ON $\alpha$-QUASI SHORT MODULES

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Abstract. We introduce and study the concept of $\alpha$-quasi short modules. Using this concept we extend some of the basic results of $\alpha$-short modules to $\alpha$-quasi short modules. We observe that if $M$ is an $\alpha$-quasi short module then the Noetherian dimension of $M$ is $\alpha$ or $\alpha + 1$ or $\alpha + 2$.

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1. Introduction

Lemonnier [18] has introduced the concept of deviation (resp., codeviation) of an arbitrary poset, which in particular, when applied to the lattice of all submodules of a module $M_R$ give the concept of Krull dimension, see [9], [10] and [20] (resp., the concept of dual Krull dimension of $M$. The dual Krull dimension in [7,8,11,12,13, 14,15,16,17] is called Noetherian dimension and in [5] is called N-dimension. This dimension is called Krull dimension in [21]. The name of dual Krull dimension is also used by some authors, see [1], [2] and [3]). The Noetherian dimension of an $R$-module $M$ is denoted by $n$-dim $M$ and by $k$-dim $M$ we denote the Krull dimension of $M$. We recall that if an $R$-module $M$ has Noetherian dimension and $\alpha$ is an ordinal number, then $M$ is called $\alpha$-atomic if $n$-dim $M = \alpha$ and $n$-dim $N < \alpha$, for all proper submodule $N$ of $M$. An $R$-module $M$ is called atomic if it is $\alpha$-atomic for some ordinal $\alpha$ (note, atomic modules are also called conotable, dual critical and N-critical in some other articles; see for example [2], [5] and [19]). We introduced and extensively investigated quasi-Krull dimension and quasi-Noetherian dimension of an $R$-module $M$, see [6]. The quasi-Noetherian dimension (resp., quasi-Krull dimension), which is denoted by $qn$-dim $M$ (resp., $qk$-dim $M$) is defined to be the codeviation (resp., deviation) of the poset of the non-finitely generated submodules of $M$. We recall that an $R$-module $M$ is called $\alpha$-quasi-atomic, where $\alpha$ is an ordinal, if $qn$-dim $M = \alpha$ and $qn$-dim $N < \alpha$ for any proper non-finitely generated submodule $N$ of $M$. $M$ is said to be quasi-atomic if it is $\alpha$-quasi-atomic for some $\alpha$. 


Bilhan and Smith have introduced and extensively investigated short modules and almost Noetherian modules, see [4]. Later Davoudian, Karamzadeh and Shirali undertook a systematic study of the concepts of $\alpha$-short modules and $\alpha$-almost Noetherian modules, see [8]. We recall that an $R$-module $M$ is called an $\alpha$-short module, if for each submodule $N$ of $M$, either $n\dim N \leq \alpha$ or $n\dim \frac{M}{N} \leq \alpha$ and $\alpha$ is the least ordinal number with this property. We shall call an $R$-module $M$ to be $\alpha$-quasi short, if for each non-finitely generated submodule $N$ of $M$, either $qn\dim N \leq \alpha$ or $qn\dim \frac{M}{N} \leq \alpha$ and $\alpha$ is the least ordinal number with this property. Using this concept, we show that each $\alpha$-quasi short module $M$ has Noetherian dimension and $\alpha \leq n\dim M \leq \alpha + 2$. We also recall that an $R$-module $M$ is called $\alpha$-almost Noetherian, if for each proper submodule $N$ of $M$, $n\dim N < \alpha$ and $\alpha$ is the least ordinal number with this property, see [8]. We shall call an $R$-module $M$ to be $\alpha$-almost quasi Noetherian if for each proper non-finitely generated submodule $N$ of $M$, $qn\dim N < \alpha$ and $\alpha$ is the least ordinal number with this property. In Section 2, of this paper we investigate some basic properties of $\alpha$-almost quasi Noetherian and $\alpha$-quasi short modules. We show that if $M$ is an $\alpha$-quasi short module (resp., $\alpha$-almost quasi Noetherian module), then $qn\dim M = \alpha$ or $qn\dim M = \alpha + 1$ (resp., $qn\dim M \leq \alpha$). Thus we observe that if $M$ is an $\alpha$-quasi short module, then $M$ has Noetherian dimension and $\alpha \leq n\dim M \leq \alpha + 2$.

In the last section we also investigate some properties of $\alpha$-almost quasi Noetherian and $\alpha$-quasi short modules.

2. $\alpha$-quasi short modules and $\alpha$-almost quasi Noetherian modules

We recall that an $R$-module $M$ is called $\alpha$-almost Noetherian, if for each proper submodule $N$ of $M$, $n\dim N < \alpha$ and $\alpha$ is the least ordinal number with this property. In the following definition we consider a related concept.

**Definition 2.1.** An $R$-module $M$ is called $\alpha$-almost quasi Noetherian if for each proper non-finitely generated submodule $N$ of $M$, $qn\dim N < \alpha$ and $\alpha$ is the least ordinal number with this property.

It is manifest that if $M$ is an $\alpha$-almost quasi Noetherian, then each submodule and each factor module of $M$ is $\beta$-almost quasi Noetherian for some $\beta \leq \alpha$ (note, see [6, Lemmas 8, 9]).

In view of [6, Lemma 10], we have the next three trivial, but useful facts.

**Lemma 2.2.** If $M$ is an $\alpha$-almost quasi Noetherian module, then $M$ has quasi Noetherian dimension and $qn\dim M \leq \alpha$. In particular, $qn\dim M = \alpha$ if and only if $M$ is $\alpha$-quasi atomic.
Lemma 2.3. If $M$ is a module with $qn\dim M = \alpha$, then either $M$ is $\alpha$-quasi atomic, in which case it is $\alpha$-almost quasi Noetherian, or it is $\alpha + 1$-almost quasi Noetherian.

Lemma 2.4. If $M$ is an $\alpha$-almost quasi Noetherian module, then either $M$ is $\alpha$-quasi atomic or $\alpha = qn\dim M + 1$. In particular, if $M$ is $\alpha$-almost quasi Noetherian module, where $\alpha$ is a limit ordinal, then $M$ is $\alpha$-quasi atomic.

Proposition 2.5. An $R$-module $M$ has quasi-Noetherian dimension if and only if $M$ is $\alpha$-almost quasi Noetherian for some ordinal $\alpha$.

In view of Lemma 2.2 and [6, Corollary 5], we have the following result.

Corollary 2.6. If $R$-module $M$ is $\alpha$-almost quasi Noetherian, then $M$ has Noetherian dimension and $n\dim M \leq \alpha + 1$.

Next we give our definition of $\alpha$-quasi short modules.

Definition 2.7. An $R$-module $M$ is called $\alpha$-quasi short, if for each non-finitely generated submodule $N$ of $M$, either $qn\dim N \leq \alpha$ or $qn\dim \frac{M}{N} \leq \alpha$ and $\alpha$ is the least ordinal number with this property.

In view of [6, Corollary 3], we have the following results.

Remark 2.8. If $M$ is an $R$-module with $qn\dim M = \alpha$, then $M$ is $\beta$-quasi short for some $\beta \leq \alpha$.

Remark 2.9. If $M$ is an $\alpha$-quasi short module, then each submodule and each factor module of $M$ is $\beta$-quasi short for some $\beta \leq \alpha$.

We cite the following result from [6, Lemma 12].

Lemma 2.10. If $M$ is an $R$-module and for each non-finitely generated submodule $N$ of $M$, either $N$ or $\frac{M}{N}$ has quasi Noetherian dimension, then so does $M$.

The previous result and Remark 2.8, immediately yield the next result.

Corollary 2.11. Let $M$ be an $\alpha$-quasi short module. Then $M$ has quasi Noetherian dimension and $\alpha \leq qn\dim M$.

The following is now immediate.

Proposition 2.12. An $R$-module $M$ has quasi-Noetherian dimension if and only if $M$ is $\alpha$-quasi short for some ordinal $\alpha$. 
Proposition 2.13. If $M$ is an $\alpha$-quasi short $R$-module, then either $\text{qn-dim } M = \alpha$ or $\text{qn-dim } M = \alpha + 1$.

Proof. Clearly in view of Corollary 2.11, we have $\text{qn-dim } M \geq \alpha$. If $\text{qn-dim } M \neq \alpha$, then $\text{qn-dim } M \geq \alpha + 1$. Now let $M_1 \subseteq M_2 \subseteq \ldots$ be any ascending chain of non-finitely generated submodules of $M$. If there exists some $k$ such that $\text{qn-dim } \frac{M_{i+1}}{M_i} \leq \alpha$, then $\text{qn-dim } \frac{M_{i+1}}{M_i} \leq \text{qn-dim } M_{i+1} \leq \alpha$ for each $i \geq k$, see [6, Corollary 3]. Otherwise $\text{qn-dim } M_i \leq \alpha$ (M is $\alpha$-quasi short) for each $i$, hence $\text{qn-dim } \frac{M_{i+1}}{M_i} \leq \text{qn-dim } M_{i+1} \leq \alpha$ for each $i$. Thus in any case there exists an integer $k$ such that for each $i \geq k$, $\text{qn-dim } \frac{M_{i+1}}{M_i} \leq \alpha$. This shows that $\text{qn-dim } M \leq \alpha + 1$, i.e., $\text{qn-dim } M = \alpha + 1$. \hfill $\square$

In view of the previous proposition and [6, Corollary 5] we have the following result.

Corollary 2.14. If $M$ is an $\alpha$-quasi short $R$-module, then $\alpha \leq \text{n-dim } M \leq \alpha + 2$.

In view of previous corollary every $\alpha$-quasi short module has Krull dimension, for by a nice result due to Lemonnier, every module has Noetherian dimension if and only if it has Krull dimension, see [18, Corollary 6]. Thus by [20, Lemma 6.2.6], we have the following result.

Proposition 2.15. Every $\alpha$-quasi short module has finite uniform dimension.

Remark 2.16. An $R$-module $M$ is $-1$-quasi short if and only if it is either Noetherian or 1-atomic.

Proposition 2.17. Let $M$ be an $R$-module, with $\text{qn-dim } M = \alpha$, where $\alpha$ is a limit ordinal. Then $M$ is $\alpha$-quasi short.

Proof. We know that $M$ is $\beta$-quasi short for some $\beta \leq \alpha$. If $\beta < \alpha$, then by Proposition 2.13, $\text{qn-dim } M \leq \beta + 1 < \alpha$, which is a contradiction. Thus $M$ is $\alpha$-quasi short. \hfill $\square$

Proposition 2.18. Let $M$ be an $R$-module and $\text{qn-dim } M = \alpha = \beta + 1$. Then $M$ is either $\alpha$-quasi short or it is $\beta$-quasi short.

Proof. We know that $M$ is $\gamma$-quasi short for some $\gamma \leq \alpha$. If $\gamma < \beta$, then by Proposition 2.13, we have $\text{qn-dim } M \leq \gamma + 1 < \beta + 1$, which is impossible. Hence we are done. \hfill $\square$

Proposition 2.19. Let $M$ be an $\alpha$-quasi atomic $R$-module, where $\alpha = \beta + 1$, then $M$ is a $\beta$-quasi short module.
Proof. Let $N$ be a non-finitely generated submodule of $M$, therefore $qn$-$\dim N < \alpha$. This shows that for some $\beta' \leq \beta$, $M$ is $\beta'$-quasi short. If $\beta' < \beta$, then $\beta' + 1 \leq \beta < \alpha$. But $qn$-$\dim M \leq \beta' + 1 \leq \beta < \alpha$, by Proposition 2.13, which is a contradiction. Thus $\beta' = \beta$ and we are done. \hfill $\square$

The following remark, which is a trivial consequence of the previous fact, shows that the converse of Proposition 2.17, is not true in general.

Remark 2.20. Let $M$ be an $\alpha + 1$-quasi atomic $R$-module, where $\alpha$ is a limit ordinal. Then $M$ is an $\alpha$-quasi short module but $qn$-$\dim M \neq \alpha$.

Proposition 2.21. Let $M$ be an $R$-module such that $qn$-$\dim M = \alpha + 1$. Then $M$ is either $\alpha$-quasi short $R$-module or there exists a non-finitely generated submodule $N$ of $M$ such that $qn$-$\dim N = qn$-$\dim M_N = \alpha + 1$.

Proof. We know that $M$ is $\alpha$-quasi short or an $\alpha + 1$-quasi short $R$-module, by Proposition 2.18. Let us assume that $M$ is not $\alpha$-quasi short $R$-module, hence there exists a non-finitely generated submodule $N$ of $M$ such that $qn$-$\dim N \geq \alpha + 1$ and $qn$-$\dim M_N \geq \alpha + 1$. This shows that $qn$-$\dim N = \alpha + 1$ and $qn$-$\dim M_N = \alpha + 1$ and we are through. \hfill $\square$

Proposition 2.22. Let $M$ be a non-zero $\alpha$-quasi short $R$-module. Then either $M$ is $\beta$-almost quasi Noetherian for some ordinal $\beta \leq \alpha + 1$ or there exists a non-finitely generated submodule $N$ of $M$ with $qn$-$\dim M_N \leq \alpha$.

Proof. Suppose that $M$ is not $\beta$-almost quasi Noetherian for any $\beta \leq \alpha + 1$. This means that there must exist a non-finitely generated submodule $N$ of $M$ such that $qn$-$\dim N \nless \alpha$. Inasmuch as $M$ is $\alpha$-quasi short, we infer that $qn$-$\dim M_N \leq \alpha$ and we are done. \hfill $\square$

Let us cite the next result which is in [15, Theorem 2.9], see also [11, Theorem 3.2].

Theorem 2.23. For a commutative ring $R$ the following statements are equivalent.

1. Every $R$-module with finite Noetherian dimension is Noetherian.
2. Every Artinian $R$-module is Noetherian.
3. Every $R$-module with Noetherian dimension is both Artinian and Noetherian.

In view [8, Proposition 2.21], Corollary 2.14 and Corollary 2.6, we have the following result.
Proposition 2.24. The following statements are equivalent for a commutative ring $R$.

1. Every Artinian $R$-module is Noetherian.
2. Every $m$-short module is both Artinian and Noetherian for all integers $m \geq -1$.
3. Every $\alpha$-short module $M$ is both Artinian and Noetherian for all ordinal $\alpha$.
4. Every $m$-almost Noetherian module is both Artinian and Noetherian for all integers $m \geq -1$.
5. Every $\alpha$-almost Noetherian module is both Artinian and Noetherian for all integers $m \geq -1$.
6. Every $m$-quasi short module is both Artinian and Noetherian for all integers $m \geq -1$.
7. Every $\alpha$-quasi short module $M$ is both Artinian and Noetherian for all ordinal $\alpha$.
8. Every $m$-almost quasi Noetherian module is both Artinian and Noetherian for all integers $m \geq -1$.
9. Every $\alpha$-almost quasi Noetherian module $M$ is both Artinian and Noetherian for all ordinal $\alpha$.
10. No homomorphic image of $R$ can be isomorphic to a dense subring of a complete local domain of Krull dimension 1.

Finally we conclude this section by providing some examples of $\alpha$-almost quasi Noetherian (resp., $\alpha$-quasi short) modules, where $\alpha$ is any ordinal. First, we recall that given any ordinal $\alpha$ there exists an Artinian module $M$ such that $n$-dim $M = \alpha$, see [15, Example 1]. If $\alpha$ is a limit ordinal number then by [6, Corollary 5], we infer that $qn$-dim $M = \alpha$. Consequently, we may take $M$ to be an Artinian module with $n$-dim $M = \alpha$, where $\alpha$ is a limit ordinal number. Hence $qn$-dim $M = \alpha$ and for any ordinal $\beta \leq \alpha$, we take $N$ to be its $\beta$-quasi atomic submodule, see [6, Lemma 15], then by Lemma 2.3, $N$ is $\beta$-almost quasi Noetherian. We recall that the only $\alpha$-almost quasi Noetherian modules, where $\alpha$ is a limit ordinal are $\alpha$-quasi atomic module, see Lemma 2.4. Therefore to see an example of $\alpha$-almost quasi Noetherian module which is not $\alpha$-quasi atomic, the ordinal $\alpha$ must be a non-limit ordinal. Thus we may take $M$ to be a non-quasi atomic module with $qn$-dim $M = \beta$, where $\alpha = \beta + 1$, hence its follows trivially that $M$ is an $\alpha$-almost quasi Noetherian. As for examples of $\alpha$-quasi short modules, one can similarly use the facts that there are Artinian modules with Noetherian dimension equals to $\alpha$, see [15]. In view of [6, Corollary 5], we infer that $qn$-dim $M = \alpha$, where $\alpha$ is a limit ordinal number.
By [6, Lemma 15], for each $\beta \leq \alpha$ there are $\beta$-quasi atomic submodules of $M$ and then apply Propositions 2.17, 2.18, 2.19, to give various examples of $\alpha$-quasi short modules (for example, by Proposition 2.19, $\alpha + 1$-quasi atomic module is $\alpha$-quasi short).

3. Properties of $\alpha$-quasi short modules and $\alpha$-almost quasi Noetherian modules

In this section some properties of $\alpha$-quasi short modules over an arbitrary ring $R$ are investigated.

First, in view of Corollaries 2.14, 2.6, and [16, Corollary 1.8] we have the following result.

**Proposition 3.1.** If $M$ is an $\alpha$-quasi short module (resp., $\alpha$-almost quasi Noetherian module), where $\alpha$ is a countable ordinal, then every submodule of $M$ is countably generated.

**Remark 3.2.** Let $M$ be an $R$-module and $N$ be a submodule of $M$ such that $qn$-$\dim N = \alpha$ and $qn$-$\dim \frac{M}{N} = \beta$. If $\sup \{qn$-$\dim N, qn$-$\dim \frac{M}{N} \} = \gamma$, then $\gamma \leq qn$-$\dim M \leq \gamma + 1$.

**Proof.** We know that $n$-$\dim N = \alpha$ or $n$-$\dim N = \alpha + 1$ and $n$-$\dim \frac{M}{N} = \beta$ or $n$-$\dim \frac{M}{N} = \beta + 1$, see [6, Corollary 5]. Therefore $n$-$\dim M = \sup \{n$-$\dim N, n$-$\dim \frac{M}{N} \} \leq \gamma + 1$. But by [6, Remark 2], we get $qn$-$\dim M \leq n$-$\dim M \leq \gamma + 1$. In view of [6, Corollary 3], we get $\gamma \leq qn$-$\dim M$. This implies that $\gamma \leq qn$-$\dim M \leq \gamma + 1$ and we are done. □

In the following two propositions we investigate the connection between $\alpha$-short modules and $\alpha$-quasi short modules.

**Proposition 3.3.** Let $M$ be an $\alpha$-short $R$-module. Then $M$ is a $\beta$-quasi short module such that $\alpha \in \{\beta, \beta + 1, \beta + 2\}$.

**Proof.** Let $N$ be any non-finitely generated submodule of $M$, then $qn$-$\dim N \leq n$-$\dim N \leq \alpha$ or $qn$-$\dim \frac{M}{N} \leq n$-$\dim \frac{M}{N} \leq \alpha$, see [6, Remark 2]. This implies that $M$ is $\beta$-quasi short for some $\beta \leq \alpha$. If $M$ is $\beta$-quasi short, then $qn$-$\dim M = \beta$ or $qn$-$\dim M = \beta + 1$. Hence $\beta \leq n$-$\dim M \leq \beta + 2$, see [6, Corollary 5]. In other hand by [8, Proposition 1.12], we get $\alpha \leq n$-$\dim M \leq \alpha + 1$. Therefore $\beta = \alpha$ or $\alpha = \beta + 1$ or $\alpha = \beta + 2$ (note, we always have $\beta \leq \alpha$) and we are done. □

**Proposition 3.4.** Let $M$ be a $\beta$-quasi short $R$-module. Then $M$ is an $\alpha$-short $R$-module and $\alpha \in \{\beta, \beta + 1, \beta + 2\}$. 
Proof. By Proposition 2.13, $qn\dim M = \beta$ or $qn\dim M = \beta + 1$. This implies that $M$ has Noetherian dimension and $\beta \leq n\dim M \leq \beta + 2$, see [6, Corollary 5]. Thus $M$ is $\alpha$-short for some ordinal number $\alpha$, see [8, Remark 1.2]. By Proposition 3.3, we get $\alpha \in \{\beta, \beta + 1, \beta + 2\}$ and we are done. 

In view of Propositions 3.3 and 3.4 we have the following result.

Corollary 3.5. Let $M$ be an $R$-module and $\alpha$ be a limit ordinal number. Then $M$ is $\alpha$-short if and only if it is $\alpha$-quasi short.

We note that the $\mathbb{Z}$-module $\mathbb{Z} \oplus \mathbb{Z}_p^\infty$ is $0$-quasi short.

Proposition 3.6. Let $N$ be a submodule of an $R$-module $M$ such that $N$ is $\alpha$-quasi short and $M/N$ is $\beta$-quasi short. Let $\mu = \sup\{\alpha, \beta\}$, then $M$ is $\gamma$-quasi short such that $\mu \leq \gamma \leq \mu + 2$.

Proof. Since $N$ is $\alpha$-quasi short, thus by Proposition 2.13, $qn\dim N = \alpha$ or $qn\dim N = \alpha + 1$. Similarly since $M/N$ is $\beta$-quasi short, $qn\dim M/N = \beta$ or $qn\dim M/N = \beta + 1$. Let $\lambda = \sup\{qn\dim N, qn\dim M/N\}$, then $\mu \leq \lambda \leq \mu + 1$. In view of Remark 3.2, we infer that $M$ has quasi Noetherian dimension and $\lambda \leq qn\dim M \leq \lambda + 1$. Therefore $\mu \leq qn\dim M \leq \mu + 2$. But by Remark 2.12, $M$ is $\gamma$-quasi short for some ordinal number $\gamma$ and by Proposition 2.13, $\gamma \leq qn\dim M \leq \gamma + 1$. This shows that $\mu \leq \gamma \leq \mu + 2$, (note, we always have $\mu \leq \gamma$).

Using Lemma 2.2, we give the next immediate result which is the counterpart of the previous proposition for $\alpha$-almost quasi Noetherian modules.

Proposition 3.7. Let $N$ be a submodule of an $R$-module $M$ such that $N$ is $\alpha$-almost quasi Noetherian and $M/N$ is $\beta$-almost quasi Noetherian. Let $\mu = \sup\{\alpha, \beta\}$, then $M$ is $\gamma$-almost quasi Noetherian such that $\mu \leq \gamma \leq \mu + 2$.

Corollary 3.8. Let $R$ be a ring. If $M_1$ is an $\alpha_1$-quasi short (resp., $\alpha_1$-almost quasi Noetherian) $R$-module and $M_2$ is an $\alpha_2$-quasi short (resp., $\alpha_2$-almost quasi Noetherian) $R$-module and let $\alpha = \sup\{\alpha_1, \alpha_2\}$. Then $M_1 \oplus M_2$ is $\mu$-quasi short (resp., $\mu$-almost quasi Noetherian) for some ordinal number $\mu$ such that $\alpha \leq \mu \leq \alpha + 2$.

Example 3.9. If $M_1 = M_2 = \mathbb{Z}$, then $M_1$ and $M_2$ are $-1$-quasi short (resp., $-1$-almost quasi Noetherian) $\mathbb{Z}$-modules such that $M_1 \oplus M_2$ is also $-1$-quasi short (resp., $-1$-almost quasi Noetherian). Now let $M_1 = M_2 = \mathbb{Z}_p^\infty$. In this case the $\mathbb{Z}$-module $\mathbb{Z}_p^\infty$ is $-1$-quasi short (resp., $-1$-almost quasi Noetherian), but the $\mathbb{Z}$-module $\mathbb{Z}_p^\infty \oplus \mathbb{Z}_p^\infty$ is $0$-quasi short (resp., $0$-almost quasi Noetherian).
Proposition 3.10. Let $R$ be a ring and $M$ be a nonzero $\alpha$-quasi short module, which is not a quasi atomic module, then $M$ contains a non-finitely generated submodule $L$ such that $qn\dim \frac{M}{L} \leq \alpha$.

Proof. Since $M$ is not quasi atomic, we infer that there exists a non-finitely generated submodule $L \subseteq M$, such that $qn\dim L = qn\dim M$. We know that $qn\dim M = \alpha$ or $qn\dim M = \alpha + 1$, by Proposition 2.13. If $qn\dim M = \alpha$ it is clear that $qn\dim \frac{M}{L} \leq \alpha$. Hence we may suppose that $qn\dim L = qn\dim M = \alpha + 1$. If $qn\dim \frac{M}{L} = \alpha + 1$, then $M$ is $\gamma$-quasi short module for some $\gamma \geq \alpha + 1$, which is a contradiction. Consequently, $qn\dim \frac{M}{L} \leq \alpha$ and we are done. $\square$

The following example gives a module satisfying the condition of Proposition 3.10.

Example 3.11. Let $M = \mathbb{Z}_{p^\infty} \oplus \mathbb{Z}_{p^\infty} \oplus \mathbb{Z}_{p^\infty}$ and $L = \mathbb{Z}_{p^\infty} \oplus \mathbb{Z}_{p^\infty}$. By the comment which follows [6, Remark 2], we infer that $qn\dim L = 1$. Therefore $qn\dim M = 1$, see [6, Lemma 8]. Thus $M$ is not quasi atomic. But $\frac{M}{L} \simeq \mathbb{Z}_{p^\infty}$, thus $qn\dim \frac{M}{L} = 0$, see [6, Remark 1]. Clearly $M$ is a $0$-quasi short module.

Theorem 3.12. Let $\alpha$ be an ordinal number and $M$ be an $R$-module. If every proper non-finitely generated submodule of $M$ is $\gamma$-quasi short for some ordinal number $\gamma \leq \alpha$. Then $qn\dim M \leq \alpha + 2$, in particular, $M$ is $\mu$-short for some ordinal $\mu \leq \alpha + 2$.

Proof. Let $N \subseteq M$ be any non-finitely generated submodule. Since $N$ is $\gamma$-quasi short for some ordinal number $\gamma \leq \alpha$, we infer that $qn\dim N \leq \gamma + 1 \leq \alpha + 1$, by Proposition 2.13. This immediately implies that $qn\dim M \leq \alpha + 2$, see [6, Lemma 10]. The final part is now evident. $\square$

The next result is the dual of Theorem 3.12.

Theorem 3.13. Let $M$ be a nonzero $R$-module and $\alpha$ be an ordinal number. Let for each proper non-finitely generated submodule $N$ of $M$, $\frac{M}{N}$ be $\gamma$-quasi short for some ordinal number $\gamma \leq \alpha$. Then $qn\dim M \leq \alpha + 2$, in particular, $M$ is $\mu$-short for some ordinal $\mu \leq \alpha + 2$.

Proof. Let $N \subseteq M$ be any proper non-finitely generated submodule of $M$, then $\frac{M}{N}$ is $\gamma$-quasi short for some ordinal number $\gamma \leq \alpha$. In view of Proposition 2.13, we infer that $qn\dim \frac{M}{N} \leq \gamma + 1 \leq \alpha + 1$. Therefore $qn\dim M \leq \sup\{qn\dim \frac{M}{N} : N \text{ is nonfinitely generated submodule of } M\} + 1 \leq \alpha + 2$, see [6, Lemma 11]. The final part is now evident. $\square$
The next immediate result is the counterparts of Theorems 3.12, 3.13, for \(\alpha\)-almost quasi Noetherian modules.

**Proposition 3.14.** Let \(M\) be an \(R\)-module and \(\alpha\) be an ordinal number. If each proper non-finitely generated submodule \(N\) of \(M\) (resp., for each proper non-finitely generated submodule \(N\) of \(M\), \(\frac{M}{N}\)) is \(\gamma\)-almost quasi Noetherian with \(\gamma \leq \alpha\), then \(qn\)-dim \(M\) \(\leq \alpha + 1\) and \(M\) is an \(\mu\)-almost quasi Noetherian module with \(\mu \leq \alpha + 2\) (resp., \(qn\)-dim \(M\) \(\leq \alpha + 1\) and \(M\) is an \(\mu\)-almost quasi Noetherian module with \(\mu \leq \alpha + 2\)).

The following result is evident. We give the proof for the sake of completeness.

**Proposition 3.15.** If \(M\) has finite Goldie dimension, then

\[qn\text{-dim } M \leq \sup\{qn\text{-dim } \frac{M}{E} + 1 : E \subset M \text{ and } E \text{ is non-finitely generated}\}\]

if either side exists.

**Proof.** Let \(\alpha = \sup\{qn\text{-dim } \frac{M}{E} : E \text{ is essential and non-finitely generated}\}\), then it sufficient to show that \(qn\)-dim \(M\) exists and \(qn\)-dim \(M\) \(\leq \alpha\). Now let \(N_1 \subset N_2 \subset \cdots \subset N_i \subset \cdots\) be an infinite ascending chain of non-finitely generated submodule of \(M\), then by our assumption there exists some integer \(k\) such that \(N_i\) is essential in \(N_{i+1}\) for all \(i \geq k\) (note, \(M\) has finite Goldie dimension). This means that there exists a submodule \(P\) of \(M\) such that \(N_i \oplus P\) is essential in \(M\) for all \(i \geq k\). It is clear that for each \(i\), \(N_i \oplus P\) is a non-finitely generated submodule of \(M\) (note, if \(N_i \oplus P\) is finitely generated, then \(N_i\) is finitely generated which is a contradiction). But \(\frac{N_{i+1}}{N_i} \cong \frac{N_{i+1} \oplus P}{N_{i+1} \oplus P}\) for all \(i \geq k\). In view of [6, Lemma 8], we infer that \(qn\)-dim \(\frac{N_{i+1}}{N_i}\) \(= qn\)-dim \(\frac{N_{i+1} \oplus P}{N_{i+1} \oplus P}\) \(\leq qn\)-dim \(\frac{M}{N_i \oplus P}\) \(< \alpha\) for each \(i \geq k\) and hence \(qn\)-dim \(M\) \(\leq \alpha\). \(\square\)

**Proposition 3.16.** Let \(R\) be a semiprime ring. If the right \(R\)-module \(R\) is \(\alpha\)-quasi short, then \(qn\)-dim \(R\) \(= \alpha\) or \(qn\)-dim \(\frac{R}{E}\) \(\leq \alpha\) for each non-finitely generated essential right ideal \(E\) of \(R\).

**Proof.** Suppose that there exists an essential non-finitely generated right ideal \(E'\) of \(R\) such that \(qn\)-dim \(\frac{R}{E'}\) \(\leq \alpha\). Since \(R\) is \(\alpha\)-quasi short, we infer that \(qn\)-dim \(E'\) \(\leq \alpha\). In view of Corollary 2.14, \(R\) has Noetherian dimension. Therefore \(R\) is a right Goldie ring, see [10, Corollary 3.4]. Hence there exists a regular element \(c\) in \(E'\), which implies that \(qn\)-dim \(R = qn\)-dim \(cR\) \(\leq qn\)-dim \(E'\) \(\leq \alpha\). Consequently, we must have \(qn\)-dim \(R\) \(= \alpha\), by Proposition 2.13. \(\square\)
References


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