EXTENSIONS OF Σ-ZIP RINGS

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Abstract. In this note we consider a new concept, so called Σ-zip ring, which unifies zip rings and weak zip rings. We observe the basic properties of Σ-zip rings, constructing typical examples. We study the relationship between the Σ-zip property of a ring \( R \) and that of its Ore extensions and skew generalized power series extensions. As a consequence, we obtain a generalization of several known results relating to zip rings and weak zip rings.

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1. Introduction

Throughout this paper all rings \( R \) are associative with identity. The set of all nilpotent elements of \( R \) is denoted by \( \text{nil}(R) \). Recall that \( R \) is reduced if for all \( a \in R \), \( a^2 = 0 \) implies \( a = 0 \); \( R \) is reversible if for all \( a, b \in R \), \( ab = 0 \) implies \( ba = 0 \); \( R \) is an NI ring if \( \text{nil}(R) \) forms an ideal [8]. Let \( U \) and \( V \) be two nonempty subsets of \( R \). We define \( U : V = \{ x \in R \mid Vx \subseteq U \} \). If \( V \) is singleton, i.e. \( V = \{ m \} \), we use \( U : m \) in place of \( U : \{ m \} \). It is easy to see that if \( U \) and \( V \) are two right ideals of \( R \), then \( U : V \) is an ideal of \( R \) and such an ideal is usually called the quotient of \( U \) by \( V \).

For any nonempty subset \( X \) of a ring \( R \), \( r_R(X) = \{ a \in R \mid Xa = 0 \} \) denotes the right annihilator of \( X \) in \( R \). Faith in [3] called a ring \( R \) right zip if the right annihilator \( r_R(X) \) of a subset \( X \) of \( R \) is zero, then \( r_R(Y) = 0 \) for a finite subset \( Y \subseteq X \). Left zip rings are defined analogously. \( R \) is zip if it is both right and left zip. Zelmanowitz stated that any ring satisfying the descending chain condition on right annihilators is a right zip ring, but the converse does not hold [14]. Examples of right zip rings that do not satisfy the descending chain condition on right annihilators can be found in [3] and [14]. Extensions of zip rings were studied by several authors.

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Beachy and Blair [1] showed that if $R$ is a commutative zip ring, then the polynomial ring $R[x]$ over $R$ is zip. Faith in [3] proved that if $R$ is a commutative zip ring and $G$ a finite abelian group, then the group ring $R[G]$ of $G$ over $R$ is zip. Cedro in [2] proved that there exist right (left) zip rings $R$ such that $M_2(R)$ is not right (left) zip. Also, he proved that if $R$ is a commutative zip ring, then the $n \times n$ full matrix ring $M_n(R)$ over $R$ is a zip ring. For more details and properties of zip rings (see [1, 2, 3, 6, 14]).

For a nonempty subset $X$ of a ring $R$, we define $N_R(X) = \{ a \in R \mid xa \in \text{nil}(R) \text{ for all } x \in X \}$, which is called the weak annihilator of $X$ in $R$ [10]. If $X$ is a finite set, i.e. $X = \{ r_1, r_2, \ldots, r_n \}$, we use $N_R(r_1, r_2, \ldots, r_n)$ in place of $N_R(\{ r_1, r_2, \ldots, r_n \})$. Obviously, for any subset $X$ of a ring $R$, $N_R(X) = \{ a \in R \mid xa \in \text{nil}(R) \text{ for all } x \in X \}$, and $r_R(X) \subseteq N_R(X)$, $l_R(X) \subseteq N_R(X)$. If $R$ is reduced, then $r_R(X) = N_R(X) = l_R(X)$ for any subset $X$ of $R$. It is easy to see that for any subset $X \subseteq R$, $N_R(X)$ is an ideal of $R$ whenever $\text{nil}(R)$ is an ideal.

A ring $R$ is called weak zip provided that for any subset $X$ of $R$, if $N_R(X) \subseteq \text{nil}(R)$, then there exists a finite subset $Y \subseteq X$ such that $N_R(Y) \subseteq \text{nil}(R)$. L. Ouyang [10] proved that for an endomorphism $\alpha$ and an $\alpha$-derivation $\delta$ of a ring $R$, if $R$ is $(\alpha, \delta)$-compatible and reversible, then $R$ is weak zip if and only if the Ore extension $R[x; \alpha, \delta]$ is weak zip.

Motivated by the results in [1, 2, 3, 6, 14], in this article, we continue the study of $\Sigma$-zip rings. We first introduce the notion of a $\Sigma$-zip ring, which is a generalization of both zip rings and weak zip rings, and investigate their properties. We next extend the class of $\Sigma$-zip rings through various ring extensions.

2. $\Sigma$-zip rings

In this section, $U$ always denotes a proper ideal of a ring $R$ unless otherwise stated. We start this section with the following definition.

**Definition 2.1.** Let $U$ be an ideal of $R$. The ring $R$ is called $\Sigma_U$-zip provided that for any subset $X$ of $R$ with $X \not\subseteq U$, if $U : X = U$, then there exists a finite subset $Y \subseteq X$ such that $U : Y = U$.

Clearly, if $U = 0$, then for any subset $X$ of $R$, we have $U : X = r_R(X)$, and so $R$ is $\Sigma_0$-zip if and only if $R$ is right zip. Let $R$ be an $NI$ ring and $U = \text{nil}(R)$. Then for any subset $X$ of $R$, we have $\text{nil}(R) : X = N_R(X)$, and so $R$ is $\Sigma_{\text{nil}(R)}$-zip if and
only if $R$ is weak zip. So both right zip rings and weak zip rings are special $\Sigma$-zip rings.

In the following we offer some examples of $\Sigma$-zip rings.

**Example 2.2.** (1) Recall that an ideal $P$ of $R$ is completely prime if $P \neq R$, and $ab \in P$ implies $a \in P$ or $b \in P$, for $a, b \in R$. So both right zip rings and weak zip rings are special $\Sigma$-zip rings.

In the following we offer some examples of $\Sigma$-zip rings.

(2) Let $R$ be a domain and $S = R[x]/(x^n)$, where $(x^n)$ is the ideal generated by $x^n$. Denote $\mathfrak{p}$ in $S = R[x]/(x^n)$ by $\alpha$. Thus $S = R[x]/(x^n) = R[\alpha] = R + R\alpha + \cdots + R\alpha^{n-1}$, where $\alpha$ commutes with elements of $R$ and $\alpha^n = 0$. Let $U = \{\sum_{i=1}^{n-1} r_i \alpha^i | r_i \in R\}$. Then $U$ is a completely prime ideal of $S$. So $S = R[x]/(x^n) = R[\alpha]$ is $\Sigma_U$-zip.

(3) Let $k$ be any field, and consider the ring $R = \begin{pmatrix} k & 0 \\ k & k \end{pmatrix}$ of $2 \times 2$ lower triangular matrices over $k$. We can write all the proper nonzero ideals of $R$ as follows:

$$\left\{ m_1 = \begin{pmatrix} 0 & 0 \\ k & k \end{pmatrix}, m_2 = \begin{pmatrix} k & 0 \\ k & 0 \end{pmatrix}, m_3 = \begin{pmatrix} 0 & 0 \\ k & 0 \end{pmatrix} \right\}.$$

Since $m_1$ and $m_2$ are completely prime ideals of $R$, we have that $R$ are $\Sigma_{m_1}$-zip and $\Sigma_{m_2}$-zip, respectively. Now we show that $R$ is $\Sigma_{m_3}$-zip. In fact, let $X$ be any subset of $R$ with $X \nsubseteq m_3$, and $m_3 : X = m_3$. Then we consider the sets $W$ and $V$ defined as follow:

$$W = \left\{ a \in R | \begin{pmatrix} a & 0 \\ b & c \end{pmatrix} \in X \right\}, \quad V = \left\{ c \in R | \begin{pmatrix} a & 0 \\ b & c \end{pmatrix} \in X \right\}.$$

Since $m_3 : X = m_3$, we must have $W \neq \emptyset$ and $V \neq \emptyset$. Hence there exist $p = \begin{pmatrix} a & 0 \\ b & c \end{pmatrix} \in X$ with $a \neq 0$, and $q = \begin{pmatrix} x & 0 \\ y & z \end{pmatrix} \in X$ with $z \neq 0$. Let $X_0 = \{p, q\}$. Then $X_0$ is a finite subset of $X$. By a routine computation, we have $m_3 : X_0 = m_3$. So $R$ is $\Sigma_{m_3}$-zip. Note that $R$ is an NI ring and $m_3 = \text{nil}(R)$. Then by Definition 2.1, $R$ is also weak zip.

Using the same way as above, we can show that $R$ is $\Sigma_0$-zip. Then by Definition 2.1, $R$ is also right zip.

Let $U$ be an ideal of $R$, and let
Then under usual matrix operations, $DU_n$ is an ideal of $R_n$ and $LDU_n$ is also an ideal of $LR_n$. The following proposition gives more examples of $\Sigma$-zip rings.

**Proposition 2.3.** Let $U$ be an ideal of $R$. Then the following conditions are equivalent:

1. $R$ is $\Sigma_U$-zip;
2. $R_n$ is $\Sigma_{(DU_n)}$-zip;
3. $LR_n$ is $\Sigma_{(LDU_n)}$-zip.

**Proof.** (1) ⇒ (2) Suppose that $R$ is $\Sigma_U$-zip and $V$ is a subset of $R_n$ with $V \not\subseteq DU_n$ and $DU_n : V = DU_n$. Let

$$Y_i = \left\{ a_{ii} \in R \mid \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ 0 & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & a_{nn} \end{pmatrix} \in V \right\}, \quad 1 \leq i \leq n.$$ 

Then $Y_i \subseteq R$, $1 \leq i \leq n$. If $Y_i \subseteq U$ for some $1 \leq i \leq n$, then $V \cdot E_{ii} \subseteq DU_n$, where $E_{ij}$ is the usual matrix unit with 1 in the $(i, j)$ coordinate and zero elsewhere.
Thus $E_{ii} \in DU_n : V = DU_n$, and so $1 \in U$, this contradicts the fact that $U$ is a proper ideal of $R$. Hence $Y_i \not\subseteq U$ for all $1 \leq i \leq n$. Now we show that for each $1 \leq i \leq n$, $U : Y_i = U$. In fact, $U : Y_i \supseteq U$ is clear, it suffices to show the reverse inclusion. Suppose that $b \in U : Y_i$. Then

$$
\begin{pmatrix}
 a_{11} & a_{12} & \cdots & a_{1n} \\
 0 & a_{22} & \cdots & a_{2n} \\
 \vdots & \vdots & \ddots & \vdots \\
 0 & 0 & \cdots & a_{nn}
\end{pmatrix}
(bE_{ii}) \in DU_n
$$

for each

$$
\begin{pmatrix}
 a_{11} & a_{12} & \cdots & a_{1n} \\
 0 & a_{22} & \cdots & a_{2n} \\
 \vdots & \vdots & \ddots & \vdots \\
 0 & 0 & \cdots & a_{nn}
\end{pmatrix} \in V.
$$

Thus $bE_{ii} \in DU_n : V = DU_n$ and we have that $b \in U$. Hence $U : Y_i \subseteq U$ and for each $1 \leq i \leq n$, $U : Y_i = U$.

Since $R$ is $\Sigma U$-zip, there exists a finite subset $Y_i' \subseteq Y_i$ such that $U : Y_i' = U$, $1 \leq i \leq n$. For each $c \in Y_i'$, there exists $A_c = \begin{pmatrix} c_{11} & c_{12} & \cdots & c_{1n} \\
 0 & c_{22} & \cdots & c_{2n} \\
 \vdots & \vdots & \ddots & \vdots \\
 0 & 0 & \cdots & c_{nn} \end{pmatrix} \in V$ such that $c_{ii} = c$. Let $V_i'$ be a minimal subset of $V$ such that $A_c \in V_i'$ for each $c \in Y_i'$. Then $V_i'$ is a finite subset of $V$. Let $V_0 = \bigcup_{1 \leq i \leq n} V_i'$. Then $V_0$ is also a finite subset of $V$. If $B = \begin{pmatrix} b_{11} & b_{12} & \cdots & b_{1n} \\
 0 & b_{22} & \cdots & b_{2n} \\
 \vdots & \vdots & \ddots & \vdots \\
 0 & 0 & \cdots & b_{nn} \end{pmatrix} \in DU_n : V_0$, then $A'B \in DU_n$ for each $A' = \begin{pmatrix} a'_{11} & a'_{12} & \cdots & a'_{1n} \\
 0 & a'_{22} & \cdots & a'_{2n} \\
 \vdots & \vdots & \ddots & \vdots \\
 0 & 0 & \cdots & a'_{nn} \end{pmatrix} \in V_0$. Let

$$
W_i = \left\{ a'_{ii} \in R \mid \begin{pmatrix} a'_{11} & a'_{12} & \cdots & a'_{1n} \\
 0 & a'_{22} & \cdots & a'_{2n} \\
 \vdots & \vdots & \ddots & \vdots \\
 0 & 0 & \cdots & a'_{nn} \end{pmatrix} \in V_0 \right\}, \quad 1 \leq i \leq n.
$$

Clearly, $Y_i' \subseteq W_i$ for each $1 \leq i \leq n$. So $U : W_i \subseteq U : Y_i' = U$ for each $1 \leq i \leq n$. Since $A'B \in DU_n$ implies that $a'_{ii}b_{ii} \in U$ for all $1 \leq i \leq n$, we obtain
Thus $b_{ii} \in U$ for each $1 \leq i \leq n$, and hence $B \in DU_n$. Therefore $DU_n : V_0 = DU_n$, and so $R_n$ is $\Sigma(DU_n)$-zip.

(2) $\Rightarrow$ (1) Assume that $R_n$ is $\Sigma(DU_n)$-zip and $X \subseteq R$ with $X \not\subseteq U$ and $U : X = U$. Let $V = \{aI \mid a \in X\} \subseteq R_n$, where $I$ is the $n \times n$ identity matrix. If 
\[
B = \begin{pmatrix}
b_{11} & b_{12} & \cdots & b_{1n} \\
0 & b_{22} & \cdots & b_{2n} \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & b_{nn}
\end{pmatrix}
\]
$\in DU_n : V$, then $aI \cdot B \in DU_n$ for all $a \in X$. Thus $ab_{ii} \in U$ for all $1 \leq i \leq n$ and all $a \in X$, and it follows that $b_{ii} \in U : X = U$. Hence $B \in DU_n$ which implies that $DU_n : V = DU_n$. Since $R_n$ is $\Sigma(DU_n)$-zip, there exists a finite subset $V_0 = \{a_1 I, a_2 I, \ldots, a_m I\} \subseteq V$ such that $DU_n : V_0 = DU_n$. Let $X_0 = \{a_1, a_2, \ldots, a_m\} \subseteq X$. If $c \in U : X_0$, then $(a_k I) \cdot (cE_{11}) \in DU_n$ for all $k = 1, 2, \ldots, m$. Thus $cE_{11} \in DU_n : V_0 = DU_n$ and so $c \in U$. Hence $U : X_0 = U$. Therefore $R$ is $\Sigma_U$-zip.

(1) $\Leftrightarrow$ (3) is analogous to (1) $\Leftrightarrow$ (2).

**Corollary 2.4.** [10, Proposition 2.1] Let $R$ be an NI ring. Then the following conditions are equivalent:

1. $R$ is weak zip;
2. $R_n$ is weak zip;
3. $LR_n$ is weak zip.

**Proof.** Let $U = \text{nil}(R)$. Then $DU_n = \text{nil}(R_n)$, $LDU_n = \text{nil}(LR_n)$ and both $R_n$ and $LR_n$ are NI rings. Note that for any ring $R$, we have that $R$ is $\Sigma_{\text{nil}(R)}$-zip if and only if $R$ is weak zip. Therefore we complete the proof by Proposition 2.3.

Based on the preceding results, we consider the following subrings of $n \times n$ upper (lower) triangular matrix rings. Let $U$ be an ideal of $R$ and

$$S_n = \left\{ \begin{pmatrix}
a & a_{12} & \cdots & a_{1n} \\
0 & a & \cdots & a_{2n} \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & a
\end{pmatrix} \mid a, a_{ij} \in R \right\},$$

$$U_n = \left\{ \begin{pmatrix}
u & u_{12} & \cdots & u_{1n} \\
0 & u & \cdots & u_{2n} \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & u
\end{pmatrix} \mid u, u_{ii} \in U \right\}.$$
EXTENSIONS OF Σ-ZIP RINGS

\[LS_n = \begin{cases} \begin{pmatrix} a & 0 & \cdots & 0 \\ a_{21} & a & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a \end{pmatrix} & | a, a_{ij} \in R, \\ u & 0 & \cdots & 0 \\ u_{21} & u & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ u_{n1} & u_{n2} & \cdots & u \end{pmatrix} & | u, u_{ii} \in U \end{cases},\]

\[LU_n = \begin{cases} \begin{pmatrix} a_0 \\ \vdots \\ a_{n1} & a_{n2} & \cdots & a \\ u_0 \\ \vdots \\ u_{n1} & u_{n2} & \cdots & u \end{pmatrix} & | a, a_{ij} \in R, \\ u_0 \\ \vdots \\ u_{n1} & u_{n2} & \cdots & u \end{cases},\]

where \( n \geq 2 \) is a positive integer. Then we have the following.

**Proposition 2.5.** Let \( U \) be an ideal of \( R \). Then the following conditions are equivalent:

1. \( R \) is \( \Sigma_U \)-zip;
2. \( S_n \) is \( \Sigma_{U_n} \)-zip;
3. \( LS_n \) is \( \Sigma_{(LU_n)} \)-zip.

**Proof.** (1) \( \Rightarrow \) (2) Suppose that \( R \) is \( \Sigma_U \)-zip and \( V \) is a subset of \( S_n \) with \( V \not\subseteq U_n \) and \( U_n : V = U_n \). Consider the following set

\[X = \left\{ v \in R \mid \begin{pmatrix} v & v_{12} & \cdots & v_{1n} \\ 0 & v & \cdots & v_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & v \end{pmatrix} \in V \right\},\]

that is, \( X \) is the set of all elements in the ring \( R \), which occurs as diagonal entries of elements in \( V \). If \( X \subseteq U \), then \( V \cdot E_{1n} \subseteq U_n \). Thus \( E_{1n} \in U_n : V = U_n \) and so \( 1 \in U \) which contradicts the fact that \( U \) is a proper ideal of \( R \). Thus we obtain \( X \not\subseteq U \). Now we show that \( U : X = U \). Since \( U : X \supseteq U \) is clear, it suffices to show that \( U : X \subseteq U \). Suppose that \( a \in U : X \). Then \( aE_{1n} \in U_n : V = U_n \), and so \( a \in U \). Thus \( U : X = U \). Since \( R \) is \( \Sigma_U \)-zip, there exists a finite subset \( X_0 = \{ v_1, v_2, \ldots, v_k \} \subseteq X \) such that \( U : X_0 = U \). For each \( v_i \in X_0, 1 \leq i \leq k \), there exists \( A_{v_i} = \begin{pmatrix} v_i & v_{i1} & \cdots & v_{i1n} \\ 0 & v_i & \cdots & v_{i2n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & v_i \end{pmatrix} \in V \). Let \( V_0 \) be the minimal subset of \( V \) such that \( A_{v_i} \in V_0 \) for each \( v_i \in X_0 \). Then \( V_0 \) is a finite subset of \( V \). Without
loss of generality, we may write $V_0$ as follow:

$$V_0 = \begin{pmatrix} v_i & v_{i2} & \cdots & v_{in} \\ 0 & v_i & \cdots & v_{i2n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & v_i \end{pmatrix} \in V \mid v_i \in X_0, 1 \leq i \leq k \}
$$

Now we show that $U_n : V_0 = U_n$. We proceed by induction on $n$. Suppose that

$$\begin{pmatrix} v_i \\ 0 \\ \vdots \\ 0 \end{pmatrix} \text{ for all } 1 \leq i \leq k.$$

We get

then we have that for all $1 \leq i \leq k$, $v_i a_{1n} + v_{i2} a_{2n} + \cdots + v_{i(n-1)n} a_{n-1n} \in U$, $\ldots$, $v_i a_{(n-2)n} + v_{i(n-2)(n-1)n} a_{(n-1)n} \in U$ and $v_i a_{(n-1)n} \in U$. From $v_i a_{(n-1)n} \in U$ for all $1 \leq i \leq k$, we get $a_{(n-1)n} \in U : X_0 = U$. Then from $v_i a_{(n-2)n} + \ldots$, we obtain $a \in U : X_0 = U$. Then from $v_i a_{12} + v_{i2} a \in U$ for all $1 \leq i \leq k$ and $a \in U$, we get $a_{12} \in U : X_0 = U$. Hence

$$(a, a_{12}, \ldots, a_{1n}) \in U_n \quad \text{for all } 1 \leq i \leq k.$$
EXTENSIONS OF Σ-ZIP RINGS

$v^i_{(n-2)(n-1)} a_{(n-1)n} \in U$ and $a_{(n-1)n} \in U$, we get $a_{(n-2)n} \in U : X_0 = U$. Inductively, we obtain $a_{in} \in U$ for $i = 1, 2, \ldots, n - 1$, concluding that $U_n : V_0 = U_n$. Therefore $R_n$ is $\Sigma U_n$-zip.

(2) $\Rightarrow$ (1) Assume that $R_n$ is $\Sigma U_n$-zip, and $X \not\subseteq U$ with $U : X = U$. Let

$X_n = \left\{ \begin{pmatrix} x & 0 & \cdots & 0 \\ 0 & x & \cdots & 0 \\ \vdots & \vdots & \ddots & 0 \\ 0 & 0 & \cdots & x \end{pmatrix} | x \in X \right\}$ and $\begin{pmatrix} a & a_{12} & \cdots & a_{1n} \\ 0 & a & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & a \end{pmatrix} \in U_n : X_n$.

Then

$\begin{pmatrix} x & 0 & \cdots & 0 \\ 0 & x & \cdots & 0 \\ \vdots & \vdots & \ddots & 0 \\ 0 & 0 & \cdots & x \end{pmatrix} \in X_n$, and so $xa \in U$ and $xa_{ij} \in U$ for each $x \in X$.

Thus $a \in U : X = U$ and $a_{ij} \in U : X = U$, which implies that $\begin{pmatrix} a & a_{12} & \cdots & a_{1n} \\ 0 & a & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & a \end{pmatrix} \in U_n$. Hence $U_n : X_n = U_n$. Since $R_n$ is $\Sigma U_n$-zip, there exists a finite subset

$V = \left\{ \begin{pmatrix} x_i & 0 & \cdots & 0 \\ 0 & x_i & \cdots & 0 \\ \vdots & \vdots & \ddots & 0 \\ 0 & 0 & \cdots & x_i \end{pmatrix} \in X_n \right\} \subseteq X_n$ such that $U_n : V = U_n$. Let

$X_0 = \{x_1, x_2, \ldots, x_k\}$. Then $X_0 \subseteq X$ is a finite subset of $X$. If $a \in U : X_0$, then $aE_{1n} \in U_n : V = U_n$, and so $a \in U$. Hence $U : X_0 = U$. Therefore $R$ is $\sigma U$-zip.

(1) $\Leftrightarrow$ (3) is proved in the same manner.

□

Corollary 2.6. [6, Theorem 5] Let $R$ be a ring. Then the following conditions are equivalent:

1. $R$ is a right zip ring.
2. $S_n$ is a right zip ring.
3. $LS_n$ is a right zip ring.
Proof. Let \( U \) be the zero ideal of \( R \). Then \( U_n \) and \( LU_n \) are the zero ideals of \( S_n \) and \( LS_n \), respectively. Note that \( R \) is \( \Sigma_0 \)-zip if and only if \( R \) is right zip. Therefore we complete the proof by Proposition 2.5.

Corollary 2.7. Let \( U \) be an ideal of \( R \). Then we have the following:

1. \( R \) is \( \Sigma_U \)-zip if and only if the trivial extension \( T(R,R) = \left\{ \begin{pmatrix} a & b \\ 0 & a \end{pmatrix} \right\} \)
   of \( R \) by \( R \) is \( \Sigma_{T(U,U)} \)-zip, where \( T(U,U) = \left\{ \begin{pmatrix} u & v \\ 0 & u \end{pmatrix} \right\} \).

2. [6, Corollary 6] \( R \) is a right zip ring if and only if \( T(R,R) \) is right zip.

Proof. According to Proposition 2.5 and Corollary 2.6, we obtain the results.

Let \( R \) be a ring and

\[
T_3(R) = \left\{ \begin{pmatrix} a_{11} & 0 & 0 \\ a_{21} & a_{22} & a_{23} \\ 0 & 0 & a_{33} \end{pmatrix} \mid a_{ij} \in R \right\},
\]

\[
W_3(R) = \left\{ \begin{pmatrix} a & 0 & 0 \\ a_{21} & a & a_{23} \\ 0 & 0 & a \end{pmatrix} \mid a, a_{21}, a_{23} \in R \right\}.
\]

Then under usual matrix operations, \( T_3(R) \) and \( W_3(R) \) are subrings of the \( 3 \times 3 \) matrix ring \( M_3(R) \). Let \( U \) be an ideal of \( R \) and

\[
DT_3(U) = \left\{ \begin{pmatrix} u_{11} & 0 & 0 \\ a_{21} & u_{22} & a_{23} \\ 0 & 0 & u_{33} \end{pmatrix} \mid u_{11}, u_{22}, u_{33}, a_{21}, a_{23} \in R \right\},
\]

\[
W_3(U) = \left\{ \begin{pmatrix} u & 0 & 0 \\ a_{21} & u & a_{23} \\ 0 & 0 & u \end{pmatrix} \mid u, a_{21}, a_{23} \in U \right\}.
\]

Then \( DT_3(U) \) is an ideal of \( T_3(R) \) and \( W_3(U) \) is an ideal of \( W_3(R) \).

Proposition 2.8. Let \( U \) be an ideal of \( R \). Then the following conditions are equivalent:

1. \( R \) is \( \Sigma_U \)-zip.
2. \( T_3(R) \) is \( \Sigma_{(DT_3(U))} \)-zip.
3. \( W_3(R) \) is \( \Sigma_{(W_3(U))} \)-zip.
Proof. The argument for this claim is similar to that used in the proof of Proposition 2.3 and Proposition 2.5.

\bigskip

Corollary 2.9. Let \( R \) be a ring. Then we have the following:

1. If \( R \) is an NI ring, then \( R \) is weak zip if and only if \( T_3(R) \) is weak zip.
2. \( R \) is right zip if and only if \( W_3(R) \) is right zip.

Proof. (1) Let \( U = \text{nil}(R) \). Then \( DT_3(U) = \text{nil}(T_3(R)) \) and therefore we complete the proof by Proposition 2.8.

(2) Let \( U = 0 \). Then the result is an immediate consequence of Proposition 2.8 and the fact that \( R \) is \( \Sigma_0 \)-zip if and only if \( R \) is right zip.

\bigskip

Let \( R \) be an algebra over a commutative ring \( S \). Recall that the Dorroh extension of \( R \) by \( S \) is the ring \( D = R \times S \) with operations \((r_1, s_1) + (r_2, s_2) = (r_1 + r_2, s_1 + s_2)\) and \((r_1, s_1)(r_2, s_2) = (r_1r_2 + s_1r_2 + s_2r_1, s_1s_2)\), where \( r_i \in R \) and \( s_i \in S \). Let \( U \) be an ideal of \( S \). We define \( R \times U \) as follow:

\[ R \times U = \{(r, s) \in D \mid r \in R, s \in U \}. \]

Then \( R \times U \) is an ideal of \( D \).

Proposition 2.10. Let \( D \) be the Dorroh extension of \( R \) by \( S \) and \( U \) an ideal of \( S \). Then \( D \) is \( \Sigma_{(R \times U)} \)-zip if and only if \( S \) is \( \Sigma_U \)-zip.

Proof. (\( \Rightarrow \)) Suppose that \( D \) is \( \Sigma_{(R \times U)} \)-zip and \( Y \) is a subset of \( S \) with \( Y \not\subseteq U \) and \( U : Y = U \). Let \( R \times Y = \{(r, s) \in D \mid r \in R, s \in Y \} \). Then \( R \times Y \subseteq D \) and \( R \times Y \not\subseteq R \times U \). If \((u, v) \in (R \times U) : (R \times Y)\), then \((r, s)(u, v) = (ru + sv + vr, sv) \in R \times U \) for each \((r, s) \in R \times Y \). Thus \( sv \in U \) for each \( s \in Y \), and so \( v \in U : Y = U \). Hence \((u, v) \in R \times U \) and so \((R \times U) : (R \times Y) = R \times U \).

Since \( D \) is \( \Sigma_{(R \times U)} \)-zip, there exists a finite subset \((R \times Y)_0 \subseteq R \times Y \) such that \((R \times U) : (R \times Y)_0 = R \times U \). Without loss of generality, we may assume that \((R \times Y)_0 = \{(r_1, s_1), (r_2, s_2), \ldots, (r_k, s_k)\}\). Then \( Y_0 = \{s_1, s_2, \ldots, s_k\} \) is a finite subset of \( Y \). If \( r \in U : Y_0 \), then \((0, r) \in (R \times U) : (R \times Y)_0 = R \times U \), and so \( r \in U \).

Hence \( U : Y_0 = U \). Therefore \( S \) is \( \Sigma_U \)-zip.

(\( \Leftarrow \)) Assume that \( S \) is \( \Sigma_U \)-zip and \( V \) is a subset of \( D \) with \( V \not\subseteq R \times U \) and \((R \times U) : V = R \times U \). Let \( X = \{s \in S \mid (r, s) \in V \} \). Then by the condition that \( V \not\subseteq R \times U \), we have \( X \not\subseteq U \). If \( a \in U : X \), then \((0, a) \in (R \times U) : V = R \times U \) and so \( a \in U \). Thus \( U : X = U \). Since \( S \) is \( \Sigma_U \)-zip, there exists a finite subset \( X_0 = \{s_1, s_2, \ldots, s_k\} \subseteq X \) such that \( U : X_0 = U \). For each \( s_i \in X_0 \), there exists \( v_{s_i} = (r_i, s_i) \in V \). Let \( V_0 \) be the minimal subset of \( V \) such that \( v_{s_i} \in V_0 \) for each
Then $s_i \in X_0$. Then $V_0$ is a finite subset of $V$. Now we show that $(R \times U) : V_0 = R \times U$. If $(a, b) \in (R \times U) : V_0$, then $(r, s)(a, b) = (ra + sa + br, sb) \in R \times U$ for each $(r, s) \in V_0$. Then $sb \in U$ for each $s \in X_0$. Hence $b \in U : X_0 = U$, and so $(R \times U) : V_0 = R \times U$. Therefore $D$ is $\Sigma(R \times U)$-zip.

Let $R$ be a ring and $\Delta$ a multiplicatively closed subset of $R$ consisting of central regular elements. $\Delta^{-}R$ denotes the classical quotient ring of $R$. If $U$ is an ideal of $R$, then $\Delta^{-}U$ is an ideal of $\Delta^{-}R$.

**Proposition 2.11.** Let $U$ be an ideal of $R$. Then $R$ is $\Sigma_U$-zip if and only if $\Delta^{-}R$ is $\Sigma(\Delta^{-}U)$-zip.

**Proof.** (1) Let $\Sigma_U$-zip and $V$ be a subset of $\Delta^{-}R$ with $V \not\subseteq \Delta^{-}U$ and $\Delta^{-}U : V = \Delta^{-}U$. Let $X = \{a \mid u^{-1}a \in V\} \subseteq R$. Then $X \not\subseteq U$. If $r \in U : X$, then $Vr \subseteq \Delta^{-}U$. Thus $r \in \Delta^{-}U : V = \Delta^{-}U$, and so $r \in U$. Hence $U : X = U$. Since $R$ is $\Sigma_U$-zip, there exists a finite subset $X_0$ of $X$ such that $U : X_0 = U$. Let $X_0 = \{a_1, a_2, \ldots, a_n\}$. Then there exist elements $\alpha_1, \alpha_2, \ldots, \alpha_n$ in $V$ be such that $\alpha_1 = u_1^{-1}a_1$, $\alpha_2 = u_2^{-1}a_2$, $\ldots$, $\alpha_n = u_n^{-1}a_n$, where $a_1, a_2, \ldots, u_n \in \Delta$. Let $V_0 = \{\alpha_1, \alpha_2, \ldots, \alpha_n\}$. Then $V_0$ is a finite subset of $V$. Now if $\beta \in \Delta^{-}U : V_0$ and $\beta = v^{-1}b$, then $u_i^{-1}a_iv^{-1}b \in \Delta^{-}U$ for all $1 \leq i \leq n$, and so $a_ib \in U$ for all $1 \leq i \leq n$. Thus $b \in U : X_0 = U$ and so $\beta = v^{-1}b \in \Delta^{-}U$. Hence $\Delta^{-}U : V_0 = \Delta^{-}U$. Therefore $\Delta^{-}R$ is $\Sigma(\Delta^{-}U)$-zip.

(\Rightarrow) Assume that $\Delta^{-}R$ is $\Sigma(\Delta^{-}U)$-zip and $X$ is a subset of $R$ with $X \not\subseteq U$ and $U : X = U$. If $X(u^{-1}a) \subseteq \Delta^{-}U$ for some $u^{-1}a \in \Delta^{-}R$, then $Xa \subseteq U$ and so $a \in U : X = U$. Thus it is easy to see that $\Delta^{-}U : X = \Delta^{-}U$. Since $\Delta^{-}R$ is $\Sigma(\Delta^{-}U)$-zip, there exists a finite subset $X_0 \subseteq X$ such that $\Delta^{-}U : X_0 = \Delta^{-}U$. If $r \in U : X_0$, then $r \in \Delta^{-}U : X_0 = \Delta^{-}U$, and so $r \in U$. Hence $U : X_0 = U$. Therefore $R$ is $\Sigma_U$-zip.

**Corollary 2.12.** Let $R$ be a ring and $\Delta$ be a multiplicatively closed subset of $R$ consisting of central regular elements. Then we have the following:

1. [6, Proposition 12] $R$ is right zip if and only if $\Delta^{-}R$ is right zip.
2. If $R$ is an NI ring, then $R$ is weak zip if and only if $\Delta^{-}R$ is weak zip.

**Proof.** (1) Let $U = 0$. Then the result is an immediate consequence of Proposition 2.11.

(2) Let $U = nil(R)$. Then $\Delta^{-}U = nil(\Delta^{-}R)$. In view of Proposition 2.11, we obtain the result.
Let \( \phi : R \rightarrow S \) be a surjective ring homomorphism. For any subset \( V \subseteq S \), we define \( V^e = \{ r \in R \mid \phi(r) \in V \} \), and for any subset \( T \subseteq R \), we define \( T^e = \{ \phi(t) \mid t \in T \} \). Clearly, if \( V \) is an ideal of \( S \), then \( V^e \) is an ideal of \( R \).

The following proposition reveals the relationship between the \( \Sigma \)-zip property of the ring \( R \) and that of its homomorphic image.

**Proposition 2.13.** Let \( \phi : R \rightarrow S \) be a ring homomorphism, and \( M \) an ideal of \( S \). Then the following conditions are equivalent:

1. \( R \) is \( \Sigma_{M^c} \)-zip.
2. \( S \) is \( \Sigma_M \)-zip.

**Proof.** (1) \( \Rightarrow \) (2) Let \( X \subseteq S \) with \( X \not\subseteq M \) and \( M : X = M \). Now we show that \( M^c : X^c = M^c \). Suppose that \( r \in M^c : X^c \). Then \( X^c r \subseteq M^c \), and so \( X \phi(r) \subseteq M \). Hence \( \phi(r) \in M : X = M \), concluding that \( r \in M^c \). Thus \( M^c : X^c = M^c \). Since \( R \) is \( \Sigma_{M^c} \)-zip, there exists a finite subset \( V \subseteq X^c \) such that \( M^c : V = M^c \). Now we show that \( M : V^c = M \), where \( V^c \) is a finite subset of \( X \). If \( r \in M : V^c \), then \( V^c r \subseteq M \) and so \( V^c \subseteq M^c \), where \( r^c = \{ a \in R \mid \phi(a) = r \} \). Hence \( r^c \subseteq M^c : V = M^c \), and so \( r \in M \). Hence \( M : V^c = M \). Therefore \( S \) is \( \Sigma_M \)-zip.

(2) \( \Rightarrow \) (1) Assume that \( S \) is \( \Sigma_M \)-zip, and \( X \subseteq R \) with \( X \not\subseteq M^c \) and \( M^c : X = M^c \). Now we show that \( M : X^e = M \). Suppose that \( r \in M : X^e \). Then \( X^e r \subseteq M \), and so \( X r^e \subseteq M^c \). Thus \( r^e \subseteq M^c : X = M^c \), and so \( r \in M \), concluding that \( M : X^e = M \). Since \( S \) is \( \Sigma_M \)-zip, there exists a finite subset \( V \subseteq X^e \) such that \( M : V = M \). Without loss of generality, we may assume that \( V = \{ v_1, v_2, \ldots, v_k \} \). Consider the following subset

\[
W = \{ x_1, x_2, \ldots, x_k \mid x_i \in X, \phi(x_i) = v_i, 1 \leq i \leq k \} \subseteq X.
\]

Then \( W \) is a finite subset of \( X \) and \( W^e = V \). Now we show that \( M^c : W = M^c \). Suppose that \( a \in M^c : W \). Then \( Wa \subseteq M^c \), and so \( W^e \phi(a) = V \phi(a) \subseteq M \). Thus we obtain \( \phi(a) \in M : V = M \), and so \( a \in M^c \). Hence \( M^c : W = M^c \). Therefore \( R \) is \( \Sigma_{M^c} \)-zip. \( \square \)

**Corollary 2.14.** Let \( M \) be an ideal of \( R \). Then the following conditions are equivalent:

1. \( R \) is \( \Sigma_M \)-zip.
2. \( R/M \) is \( \Sigma_0 \)-zip.
3. \( R/M \) is right zip.
Proof. (1) ⇔ (2) is an immediate consequence of Proposition 2.13. (2) ⇔ (3) is trivial. □

Corollary 2.15. Let $R$ be a commutative ring and $U$ an ideal of $R$. If $R$ is $\Sigma_U$-zip, then $M_n(R)$ is $\Sigma_{M_n(U)}$-zip, where $M_n(U) = \{(a_{ij})_{n \times n} \in M_n(R) \mid a_{ij} \in U$ for all $i, j = 1, 2, \ldots, n\}$.

Proof. Suppose that $R$ is $\Sigma_U$-zip. Then by Corollary 2.14, we have that $R/U$ is $\Sigma_U$-zip, and so by [2, Proposition 1], $M_n(R/U) \cong M_n(R)/M_n(U)$ is zip. Hence the result follows from Corollary 2.14. □

Rege and Chhawchharia in [11] introduced the notion of an Armendariz ring. A ring $R$ is called Armendariz if whenever polynomials $\sum_{i=0}^{m} a_i x^i, \sum_{j=0}^{n} b_j x^j \in R[x]$ satisfy $f(x)g(x) = 0$, then $a_i b_j = 0$ for each $0 \leq i \leq m$ and $0 \leq j \leq n$. Hong [6] showed that if $R$ is an Armendariz ring, then $R$ is right zip if and only if the polynomial ring $R[x]$ is right zip, if and only if the Laurent polynomial ring $R[x, x^{-1}]$ is right zip. Let $U$ be an ideal of $R$. Let $U[x]$ and $U[x, x^{-1}]$ denote the subsets $U[x] = \{f(x) = \sum_{i=0}^{m} a_i x^i \in R[x] \mid a_i \in U, 0 \leq i \leq m\}$ and $U[x, x^{-1}] = \{f(x) = \sum_{i=m}^{n} a_i x^i \in R[x, x^{-1}] \mid a_i \in U, m \leq i \leq n\}$, respectively. Then we have the following proposition.

Proposition 2.16. Let $U$ be an ideal of $R$ and $R/U$ an Armendariz ring. Then the following conditions are equivalent:

1. $R$ is $\Sigma_U$-zip.
2. $R[x]$ is $\Sigma_U[x]$-zip.
3. $R[x, x^{-1}]$ is $\Sigma_{U[x, x^{-1}]}$-zip.

Proof. (1) ⇔ (2) Since $R/U$ is Armendariz, by [6, Theorem 11], we have that $R/U$ is right zip if and only if $(R/U)[x] \cong R[x]/U[x]$ is right zip, and therefore we complete the proof by Corollary 2.14.

(1) ⇔ (3) is proved in the same manner. □

3. Ore extension of $\Sigma$-zip rings

In this section we always denote the Ore extension ring by $R[x; \alpha, \delta]$, where $\alpha : R \rightarrow R$ is an endomorphism and $\delta : R \rightarrow R$ is an $\alpha$-derivation. Recall that an $\alpha$-derivation $\delta$ is an additive operator on $R$ with the property that $\delta(ab) = \alpha(a)\delta(b) + \delta(a)b$ for all $a, b \in R$. The elements of $R[x; \alpha, \delta]$ are polynomials in $x$ with coefficients written on the left. Multiplication in $R[x; \alpha, \delta]$ is given by the multiplication in $R$ and the condition $xa = \alpha(a)x + \delta(a)$ for all $a \in R$. 


For any $0 \leq i \leq j$, $f_i^j \in End(R, +)$ will denote the map which is the sum of all possible words in $\alpha$ and $\delta$ built with $i$ letters $\alpha$ and $j - i$ letters $\delta$.

Using recursive formulas for the $f_i^j$'s and induction (see [5]), one can show with a routine computation that

$$x^i a = \sum_{i=0}^{j} f_i^j(a)x^i.$$ 

This formula uniquely determines a general product of polynomials in $R[x; \alpha, \delta]$ and will be used freely in what follows.

Let $I$ be a subset of $R$, $I[x; \alpha, \delta]$ means the set $\{u_0 + u_1x + \cdots + u_nx^n \in R[x; \alpha, \delta] | u_i \in I, 0 \leq i \leq n\}$, that is, for any skew polynomial $f(x) = u_0 + u_1x + \cdots + u_nx^n \in R[x; \alpha, \delta]$, $f(x) \in I[x; \alpha, \delta]$ if and only if $u_i \in I$ for all $0 \leq i \leq n$.

Let $\alpha$ be an endomorphism and $\delta$ an $\alpha$-derivation of $R$. Following Hashemi and Moussavi [5], a ring $R$ is said to be $\alpha$-compatible if for each $a, b \in R, ab = 0 \Leftrightarrow a\alpha(b) = 0$. Moreover, $R$ is called to be $\delta$-compatible if for each $a, b \in R, ab = 0 \Rightarrow a\delta(b) = 0$. If $R$ is both $\alpha$-compatible and $\delta$-compatible, then $R$ is said to be $(\alpha, \delta)$-compatible.

Let $I$ be an ideal of $R$. Due to Hashemi [4], $I$ is said to be $\alpha$-compatible if for each $a, b \in R, ab \in I \Leftrightarrow a\alpha(b) \in I$. Moreover, $I$ is called to be $\delta$-compatible if for each $a, b \in R, ab \in I \Rightarrow a\delta(b) \in I$. If $I$ is both $\alpha$-compatible and $\delta$-compatible, then $I$ is said to be $(\alpha, \delta)$-compatible. Clearly, a ring $R$ is an $(\alpha, \delta)$-compatible ring if and only if the zero ideal is an $(\alpha, \delta)$-compatible ideal. Let $U$ be an ideal of $R$, we say that $U$ is a semiprime ideal if for any $a \in R$, $a^2 \in U$ implies $a \in U$.

The following lemma appears in [4].

**Lemma 3.1.** [4, Proposition 2.3] Let $I$ be an $(\alpha, \delta)$-compatible ideal, and $a, b \in R$.

1. If $ab \in I$, then $a\alpha^n(b) \in I$ and $\alpha^n(a)b \in I$ for every positive integer $n$. Conversely, if $a\alpha^k(b)$ or $\alpha^k(a)b \in I$ for some positive integer $k$, then $ab \in I$.

2. If $ab \in I$, then $\alpha^m(a)\delta^n(b) \in I$ and $\delta^m(a)\alpha^n(b) \in I$ for any nonnegative integers $m, n$.

**Lemma 3.2.** Let $I$ be an $(\alpha, \delta)$-compatible ideal and $a, b \in R$. If $ab \in I$, then $af_i^j(b) \in I$ and $f_i^j(a)b \in I$ for all $0 \leq i \leq j$.

**Proof.** It is clear by Lemma 3.1. ☐

**Lemma 3.3.** Let $U$ be an $(\alpha, \delta)$-compatible ideal of $R$. Then for each $Y \subseteq R$, we have $(U[x; \alpha, \delta] : Y) \cap R = U : Y$.

**Proof.** It is trivial. ☐
Proposition 3.4. Let \( \alpha \) be an endomorphism and \( \delta \) an \( \alpha \)-derivation of \( R \). If \( U \) is an \((\alpha, \delta)\)-compatible semiprime ideal, then the following conditions are equivalent:

1. \( R \) is \( \Sigma_U \)-zip.
2. \( R[\alpha, \delta] \) is \( \Sigma_{U[\alpha, \delta]} \)-zip.

Proof. (1) \( \Rightarrow \) (2) Suppose that \( R \) is \( \Sigma_U \)-zip and \( V \) is a subset of \( R[\alpha, \delta] \) with \( V \not\subseteq U[\alpha, \delta] \) and \( U[\alpha, \delta] : V = U[\alpha, \delta] \). For a skew polynomial \( f(x) = \sum_{i=0}^{n} a_i x^i \in R[\alpha, \delta] \), \( C_f \) denotes the set of coefficients of \( f(x) \), and for a subset \( X \) of \( R[\alpha, \delta] \), \( C_X \) denotes the set \( \bigcup_{f \in X} C_f \). Then \( C_V \subseteq R \) and \( C_V \not\subseteq U \). Now we show that \( U : C_V = U \). If \( r \in U : C_V \), then \( ar \in U \) for any \( a \in C_V \). So by Lemma 3.2, we obtain

\[
 f(x)r = \left( \sum_{i=0}^{n} a_i x^i \right)r = \sum_{k=0}^{n} \left( \sum_{s=k}^{n} a_s f_s^*(r)x^k \right) \in U[\alpha, \delta]
\]

for any skew polynomial \( f(x) = \sum_{i=0}^{n} a_i x^i \in V \). Hence \( r \in U[\alpha, \delta] : V = U[\alpha, \delta] \), and so \( r \in U \). Thus \( U : C_V = U \). Since \( R \) is \( \Sigma_U \)-zip, there exists a finite subset \( Y_0 \subset C_V \) such that \( U : Y_0 = U \). For each \( a \in Y_0 \), there exists \( g_a(x) \in V \) such that some of the coefficients of \( g_a(x) \) are \( a \). Let \( V_0 \) be a minimal subset of \( V \) such that \( g_a(x) \in V_0 \) for each \( a \in Y_0 \). Then \( V_0 \) is a finite subset of \( V \). Let \( Y_1 = \bigcup_{g_a(x) \in V_0} C_{g_a(x)} \). Then \( Y_0 \subseteq Y_1 \), and so \( U : Y_1 \subseteq U : Y_0 = U \). If \( g(x) = \sum_{j=0}^{n} b_j x^j \in U[\alpha, \delta] : V_0 \), then \( f(x)g(x) \in U[\alpha, \delta] \) for each \( f(x) = \sum_{i=0}^{m} a_i x^i \in V_0 \). We have

\[
 f(x)g(x) = \left( \sum_{i=0}^{m} a_i x^i \right) \left( \sum_{j=0}^{n} b_j x^j \right) = \sum_{k=0}^{m+n} \left( \sum_{s+t=k}^{m} a_s f_s^*(b_t) \right)x^k \in U[\alpha, \delta].
\]

Thus we obtain

\[
 \sum_{s+t=k}^{m} \left( \sum_{i=s}^{m} a_i f_i^*(b_t) \right) \in U, \quad k = 0, 1, \ldots, m + n, 0 \leq s \leq m, 0 \leq t \leq n.
\]

Set \( k = m + n \). Then \( a_m \alpha^m(b_n) \in U \). By Lemma 3.1, we obtain \( a_m b_n \in U \), and so \( b_n a_m \in U \) since \( U \) is a semiprime ideal.

Set \( k = m + n - 1 \). We have

\[
 a_m \alpha^m(b_{n-1}) + a_{m-1} \alpha^{m-1}(b_n) + a_m f_{m-1}^m(b_n) \in U.
\]

Then

\[
 b_n a_m \alpha^m(b_{n-1}) + b_n a_{m-1} \alpha^{m-1}(b_n) + b_n a_m f_{m-1}^m(b_n) \in U,
\]
and so \( b_n a_{m-1} a^{m-1} (b_n) \in U \). By using Lemma 3.1 again, we obtain \( b_n a_{m-1} b_n \in U \), and so \( (b_n a_{m-1})^2 \in U \), \((a_{m-1} b_n)^2 \in U \). Since \( U \) is semiprime, we obtain \( b_n a_{m-1} \in U \) and \( a_{m-1} b_n \in U \).

Continuing this procedure yields that \( a_i b_n \in U \) for all \( 0 \leq i \leq m \), and so \( a_i f_s^t (b_n) \in U \) for every \( t \geq s \geq 0 \) and every \( 0 \leq i \leq m \). Thus it is easy to verify that \( \left( \sum_{i=0}^{m} a_i x^i \right) \left( \sum_{j=0}^{n-1} b_j x^j \right) \in U[x; \alpha, \delta] \). Applying the preceding method repeatedly, we obtain \( a_i b_j \in U \) for each \( 0 \leq i \leq m \) and \( 0 \leq j \leq n \). Thus \( b_j \in U : Y_1 \subseteq U : Y_0 = U \) for all \( 0 \leq j \leq n \), and so \( g(x) \in U[x; \alpha, \delta] \). Hence \( U[x; \alpha, \delta] : V_0 = U[x; \alpha, \delta] \). Therefore \( R[x; \alpha, \delta] \) is \( \Sigma_{U[x; \alpha, \delta]} \) zip.

(\( \Rightarrow \)) Conversely, assume that \( R[x; \alpha, \delta] \) is \( \Sigma_{U[x; \alpha, \delta]} \) zip. Let \( Y \) be a subset of \( R \) with \( Y \not\subseteq U \) and \( U : Y = U \). If \( f(x) = \sum_{i=0}^{n} a_i x^i \in U[x; \alpha, \delta] : Y \), then for each \( r \in Y \),

\[
rf(x) = r \left( \sum_{i=0}^{n} a_i x^i \right) = \sum_{i=0}^{n} ra_i x^i \in U[x; \alpha, \delta].
\]

Thus for each \( 0 \leq i \leq n \), we obtain \( a_i \in U : Y = U \), and it follows that \( f(x) \in U[x; \alpha, \delta] \). Thus we obtain \( U[x; \alpha, \delta] : Y = U[x; \alpha, \delta] \). Since \( R[x; \alpha, \delta] \) is \( \Sigma_{U[x; \alpha, \delta]} \) zip, there exists a finite subset \( Y_0 \subset Y \) such that \( U[x; \alpha, \delta] : Y_0 = U[x; \alpha, \delta] \). By Lemma 3.3, we obtain \( U : Y_0 = (U[x; \alpha, \delta] : Y_0) \cap R = U \). Therefore \( R \) is \( \Sigma_{U} \) zip.

**Corollary 3.5.** Let \( R \) be an \((\alpha, \delta)\)-compatible reduced ring. Then the following conditions are equivalent:

1. \( R \) is right zip.
2. \( R[x; \alpha, \delta] \) is right zip.

**Proof.** Note that the zero ideal of \( R \) is an \((\alpha, \delta)\)-compatible semiprime ideal if and only if \( R \) is an \((\alpha, \delta)\)-compatible reduced ring. Hence the result follows from Proposition 3.4. \( \square \)

**Corollary 3.6.** Let \( U \) be a semiprime ideal of \( R \). Then we have the following:

1. If \( U \) is an \( \alpha \)-compatible ideal, then the skew polynomial ring \( R[x; \alpha] \) is \( \Sigma_{U[x; \alpha]} \) zip if and only if \( R \) is \( \Sigma_{U} \) zip.
2. If \( U \) is an \( \delta \)-compatible ideal, then the differential polynomial ring \( R[x; \delta] \) is \( \Sigma_{U[x; \delta]} \) zip if and only if \( R \) is \( \Sigma_{U} \) zip.
3. the polynomial ring \( R[x] \) is \( \Sigma_{U[x]} \) zip if and only if \( R \) is \( \Sigma_{U} \) zip.
4. Skew generalized power series extension of $\Sigma$-zip rings

Let $(S, \leq)$ be an ordered set. Recall that $(S, \leq)$ is artinian if every strictly decreasing sequence of elements of $S$ is finite, and that $(S, \leq)$ is narrow if every subset of pairwise order-incomparable elements of $S$ is finite. Let $S$ be a commutative monoid. Unless stated otherwise, the operation of $S$ shall be denoted additively, and the neutral element by 0. The following definition is due to [7], [9], [12] and [13].

Let $R$ be a ring, $(S, \leq)$ a strictly ordered monoid (that is, $(S, \leq)$ is an ordered monoid satisfying the condition that, if $s, s', t \in S$ and $s < s'$, then $s + t < s' + t$), and $\omega : S \to \text{End}(R)$ a monoid homomorphism with $\omega(0)$ the identity map of $R$. For any $s \in S$, let $\omega_s$ denote the image of $s$ under $\omega$, that is, $\omega_s = \omega(s)$, and $1 = \omega_0 = \omega(0)$. Consider the set $A$ of all maps $f : S \to R$ whose support $\text{supp}(f) = \{s \in S \mid f(s) \neq 0\}$ is artinian and narrow. Then for any $s \in S$ and $f, g \in A$, the set

$$X_s(f, g) = \{(u, v) \in S \times S \mid u + v = s, f(u) \neq 0, g(v) \neq 0\}$$

is finite [13]. This fact allows to define the operation of convolution as follows:

$$(fg)(s) = \sum_{(u, v) \in X_s(f, g)} f(u)\omega_u(g(v)), \text{ if } X_s(f, g) \neq \emptyset,$$

and $(fg)(s) = 0$ if $X_s(f, g) = \emptyset$. With this operation of convolution, and pointwise addition, $A$ becomes a ring, which is called the ring of skew generalized power series with coefficients in $R$ and exponents in $S$, and we denote by $[[R^S, \leq, \omega]]$.

The skew generalized power series construction embraces a wide range of classical ring-theoretic extensions, including skew polynomial rings, skew power series rings, skew Laurent polynomial rings, skew group rings, Malcev-Neumann Laurent series rings and of courses the untwisted versions of all of these.

If $(S, \leq)$ is a strictly totally ordered monoid and $0 \neq f \in [[R^S, \leq, \omega]]$, then $\text{supp}(f)$ is a nonempty well-ordered subset of $(S, \leq)$. For any $r \in R$ and any $s \in S$, we define $\lambda^r_s \in [[R^S, \leq, \omega]]$ via

$$\lambda^r_s(t) = \begin{cases} r & t = s \\ 0 & t \neq s \end{cases}, \quad t \in S.$$

It is clear that $r \mapsto \lambda^r_s$ is a ring embedding of $R$ into $[[R^S, \leq, \omega]]$, and for any $r \in R, f \in [[R^S, \leq, \omega]]$, we have $rf = \lambda^r_s f$.

Let $U$ be a nonempty subset of $R$. We define $[[U^S, \leq, \omega]] = \{f \in [[R^S, \leq, \omega]] \mid f(s) \in U \cup \{0\} \text{ for all } s \in S\}$. In particular, we have $[[\text{nil}(R)^S, \leq, \omega]] = \{f \in [[R^S, \leq, \omega]] \mid f(s) \in \text{nil}(R) \text{ for all } s \in S\}$. 
Lemma 4.2. Let $\omega : S \rightarrow \text{End}(R)$ be a monoid homomorphism and $U$ an ideal of $R$. We say that $U$ is $\Sigma$-compatible if for each $a, b \in R$ and each $s \in S$, $ab \in U \iff a\omega_s(b) \in U$.

Proposition 4.3. Let $(S, \leq)$ be a strictly totally ordered monoid, and $U$ a $\Sigma$-compatible semiprime ideal of $R$. Then the following condition are equivalent:

1. $R$ is $\Sigma_U$-zip.
2. The skew generalized power series ring $[[R^{S,\leq},\omega]]$ is $\Sigma[[U^{S,\leq},\omega]]$-zip.

Proof. (1) $\Rightarrow$ (2) Suppose that $R$ is $\Sigma_U$-zip and $X$ is a subset of $[[R^{S,\leq},\omega]]$ with $X \not\subseteq [[U^{S,\leq},\omega]]$ and $[[U^{S,\leq},\omega]] : X = [[U^{S,\leq},\omega]]$. For any $f \in [[R^{S,\leq},\omega]]$, let $C_f$ denote the subset $\{f(s) \mid s \in S\}$ and for any subset $V \subseteq [[R^{S,\leq},\omega]]$, let $C_V$ denote the subset $\bigcup_{f \in V} C_f$. Now we show that $U : C_X = U$. If $r \in U : C_X$, then $ar \in U$ for all $a \in C_X$. By the condition that $U$ is $\Sigma$-compatible, we have that for any $f \in X$ and any $s \in S$,

$$(fr)(s) = (fx^0)(s) = f(s)\omega_s(r) \in U.$$  

So $fr \in [[U^{S,\leq},\omega]]$ and hence $r \in [[U^{S,\leq},\omega]] : X = [[U^{S,\leq},\omega]]$. Thus $r \in U$ and so $U : C_X = U$. Since $R$ is $\Sigma_U$-zip, there exists a finite subset $Y_0 = \{q_1, q_2, \ldots, q_k\} \subseteq C_X$ such that $U : Y_0 = U$. For each $q_i \in Y_0$, there exists $f_i \in X$ such that $f(s_i) = q_i$ for some $s_i \in \text{supp}(f_i)$. Let $X_0$ be a minimal subset of $X$ such that for each $q_i \in Y_0$, $f_i \in X_0$. Then $X_0$ is a finite subset of $X$. Since $C_{X_0} \supseteq Y_0$, we have $U : C_{X_0} \subseteq U : Y_0 = U$. Now we show that $[[U^{S,\leq},\omega]] : X_0 = [[U^{S,\leq},\omega]]$. Since $[[U^{S,\leq},\omega]] : X_0 \supseteq [[U^{S,\leq},\omega]]$ is clear, it suffices to show that $[[U^{S,\leq},\omega]] : X_0 \subseteq [[U^{S,\leq},\omega]]$. Let $g \in [[U^{S,\leq},\omega]] : X_0$. Then $fg \in [[U^{S,\leq},\omega]]$ for each $f \in X_0$. We proceed by transfinite induction on the strictly totally set $(S, \leq)$ to show that $f(u)g(v) \in U$ for any $u \in \text{supp}(f)$ and $v \in \text{supp}(g)$. Let $s$ and $t$ denote the minimal elements of $\text{supp}(f)$ and $\text{supp}(g)$ in the $\leq$ order, respectively. Thus

$$(fg)(s + t) = \sum_{(u,v) \in X_{s+t}(f,g)} f(u)\omega_u(g(v)) = f(s)\omega_s(g(t)) \in U,$$

and so $f(s)g(t) \in U$ since $U$ is $\Sigma$-compatible.
Now suppose that \( w \in S \) is such that for any \( u \in \text{supp}(f) \) and \( v \in \text{supp}(g) \) with \( u + v < w \), \( f(u)g(v) \in U \). We will show that \( f(u)g(v) \in U \) for any \( u \in \text{supp}(f) \) and \( v \in \text{supp}(g) \) with \( u + v = w \). We write
\[
X_w(f, g) = \{(u, v) \mid u + v = w, u \in \text{supp}(f), v \in \text{supp}(g)\},
\]
as \( \{(u_i, v_i) \mid i = 1, 2, \ldots, n\} \) such that
\[
u_1 < u_2 < \cdots < u_n.
\]
Since \((S, \leq)\) is a strictly totally ordered monoid, we have
\[
v_n < v_{n-1} < \cdots < v_2 < v_1.
\]
Now
\[
(fg)(w) = \sum_{(u, v) \in X_w(fg)} f(u)\omega_u(g(v)) = \sum_{i=1}^{n} f(u_i)\omega_{u_i}(g(v_i)) = a_1 \quad (1)
\]
where \( a_1 \in U \). For any \( i \geq 2 \), \( u_1 + v_i < u_i + v_i = w \), and thus, by induction hypothesis, we have \( f(u_1)g(v_i) \in U \). Since \( U \) is semiprime, we also have \( g(v_i)f(u_1) \in U \).

Since \( U \) is \( \Sigma \)-compatible, by Lemma 4.2, we have \( \omega_{u_1}(g(v_i))f(u_1) \in U \). Hence multiplying (1) on the right by \( f(u_1) \), we obtain \( f(u_1)\omega_{u_1}(g(v_i))f(u_1) \in U \), and so
\[
f(u_1)\omega_{u_1}(g(v_i))f(u_1) = f(u_1)\omega_{u_1}(g(v_i)f(u_1)) \in U.
\]
Thus we obtain \( f(u_1)g(v_1)f(u_1) \in U \). Since \( U \) is semiprime, we have \( f(u_1)g(v_i) \in U \), and \( g(v_i)f(u_1) \in U \). Now (1) becomes
\[
\sum_{i=2}^{n} f(u_i)\omega_{u_i}(g(v_i)) = a_1 - f(u_1)\omega_{u_1}(g(v_1)) = a_2, \quad \text{where } a_2 \in U. \quad (2)
\]
Multiplying (2) on the right by \( f(u_2) \), we obtain \( f(u_2)g(v_2) \in U \), \( g(v_2)f(u_2) \in U \) by the same way as above. Continuing this procedure yields that \( f(u_i)g(v_i) \in U \) for all \( 1 \leq i \leq n \). Thus \( f(u)g(v) \in U \) for any \( u \in \text{supp}(f) \) and \( v \in \text{supp}(g) \) with \( u + v = w \). Therefore by transfinite induction, \( f(u)g(v) \in U \) any \( u \in \text{supp}(f) \) and \( v \in \text{supp}(g) \). So for any \( s \in S \), \( g(s) \in U : CX_0 \subseteq U \). Thus \( g \in \[[U^{S, \leq}, \omega] \) and so \( \[[U^{S, \leq}, \omega] : X_0 \subseteq \[[U^{S, \leq}, \omega] \). Hence \( \[[U^{S, \leq}, \omega] : X_0 = \[[U^{S, \leq}, \omega] \). Therefore \( [[R^{S, \leq}, \omega] \) is \( \Sigma_{[[U^{S, \leq}, \omega]]}-\text{zip} \).

(2) \( \Rightarrow \) (1) Assume that \( [[R^{S, \leq}, \omega] \) is \( \Sigma_{[[U^{S, \leq}, \omega]]}-\text{zip} \). We will show that \( R \) is \( \Sigma_U \)-zip. Let \( Y \subseteq R \) with \( Y \not\subseteq U \) and \( U : Y = U \). If \( f \in \[[U^{S, \leq}, \omega] : Y \), then \( yf = \lambda_y^0 f \in [[U^{S, \leq}, \omega] \) for each \( y \in Y \), and so for any \( s \in S \), \( (yf)(s) = yf(s) \in U \). Thus for any \( s \in S \), \( f(s) \in U : Y = U \), and so \( f \in [[U^{S, \leq}, \omega] \). Hence \( [[U^{S, \leq}, \omega] : Y = [[U^{S, \leq}, \omega] \). Since \( [[R^{S, \leq}, \omega] \) is \( \Sigma_{[[U^{S, \leq}, \omega]]}-\text{zip} \), there exists a finite
subset \( Y_0 \subseteq Y \) such that \( [[U^{S,\leq},\omega]] : Y_0 = [[U^{S,\leq},\omega]] \). Then it is easy to see that
\[
U : Y_0 = \left( [[U^{S,\leq},\omega]] : Y_0 \right) \cap R = [[U^{S,\leq},\omega]] \cap R = U.
\]
Therefore \( R \) is \( \Sigma U \)-zip.

**Proposition 4.4.** Let \((S, \leq)\) be a strictly totally ordered monoid, and the zero ideal of \( R \) is \( \Sigma \)-compatible semiprime. Then the following condition are equivalent:

1. \( R \) is right zip.
2. the skew generalized power series ring \([[R^{S,\leq},\omega]]\) is right zip.

**Proof.** Let \( U = 0 \). Then we complete the proof by Proposition 4.3.

Let \( \alpha \) be a ring endomorphism of \( R \). Let \( S = \mathbb{N} \cup \{0\} \) be endowed with the usual order, and define \( \omega : S \rightarrow \text{End}(R) \) via \( \omega(0) = 1 \), the identity map of \( R \), and \( \omega(k) = \alpha^k \) for \( k \in \mathbb{N} \). Then \( [[R^{S,\leq},\omega]] \cong R[[x;\alpha]] \), the usual skew power series rings.

Let \( \alpha \) be a ring automorphism of \( R \). Let \( S = \mathbb{Z} \) be endowed with the usual order, and define \( \omega : S \rightarrow \text{End}(R) \) via \( \omega(s) = \alpha^s \). Then \( [[R^{S,\leq},\omega]] \cong R[[x,x^{-1};\alpha]] \), the usual skew Laurent power series rings.

As an immediate consequence of Proposition 4.3, we obtain the following corollary.

**Corollary 4.5.** Let \( U \) be an \( \alpha \)-compatible semiprime ideal. Then the following conditions are equivalent:

1. \( R \) is \( \Sigma U \)-zip.
2. The skew power series ring \( R[[x;\alpha]] \) is \( \Sigma U[[x,\alpha]] \)-zip.
3. The skew Laurent power series ring \( R[[x,x^{-1};\alpha]] \) is \( \Sigma U[[x,1,\alpha]] \)-zip.

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**References**


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