such that $m_i = \sum_{j \in J_1} s_{ij}m_j$. Define a mapping $\varphi_1 : X \to X$ as follows: for each element (u_i) in X, $\varphi_1(u_i) = (v_i)$ where $v_i = u_i$ if $i \in J_1$ and $v_i = u_i - \sum_{j \in J_1} s_{ij}u_j$ if $i \in I \backslash J_1$. It is not difficult to check that φ_1 is an R-isomorphism from X to X and that $\varphi_1(x) \in M^{J_1}$.

Let $I_1 = I \backslash J_1$, let $X_1 = M^{J_1}$ and let $X_2 = M^{I_1}$ so that $X = X_1 \bigoplus X_2$. For each $2 \leq i \leq k$ there exist elements $y_i \in X_1$ and $z_i \in X_2$ such that $\varphi_1(x_i) = y_i + z_i$. By induction on k there exists a finite subset J_2 of I_1 and an R-isomorphism $\varphi_2 : X_2 \to X_2$ such that $\varphi_2(z_2R + \cdots + z_kR) \subseteq M^{J_2}$. Now $\varphi_3 = \iota + \varphi_2$ is an R-isomorphism from X to X where ι is the identity mapping on X_1. Finally note that $\varphi = \varphi_3\varphi_1$ is an R-isomorphism from X to X such that $\varphi(F) \subseteq M^J$ where J is the finite subset $J_1 \bigcup J_2$ of I. $\square$

Theorem 5. *Let R and S be rings and let M be a left S-, right R-bimodule such that the left S-module M is Noetherian and the right R-module M is finitely generated. Then the right R-module M^I satisfies the direct sum condition for every index set I.*

Proof. Let F be any finitely generated submodule of the right R-module $X = M^I$. By Lemma 4 there exist a finite subset J of I and an R-isomorphism $\varphi : X \to X$ such that $\varphi(F) \subseteq M^J$. Let $J' = I \backslash J$, let $X_1 = \varphi^{-1}(M^J)$ and let $X_2 = \varphi^{-1}(M^{J'})$. Then the right R-module $X = X_1 \bigoplus X_2$ is a direct sum of the submodules X_1 and X_2, X_1 is a finitely generated right R-module and $F \subseteq X_1$. By Lemma 3 X satisfies the direct sum condition. $\square$

Next we give an example of a module M which satisfies the direct sum condition but which is not itself a direct sum of finitely generated submodules.

Example 6. *Let* $\mathbf{Z}$ *denote the ring of integers and let* M *denote the direct product* $\mathbf{Z}^I$ *for any infinite index set* I. *Then the* $\mathbf{Z}$*-module* M *satisfies the direct sum condition but* M *is not a direct sum of finitely generated submodules.*

Proof. By Theorem 5 the $\mathbf{Z}$-module M satisfies the direct sum condition. However M is not a direct sum of finitely generated submodules because M is not projective (see [8, Vol I Theorem 19.2]). $\square$

Let R be a ring and let M be a non-zero module. Then M has *finite uniform dimension* provided M does not contain an infinite direct sum of non-zero submodules. In this case there exists a positive integer n such that n is the maximum number of submodules of M which form a direct sum. The integer n is called the *uniform dimension* of M and is denoted by $u(M)$. In case $M = 0$ we say that M is *zero dimensional* and write $u(M) = 0$. The ring R has *finite right uniform dimension* in case the right R-module R has finite uniform dimension Note that every Noetherian module has finite uniform dimension. The next two results concern rings with finite right uniform dimension.

Lemma 7. *Let* R *be a ring with finite right uniform dimension, let* n *be a positive integer and let* M *be a nonsingular* n*-generated* R*-module. Then* M *has finite uniform dimension and* $u(M) \leq nu(R)$.

Proof. There exists an epimorphism from $F = R^{(n)}$ to M with kernel K. Because M is nonsingular, K is an essentially closed submodule of F and hence $u(M) = u(F/K) \leq u(F) = nu(R)$ by [9, Exercise 4N]. $\square$

Let R be a ring and let M be any R-module. Then the *singular submodule* $Z(M)$ of M is defined to be the set of elements m in M such that $mE = 0$ for some essential right ideal E of R. The *second singular submodule* of M is the submodule $Z_2(M)$ of M containing $Z(M)$ such that $Z_2(M)/Z(M)$ is the singular submodule of the module $M/Z(M)$. In the Goldie torsion theory, a module M is torsion if $M = Z_2(M)$ and is torsion-free if it is nonsingular, i.e. $Z(M) = 0$ (see [15] for more details).

Let R be a ring and let M be an R-module such that $MA = 0$ for some ideal A of R. Then M is both an R-module and an (R/A)-module. The singular submodule of the R-module M need not coincide with the singular submodule of the (R/A)-module M and we shall denote these submodules by $Z(M_R)$ and $Z(M_{R/A})$, respectively. Similarly we denote by $Z_2(M_R)$ and $Z_2(M_{R/A})$ the second singular submodules of M considered as an R-module and as an (R/A)-module, respectively. When there is no ambiguity we shall use $Z(M)$ and $Z_2(M)$, as indicated above. We want to make one further observation at this point, namely if R is a prime (or even semiprime) right Noetherian ring then $Z_2(M) = Z(M)$ for every R-module M by [9, Proposition 6.10].

Given a ring R and an R-module M, if N is a submodule of M then Zorn's Lemma gives a submodule K of M maximal among the submodules H of M such that $N \bigcap H = 0$. In this case, K is called a *complement* of N (in M). Note that K is essentially closed in M in the sense of [9]. The next result is crucial for the remainder of this paper.

Theorem 8. *Let R be a right Noetherian ring, let M be a right R-module which satisfies the direct sum condition and let n be a positive integer. Then M satisfies n-acc if and only if $Z_2(M)$ satisfies n-acc.*

Proof. The necessity is clear. Conversely, suppose that $Z_2(M)$ satisfies n-acc. Let $L_1 \subseteq L_2 \subseteq L_3 \subseteq \dots$ be any ascending chain of n-generated submodules of M. Let $L = \bigcup_{i \geq 1} L_i$. Let K be a complement of $Z_2(L)$ in L. Note that for all $i \geq 1$, $L_i/Z_2(L_i)$ is an n-generated nonsingular module and hence $u(L_i/Z_2(L_i)) \leq nu(R)$ by Lemma 7. Moreover note that for all $i \geq 1$, $L_i \bigcap K$ embeds in $L_i/Z_2(L_i)$ so that $u(L_i \bigcap K) \leq nu(R)$. Now the ascending chain $L_1 \bigcap K \subseteq L_2 \bigcap K \subseteq \dots$ gives that $L_i \bigcap K$ is essential in $L_{i+1} \bigcap K$ for all $i \geq k$, for some positive integer k, and hence $L_k \bigcap K$ is essential in K. By hypothesis, there exists a submodule H of M such that $L \subseteq H$, $H = H_1 \bigoplus H_2$ for some submodules H_1 and H_2, H_1 is finitely generated and $L_k \subseteq H_1$. Let $\pi : H \to H_2$ denote the canonical projection. Let $x \in L$. Because $Z_2(L) \bigoplus K$ is essential in L, we have $(xR + K)/K$ is Goldie torsion and hence so too is $[xR + (L_k \bigcap K)]/(L_k \bigcap K)$. It follows that $\pi(xR)$ is Goldie torsion. Thus $\pi(L_1) \subseteq \pi(L_2) \subseteq \dots$ is an ascending chain of n-generated submodules of $Z_2(M)$. There exists a positive integer t such that $\pi(L_t) = \pi(L_{t+1}) = \dots$. But H_1 is Noetherian and hence, without loss of generality, $L_t \bigcap H_1 = L_{t+1} \bigcap H_1 = \dots$. It follows that $L_t = L_{t+1} = \dots$, as required. □

Corollary 9. *Let R be a right Noetherian ring and let M be a nonsingular right R-module which satisfies the direct sum condition. Then M satisfies pan-acc.*

Proof. By the theorem. □

The next result is adapted from [13].

Lemma 10. *Let R be a right Noetherian ring and let M be a right R-module which satisfies the direct sum condition but does not satisfy pan-acc. Let P be an ideal of R which is maximal in the collection of ideals A of R such that there exist a positive integer k and a properly ascending chain $H_1 \subseteq H_2 \subseteq H_3 \dots$ of k-generated submodules H_i $(i \geq 1)$ of M with $H_i A = 0$ for all $i \geq 1$. Then P is a prime ideal of R.*

Proof. There exist a positive integer n and a properly ascending chain $L_1 \subseteq L_2 \subseteq L_3 \subseteq \dots$ of n-generated submodules L_i $(i \geq 1)$ such that $L_i P = 0$ for all $i \geq 1$. Suppose that P is not a prime ideal of R. Then there exist ideals A and B of R, each properly containing P, such that $AB \subseteq P$. Note that A is a q-generated right ideal of R, for some positive integer q, and hence $L_i A$ is an nq-generated submodule of M for each $i \geq 1$. By the choice of P, the ascending chain $L_1 A \subseteq L_2 A \subseteq \dots$ must terminate and hence there exists a positive integer s such that $L_s A = L_{s+1} A = L_{s+2} A = \dots$. Let L denote the countably generated submodule $\bigcup_{i \geq 1} L_i$. By hypothesis there exists a submodule K of M such that $L \subseteq K$, $K = K_1 \bigoplus K_2$ for some submodules K_1 and K_2, K_1 is finitely generated and $L_s \subseteq K_1$. Let $\pi : K \to K_2$ denote the canonical projection. Note that ker $\pi = K_1$ which is a Noetherian module. Moreover, for each $i \geq s$, $\pi(L_i)$ is an n-generated submodule of M such that $\pi(L_i)A \subseteq \pi(L_s) = 0$. By the choice of P, there exists an integer $t \geq s$ such that $\pi(L_t) = \pi(L_{t+1}) = \dots$. But K_1 is Noetherian so that without loss of generality we can suppose that $L_t \bigcap K_1 = L_{t+1} \bigcap K_1 = \dots$. It follows that $L_t = L_{t+1} = \dots$, a contradiction. Thus P is a prime ideal of R. □

There is a stronger version of Lemma 10 in the case of commutative Noetherian rings, namely:

Lemma 11. *Let R be a commutative Noetherian ring and let M be an R-module which satisfies the direct sum condition but does not satisfy n-acc for some positive integer n. Let P be an ideal of R which is maximal in the collection of ideals A of R such that there exists a properly ascending chain $H_1 \subseteq H_2 \subseteq H_3 \subseteq \dots$ of n-generated submodules H_i $(i \geq 1)$ of M with $H_i A = 0$ for all $i \geq 1$. Then P is a prime ideal of R.*

Proof. We adapt the proof of Lemma 10. In this case we can replace the ideals A and B by elements a and b. Note that $L_i a$ is an n-generated submodule of M for each $i \geq 1$ and the proof proceeds as before. □

Let R be a ring and let M be an R-module. Given a non-empty set W in M, the *annihilator* of W in R will be denoted by $\mathrm{ann}(W)$, i.e. $\mathrm{ann}(W)$ is the set of elements r in R such that $wr = 0$ for all $w \in W$. Note that ann(W) is a right ideal of R and in case W is a submodule of M then ann(W) is an ideal of R.

We now derive some consequences of Theorem 8 using Lemmas 10 and 11. The first consequence is the following result.

Theorem 12. *Let R be a commutative Noetherian ring, let M be an R-module which satisfies the direct sum condition and let n be a positive integer. Then M satisfies n-acc if and only if for each ascending chain $L_1 \subseteq L_2 \subseteq L_3 \subseteq \dots$ of n-generated submodules L_i $(i \geq 1)$ of M there exists a positive integer k such that ann(L_k) = ann$(L_{k+1}) = \dots$.*

Proof. The necessity is clear. Conversely, suppose that M satisfies the stated condition but that M does not satisfy n-acc. By Lemma 11 there exists a prime ideal P of R such that P is maximal with the property that $L_iP = 0$ for all submodules L_i $(i \geq 1)$ such that $L_1 \subseteq L_2 \subseteq \dots$ is a proper ascending chain of n-generated submodules of M. Let N denote the set of elements $m \in M$ such that $mP = 0$. Then the right (R/P)-module N does not satisfy n-acc. By Theorem 8, $Z(N_{R/P})$ does not satisfy n-acc. Therefore there exist a properly ascending chain $H_1 \subseteq H_2 \subseteq \dots$ of n-generated submodules of $Z(N_{R/P})$. By hypothesis, there exists a positive integer k such that ann(H_k) = ann$(H_{k+1}) = \dots$. Because H_k is finitely generated there exists an ideal A of R, properly containing P, such that $H_kA = 0$. But then $H_iA = 0$ for all $i \geq k$, which contradicts the choice of P. The result follows. $\square$

A prime ring is called *right bounded* if every essential right ideal contains a non-zero two-sided ideal. Also a ring R is called *fully right bounded* if every prime homomorphic image of R is right bounded. A ring R is called a *right FBN ring* if R is a right Noetherian right fully bounded ring. Clearly commutative Noetherian rings are FBN rings and so too are right Noetherian rings which satisfy a polynomial identity (see, for example [11, Corollary 13.6.6]). We have the following result for right FBN rings.

Theorem 13. *Let R be a right FBN ring and let M be a right R-module which satisfies the direct sum condition. Then M satisfies pan-acc if and only if for each positive integer n and each ascending chain $L_1 \subseteq L_2 \subseteq \dots$ of n-generated submodules L_i $(i \geq 1)$ of M there exists a positive integer k such that ann(L_k) = ann $(L_{k+1}) = \dots$.*

Proof. The necessity is clear. Conversely, to prove the sufficiency we can adapt the proof of Theorem 12. In this case we use Lemma 10 instead of Lemma 11. Also at the end of the proof when we consider $\operatorname{ann}(H_k)$ we need to apply [9, Lemma 8.2] to obtain the ideal A strictly containing P. $\square$

Another consequence of Theorem 8 is given in the next lemma.

Lemma 14. *Let R be a right Noetherian ring and let M be a right R-module which satisfies the direct sum condition such that for each prime ideal P of R for which $L_iP = 0$ for all submodules L_i $(i \geq 1)$ of M such that $L_1 \subseteq L_2 \subseteq \ldots$ is an ascending chain of n-generated submodules of M there exists a finite subset F of $\bigcup_{i\geq 1} L_i$ with $P = ann(F)$. Then M satisfies pan-acc.*

Proof. Again we adapt the proof of Theorem 12. Suppose that M does not satisfy pan-acc. With the notation of that proof we obtain an ascending chain $H_1 \subseteq H_2 \subseteq \ldots$ of n-generated submodules of $Z(N_{R/P})$, where N is the set of $m \in M$ such that $mP = 0$. By hypothesis, there exists a finite subset F of N such that $P = \operatorname{ann}(F)$. But for each f in F there exists a right ideal E of R, containing P, such that E/P is an essential right ideal of R/P and $fE = 0$. Thus there exists a right ideal E' of R, containing P, such that E'/P is an essential right ideal of R/P and $gE' = 0$ for all $g \in F$. Thus $E' \subseteq P$, a contradiction. It follows that M satisfies pan-acc. $\square$

Before proving our promised generalization of the theorems of Renault and Frohn we first establish a simple lemma.

Lemma 15. *Let S and R be rings and let M be a left S-, right R-bimodule such that M is a finitely generated left S-module. Let X denote the direct product M^I and let A be an ideal of R such that $A = ann(Y)$ for some submodule Y of the right R-module X. Then $A = ann(F)$ for some finite subset F of Y.*

Proof. Let L denote the set of elements m in M such that m is a component of some element of Y. Clearly $uA = 0$ for all $u \in L$. Now SL is a submodule of the left S-module M so that $SL = Sx_1 + \cdots + Sx_n$ for some positive integer n and elements $x_i \in L$ $(1 \leq i \leq n)$. There exists a finite subset F of elements of Y such that for each $1 \leq i \leq n$, x_i is a component of an element of F. It is now clear that if an element r in R satisfies $fr = 0$ for all $f \in F$ then $x_ir = 0$ for all $1 \leq i \leq n$ so that $SLr = 0$ and hence $Yr = 0$. It follows that $A = \operatorname{ann}(F)$. $\square$

Theorem 16. *Let S and R be rings and let M be a left S-, right R-bimodule such that the left S-module M is Noetherian and the right R-module M is Noetherian. The the right R-module M^I satisfies pan-acc, for every index set I.*

Proof. Let $A = \mathrm{ann}(M_R)$. Note that $M = Sm_1 + \cdots + Sm_k$ for some positive integer k and elements $m_i \in M$ $(1 \leq i \leq k)$. Define a mapping $\varphi : R \to M^{(k)}$ by $\varphi(r) = (m_1 r, \ldots, m_k r)$ for all $r \in R$. Then φ is an R-homomorphism with kernel A so that the ring R/A is right Noetherian. Without loss of generality we can suppose that $A = 0$. The result now follows by Theorem 5 and Lemmas 14 and 15. □

Corollary 17. *Let S and R be rings and let M be a left S-, right R-bimodule such that the left S-module M is Noetherian and the right R-module M is Noetherian. Let N_i $(i \in I)$ be any non-empty collection of submodules of the right R-module M. Then the right R-module $\prod_{i \in I} N_i$ satisfies pan-acc.*

Proof. By the theorem. □

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