A NOTE ON GROUP INVARIANT INCIDENCE FUNCTIONS

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Abstract. Partially ordered sets \((X, \leq)\) and the corresponding incidence algebra \(I(X, F)\) are important algebraic structures also playing a crucial role for the enumeration, construction and the classification of many discrete structures. In this paper we consider partially ordered sets \(X\) on which some group \(G\) acts via the mapping \(X \times G \to X, (x, g) \mapsto x^g\) and investigate such incidence functions \(\phi: X \times X \to F\) of the incidence algebra \(I(X, F)\) which are invariant under the group action, i.e. which satisfy the condition \(\phi(x, y) = \phi(x^g, y^g)\) for all \(x, y \in X\) and \(g \in G\). Within these considerations we define for such incidence functions \(\phi\) the matrices \(\phi^\wedge\) respectively \(\phi^\vee\) by summation of entries of \(\phi\) and we investigate the structure of these matrices and generalize the results known from group actions on posets.

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1. Introduction

A partially ordered set, for short poset, \((X, \leq)\) is a set \(X\) together with a reflexive, antisymmetric and transitive binary relation \(\leq\). Instead of \(x \leq y\) and \(x \neq y\) the notation \(x < y\) is also used. The poset is said to be locally finite if and only if all its intervals \([x, y] := \{z \in X \mid x \leq z \leq y\}\) are finite. In the following we consider locally finite posets. Let \(F\) be a field. The set \(I(X, F)\) consisting of all mappings \(\phi: X \times X \to F\) with the property that \(\phi(x, y) = 0\) unless \(x \leq y\) yields an \(F\)-algebra with respect to the addition

\[(\phi + \psi)(x, y) := \phi(x, y) + \psi(x, y),\]

the scalar multiplication

\[(f \cdot \phi)(x, y) := f \cdot \phi(x, y), \quad f \in F,\]

and the convolution product

\[(\phi * \psi)(x, y) := \sum_{z \in X} \phi(x, z) \cdot \psi(z, y) = \sum_{z \in [x, y]} \phi(x, z) \cdot \psi(z, y),\]
the so-called *incidence algebra* over $F$ on $X$. The identity element with respect to the convolution product is defined by the Kronecker function:

$$
\delta(x, y) := \begin{cases} 
1 & \text{if } x = y \\
0 & \text{otherwise}.
\end{cases}
$$

An important element of the incidence algebra is the well-known Zeta-function which characterizes the poset completely:

$$
\zeta(x, y) := \begin{cases} 
1 & \text{if } x \preceq y \\
0 & \text{otherwise}.
\end{cases}
$$

An incidence function $\phi$ is invertible with respect to the convolution product if and only if the values $\phi(x, x)$ are non-zero. In that case we can construct the inverse incidence function $\phi^{-1}$ recursively:

$$
\phi^{-1}(x, x) = \phi(x, x)^{-1}
$$

for all $x \in X$, and

$$
\phi^{-1}(x, y) = -\phi(x, x)^{-1} \sum_{z : x \preceq z \preceq y} \phi(x, z) \cdot \phi^{-1}(z, y)
$$

$$
= -\phi(y, y)^{-1} \sum_{z : x \preceq z \preceq y} \phi^{-1}(x, z) \cdot \phi(z, y)
$$

for all different $x, y \in X$.

Since $\zeta(x, x) = 1$ for all $x \in X$, the Zeta-function is invertible over $F$ and its inverse is called *Moebius-function* and is denoted by $\mu$.

2. Group invariant incidence functions

From now one we assume a (multiplicatively written) group $G$ with neutral element $1_G$ acting on a poset $X$ via the mapping $X \times G \to X, (x, g) \mapsto x^g$ from the right, i.e. this mapping satisfies $(x^g)^h = x^{gh}$ and $x^{1_G} = x$ for all $x \in X$ and $g, h \in G$. In the following we consider such $\phi \in I(X, F)$ satisfying the equation

$$
\phi(x, y) = \phi(x^g, y^g)
$$

for all $x, y \in X$ and $g \in G$. We call such incidence functions $G$-invariant and we use the symbol $I(X, F)_G$ for the set of all these functions.

A well-known situation occurs if $\zeta \in (X, F)_G$. This is equivalent to

$$
x \prec y \iff x^g \prec y^g
$$

for all $x, y \in X$ and $g \in G$. In this case we say that $G$ acts as a group of automorphisms on the poset $X$ [3].
Important properties of $I(X, F)_G$ are described in the following lemma:

**Lemma 1.** Let $G$ be a group acting on a locally finite poset $X$ and $F$ be a field. Then $I(X, F)_G$ is a subalgebra of $I(X, F)$. In addition $I(X, F)_G$ is a monoid with respect to the convolution product, i.e. $\delta \in I(X, F)_G$. Furthermore, if $\phi \in I(X, F)_G$ is an invertible incidence function in $I(X, F)$ and $\zeta \in I(X, F)_G$, then $\phi^{-1} \in I(X, F)_G$.

**Proof.** (i) Let $\phi, \psi \in I(X, F)_G$, $f \in F$ and $g \in G$. We now show that the functions $\phi + \psi$, $f \cdot \phi$ and $\phi \ast \psi$ are also $G$-invariant. This implies that $I(X, F)_G$ is a subalgebra of $I(X, F)$:

\[
\begin{align*}
(\phi + \psi)(x, y) &= \phi(x, y) + \psi(x, y) \\
&= \phi(x^g, y^g) + \psi(x^g, y^g) \\
&= (\phi + \psi)(x^g, y^g),
\end{align*}
\]

\[
\begin{align*}
(f \cdot \phi)(x, y) &= f \cdot \phi(x, y) \\
&= f \cdot \phi(x^g, y^g) \\
&= (f \cdot \phi)(x^g, y^g),
\end{align*}
\]

\[
\begin{align*}
(\phi \ast \psi)(x, y) &= \sum_{z \in X} \phi(x, z) \cdot \psi(z, y) \\
&= \sum_{z \in X} \phi(x^g, z^g) \cdot \psi(z^g, y^g) \\
&= \sum_{z \in X} \phi(x^g, z^g) \cdot \psi(z^g, y^g) \\
&= \sum_{z' \in X} \phi(x^g, z') \cdot \psi(z', y^g) \\
&= (\phi \ast \psi)(x^g, y^g).
\end{align*}
\]

(ii) Furthermore, the equivalence $x^g = y^g \Leftrightarrow x = y$ for all $x, y \in X$ and $g \in G$ implies $\delta(x, y) = \delta(x^g, y^g)$, i.e. $I(X, F)_G$ is a monoid.

(iii) Now, let $\phi \in I(X, F)_G$ be an invertible incidence function and let $\zeta \in I(X, F)_G$. We show that $\phi^{-1}(x, y) = \phi^{-1}(x^g, y^g)$ for all $x, y \in X$ and $g \in G$. First we consider the case that $x \nleq y$. Then we also get $x^g \nleq y^g$ since $\zeta \in I(X, F)_G$ and hence we have $\phi^{-1}(x, y) = 0 = \phi^{-1}(x^g, y^g)$. Now we consider the second case $x \preceq y$. There exist chains between $x$ and $y$. Let $\ell(x, y)$ denote the length of a maximal chain between $x$ and $y$. We prove $\phi^{-1}(x, y) = \phi^{-1}(x^g, y^g)$ by induction on $n = \ell(x, y)$:
I. $n = 0$. First $\ell(x, y) = 0$, i.e. $x = y$. Then

$$\phi^{-1}(x, x) = \phi(x, x)^{-1} = \phi(x^g, x^g)^{-1} = \phi^{-1}(x^g, x^g).$$

II. $n - 1 \rightarrow n$. Then

$$\phi^{-1}(x, y) = -\phi(y, y)^{-1} \sum_{z : x \leq z < y} \phi^{-1}(x, z) \cdot \phi(z, y)$$
$$= -\phi(y^g, y^g)^{-1} \sum_{z : x^g \leq z^g < y^g} \phi^{-1}(x^g, z^g) \cdot \phi(z^g, y^g)$$
$$= -\phi(y^g, y^g)^{-1} \sum_{z : x^g \leq z^g < y^g} \phi^{-1}(x^g, z^g) \cdot \phi(z^g, y^g)$$
$$= -\phi(y^g, y^g)^{-1} \sum_{z : x^g \leq z^g < y^g} \phi^{-1}(x^g, z^g) \cdot \phi(z^g, y^g)$$
$$= \phi^{-1}(x^g, y^g).$$

\[\square\]

From now on let $X$ be a finite poset and let $y^G := \{y^g \mid g \in G\}$ denote the orbit of $y \in X$. Then we define for a $G$-invariant incidence function $\phi \in I(X, F)_G$ the values

$$\phi(x, y^G) := \sum_{z \in y^G} \phi(x, z)$$

and

$$\phi(y^G, x) := \sum_{z \in y^G} \phi(z, x)$$

for $x, y \in X$.

**Lemma 2.** Let $G$ be a group acting on the finite poset $X$ and $F$ be a field. Let $\phi \in I(X, F)_G$. Then the equations

$$\phi(x, y^G) = \phi(x^g, y^G)$$

and

$$\phi(y^G, x) = \phi(y^G, x^g)$$

hold for all $x, y \in X$ and $g \in G$.

**Proof.** We prove the first equation, the proof of the second one is analogous. Let $x, y \in X$, $g \in G$ and $\phi \in I(X, F)_G$. Then we have

$$\phi(x, y^G) = \sum_{z \in y^G} \phi(x, z) = \sum_{z \in y^G} \phi(x^g, z^g) = \sum_{z' \in y^G} \phi(x^g, z') = \phi(x^g, y^G).$$

\[\square\]
Let $O_1, \ldots, O_n$ denote the orbits of $G$ on the poset $X$ and let $x_i \in O_i$ denote a representative of the $i$th orbit. Now we can define two $n \times n$ matrices $\phi^\wedge = (\phi^\wedge_{ij})$ and $\phi^\vee = (\phi^\vee_{ij})$ with entries

$$\phi^\wedge_{ij} := \phi(x_i, O_j)$$

and

$$\phi^\vee_{ij} := \phi(O_i, x_j).$$

The following lemma shows the connection between $\phi^\wedge$ and $\phi^\vee$.

**Lemma 3.** Let $G$ be a group acting on the finite poset $X$ with corresponding orbits $O_1, \ldots, O_n$ and let $F$ be a field. Let $\phi \in I(X, F)_G$ and let

$$\Delta := \begin{pmatrix} |O_1| & 0 \\ \vdots & \ddots \\ 0 & \cdots & |O_n| \end{pmatrix}.$$

Then the following equation holds

$$\phi^\vee \cdot \Delta = \Delta \cdot \phi^\wedge.$$

Furthermore, if the characteristic of the field $F$ does not divide the orbit sizes $|O_1|, \ldots, |O_n|$, then

$$\phi^\vee = \Delta \cdot \phi^\wedge \cdot \Delta^{-1}.$$

**Proof.** Let $M = (m_{ij}) = \phi^\vee \cdot \Delta$ and let $N = (n_{ij}) = \Delta \cdot \phi^\wedge$. In the following we show the equality of these two matrices $M = N$:

$$m_{ij} = \phi^\vee_{ij} \cdot |O_j| = \phi(O_i, x_j) \cdot |O_j|$$

$$= \sum_{y \in O_j} \phi(O_i, x_j) = \sum_{y \in O_j} \phi(O_i, y)$$

$$= \sum_{x \in O_i} \sum_{y \in O_j} \phi(x, y) = \sum_{x \in O_i} \sum_{y \in O_j} \phi(x, y)$$

$$= \sum_{x \in O_i} \phi(x, O_j) = \sum_{x \in O_i} \phi(x, O_j)$$

$$= |O_i| \cdot \phi(x_i, O_j) = |O_i| \cdot \phi^\wedge_{ij}$$

$$= n_{ij}.$$
Lemma 4. Let $G$ be a group acting on the finite poset $X$ and $\mathbb{F}$ be a field. Then $\delta^\wedge$ is the $n \times n$ unit matrix, where the dimension $n$ is the number of orbits of $G$ on the poset $X$.

**Proof.** For all $i, j \in \{1, \ldots, n\}$ with $i \neq j$ we obtain

$$\delta^\wedge_{ij} = \sum_{y \in O_j} \delta(x_i, y) = 0$$

and

$$\delta^\wedge_{ii} = \delta(x_i, x_i) + \sum_{y \in O_i : y \neq x_i} \delta(x_i, y) = 1 + 0 = 1.$$

\[\square\]

Theorem 5. Let $G$ be a group acting on the finite poset $X$ and $\mathbb{F}$ be a field. Then the equations

$$(f \cdot \phi)^\wedge = f \cdot \phi^\wedge,$$
$$(\phi + \psi)^\wedge = \phi^\wedge + \psi^\wedge,$$
$$(\phi \ast \psi)^\wedge = \phi^\wedge \cdot \psi^\wedge$$

hold for all $\phi, \psi \in I(X, \mathbb{F})_G$ and $f \in \mathbb{F}$.

**Proof.** (i)

$$(f \cdot \phi)^\wedge_{ij} = (f \cdot \phi)(x_i, O_j) = \sum_{y \in O_j} (f \cdot \phi)(x_i, y)$$

$$= \sum_{y \in O_j} f \cdot \phi(x_i, y) = f \cdot \sum_{y \in O_j} \phi(x_i, y)$$

$$= f \cdot \phi(x_i, O_j)$$

$$= f \cdot \phi^\wedge_{ij}$$

(ii)

$$(\phi + \psi)^\wedge_{ij} = (\phi + \psi)(x_i, O_j) = \sum_{y \in O_j} (\phi + \psi)(x_i, y)$$

$$= \sum_{y \in O_j} [\phi(x_i, y) + \psi(x_i, y)] = \sum_{y \in O_j} \phi(x_i, y) + \sum_{y \in O_j} \psi(x_i, y)$$

$$= \phi(x_i, O_j) + \psi(x_i, O_j)$$

$$= \phi^\wedge_{ij} + \psi^\wedge_{ij}$$
(iii)

\[(\phi \ast \psi)^\wedge_{ij} = (\phi \ast \psi)(x_i, O_j) = \sum_{y \in O_j} (\phi \ast \psi)(x_i, y)\]

\[= \sum_{y \in O_j} \sum_{z \in X} \phi(x_i, z) \cdot \psi(z, y) = \sum_{z \in X} \sum_{y \in O_j} \phi(x_i, z) \cdot \psi(z, y)\]

\[= \sum_{z \in X} \sum_{y \in O_j} \phi(x_i, z) \cdot \psi(z, y) = \sum_{z \in X} \phi(x_i, z) \cdot \psi(z, O_j)\]

\[= \sum_{k} \psi(x_k, O_j) \sum_{z \in O_k} \phi(x_i, z) = \sum_{k} \psi(x_k, O_j) \cdot \phi(x_i, O_k)\]

\[= \sum_{k} \phi(x_i, O_k) \cdot \psi(x_k, O_j)\]

\[= \sum_{k} \phi^\wedge_{ik} \cdot \psi^\wedge_{kj}\]

\[\square\]

**Corollary 6.** Let \(G\) be a group acting on the finite poset \(X\) and \(\mathbb{F}\) be a field. Let \(\zeta \in I(X, \mathbb{F})_G\) and let \(\phi \in I(X, \mathbb{F})_G\) be an invertible incidence function. Then \(\phi^\wedge\) is invertible and for its inverse holds the following equation

\[(\phi^\wedge)^{-1} = (\phi^{-1})^\wedge.\]

**Proof.** Let \(\phi \in I(X, \mathbb{F})_G\) be invertible. Since \(\zeta\) is \(G\)-invariant we obtain from Lemma 1 that \(\phi^{-1} \in I(X, \mathbb{F})_G\). Hence we can apply Theorem 5 and get

\[\phi^\wedge \cdot (\phi^{-1})^\wedge = (\phi \ast \phi^{-1})^\wedge = \delta^\wedge\]

which means that \((\phi^\wedge)^{-1} = (\phi^{-1})^\wedge\) since \(\delta^\wedge\) is the unit matrix. \(\square\)

3. Examples

3.1. Binomial coefficients. We consider for a natural number \(n\) the matrix \(B = (b_{ij}), 0 \leq i, j \leq n\), where \(b_{ij} = \binom{j}{i}\) is the number of \(i\)-subsets which are contained in a set with \(j\) elements. The aim is to compute the inverse matrix \(B^{-1}\). We take a set \(X\) with \(n\) elements and consider the action of the symmetric group \(S_X := \{\pi : X \to X | \pi \text{ bijectively}\}\) on the power set \(P(X) := \{S | S \subseteq X\}\) via the mapping

\[P(X) \times S_X \to P(X), (S, \pi) \mapsto S^\pi := \{x^\pi | x \in S\}.\]
It is obvious that \( S_X \) acts as a group of automorphisms on \( P(X) \). If \( \binom{X}{k} \) denotes the set of \( k \)-subsets of \( X \), the orbits of this action are exactly the sets \( \mathcal{O}_0 = \binom{X}{0}, \mathcal{O}_1 = \binom{X}{1}, \ldots, \mathcal{O}_n = \binom{X}{n} \). As \( S_X \)-invariant incidence function we take the Zeta-function
\[
\zeta(T, K) := \begin{cases} 
1 & \text{if } T \subseteq K \\
0 & \text{otherwise}
\end{cases}
\]
together with its inverse \( \mu(T, K) = (-1)^{|K|-|T|} \zeta(T, K) \). Then we consider the matrix \( \zeta^\wedge \) whose entries are
\[
\zeta^\wedge_{ij} = \zeta(\mathcal{O}_i, \mathcal{S}_j) = \sum_{S \in \binom{X}{i}} \zeta(S, S_j) = \binom{j}{i}, \quad \text{where } S_j \in \mathcal{O}_j = \binom{X}{j}.
\]
i.e. we have \( B = \zeta^\wedge \). Because of the equation \( (\zeta^\wedge)^{-1} = \mu^\wedge \) we obtain for the inverse of \( B \) the matrix \( \mu^\wedge \) that is given by the following entries:
\[
\mu^\wedge_{ij} = \mu(\mathcal{O}_i, \mathcal{S}_j) = \sum_{S \in \mathcal{O}_i} \mu(S, S_j) = \sum_{S \in \binom{X}{i}} (-1)^{j-i} \zeta(S, S_j)
\]
\[
= (-1)^{j-i} \sum_{S \in \binom{X}{i}} \zeta(S, S_j) = (-1)^{j-i} \binom{j}{i}.
\]
Finally we have that the matrix \( B^{-1} = (b^{-1})_{ij} \), \( b^{-1}_{ij} = (-1)^{j-i} \binom{j}{i} \) is the inverse of \( B = (b_{ij}), \ b_{ij} = \binom{j}{i} \).

### 3.2. Table of Marks and Burnside matrix.

The table of marks of a group, introduced by Burnside (see [1]), plays an important role for the enumeration, construction and classification of discrete structures as groups, graphs and \( t \)-designs (see [3,4,5]). Especially the combinatorial chemistry (see [2]) uses the table of marks as a tool for the enumeration of chemical compounds. Now we show here that the table of marks is a matrix \( \phi^\wedge \) with a certain group invariant incidence function \( \phi \).

Let \( G \) be a finite group, and let \( L(G) := \{ S \mid S \leq G \} \) denote the set of all subgroups of \( G \). This set together with the inclusion relation forms a finite poset, the so-called subgroup lattice of \( G \). The group \( G \) acts on \( L(G) \) by conjugation
\[
L(G) \times G \rightarrow L(G), (g, S) \mapsto g^{-1} S g := \{ g^{-1} s g \mid s \in S \}
\]
such that \( G \) acts on \( L(G) \) as a group of automorphisms, i.e. the equivalence
\[
S < T \iff g^{-1} S g < g^{-1} T g
\]
holds for all \( S, T \in L(G) \) and \( g \in G \). The orbits of this action are the conjugacy classes of subgroups
\[
\widetilde{S} := \{ g^{-1} S g \mid g \in G \}.
\]
Now if $G$ acts on a set $X$ and if $N_G(x) := \{ g \in G \mid x^g = x \}$ denotes the stabilizer of an element $x \in X$, the conjugacy class of $N_G(x)$ is

$$\tilde{N}_G(x) = \{ g^{-1} N_G(x) g \mid g \in G \} = \{ N_G(y) \mid y \in x^G \}$$

where $x^G := \{ x^g \mid g \in G \}$ is the orbit of $x$, i.e. the elements of an orbit have as their stabilizers a complete conjugacy class of subgroups of $G$. We say $\tilde{N}_G(x)$ is the type of the orbit $x^G$. For a given subgroup $S \in L(G)$ we define

$$\Omega(G, X)_S := \{ x^G \mid N_G(x) \in \tilde{S} \}$$

to be the set of orbits of $G$ on $X$ of type $\tilde{S}$. The task is now to determine the cardinality of this set. In order to determine this number we consider the set of $S$-invariants:

$$X_S := \{ x \in X \mid \forall g \in S : x^g = x \}.$$  

The cardinality of $X_S$ is called the mark of $S$ on $X$ and we get the following well-known connection (see [3]):

$$|X_S| = \sum_{T \in L(G)} \zeta(S, T) \frac{|T \setminus G|}{|T|} |\Omega(G, X)_T|$$

If we substitute

$$\phi(S, T) := \zeta(S, T) \frac{|T \setminus G|}{|T|}$$

we obtain a mapping $\phi$ which is obviously an element of $I(L(G), \mathbb{Q})$. Moreover, $\phi$ is an invertible function. Therefore, if $\tilde{S}_1, \ldots, \tilde{S}_n$ denote the orbits of $G$ on $L(G)$, we obtain the equation

$$\begin{pmatrix}
|X_{\tilde{S}_1}| \\
\vdots \\
|X_{\tilde{S}_n}|
\end{pmatrix} = \phi^\wedge \cdot 
\begin{pmatrix}
|\Omega(G, X)_{\tilde{S}_1}| \\
\vdots \\
|\Omega(G, X)_{\tilde{S}_n}|
\end{pmatrix},$$

respectively after multiplication with $(\phi^{-1})^\wedge$ from the left

$$\begin{pmatrix}
|\Omega(G, X)_{\tilde{S}_1}| \\
\vdots \\
|\Omega(G, X)_{\tilde{S}_n}|
\end{pmatrix} = (\phi^{-1})^\wedge \cdot 
\begin{pmatrix}
|X_{\tilde{S}_1}| \\
\vdots \\
|X_{\tilde{S}_n}|
\end{pmatrix}.$$

The matrix

$$M(G) := \phi^\wedge$$

is known as the table of marks of $G$ and its inverse

$$B(G) := (\phi^{-1})^\wedge$$

is called the Burnside matrix of $G$. 
3.3. Plesken matrices. The Plesken matrices [6] provide another application of group invariant incidence functions. If a group $G$ acts on a finite poset $X$ as a group of automorphisms, i.e. $x \prec y \iff x^g \prec y^g$ and if $O_1, \ldots, O_n$ are the corresponding orbits with representative $x_i \in O_i$, then Plesken defined the matrices $A^\wedge = (a^\wedge_{ij})$ and $A^\vee = (a^\vee_{ij})$ by
\[ a^\wedge_{ij} := |\{ y \in O_j \mid x_i \preceq y \}| \]
and
\[ a^\vee_{ij} := |\{ y \in O_i \mid y \preceq x_j \}|. \]
These matrices play an important role for the determination of the number of solutions of equations of the form $x \wedge y = z$, respectively $x \vee y = z$. There is the following correspondence to the group invariant incidence functions:

**Corollary 7.** Let $G$ be a group acting on a finite poset $X$ as a group of automorphisms. Then $A^\wedge = \zeta^\wedge$ and $A^\vee = \zeta^\vee$.

**References**


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