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A GENERALIZATION OF TOTAL GRAPHS OF MODULES

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Abstract. Let R be a commutative ring, and let $M \neq 0$ be an R-module with a non-zero proper submodule N, where $N^{\star} = N - \{0\}$. Let $\Gamma_{N^{\star}}(M)$ denote the (undirected) simple graph with vertices $\{x \in M - N \,|\, x + x' \in N^{\star}$ for some $x \neq x' \in M - N\}$, where distinct vertices x and y are adjacent if and only if $x + y \in N^{\star}$. We determine some graph theoretic properties of $\Gamma_{N^{\star}}(M)$ and investigate the independence number and chromatic number.

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1. Introduction

Throughout, all rings are commutative with non-zero identity and all modules are unitary. Let R be a ring, $M \neq 0$ an R-module, and N a non-zero proper submodule of M. The total graph of a commutative ring R, denoted by $T(\Gamma(R))$, was introduced by Anderson and Badawi in [3], as the graph with all elements of R as vertices, and two distinct vertices $x, y \in R$ are adjacent if and only if $x + y \in Z(R)$, where $Z(R)$ denotes the set of zero-divisors of R. The concept of total graphs is a great concept that is usually used in commutative algebra to obtain many interesting graphs in this field. In [1] and [2], A. Abbasi and S. Habibi, gave a generalization of the total graph. They studied in [2] the total graph $T(\Gamma_N(M))$ of a module M over a commutative ring with respect to a proper submodule N. It is an undirected graph with the vertex set M, where two distinct vertices m and n are adjacent if and only if $m + n \in M(N)$, where $M(N) = \{m \in M \,|\, rm \in N$ for some $r \in R - (N : M)\}$. It is easy to see that $M(N)$ is closed under multiplication by scalars. However, $M(N)$ may not be an additive subgroup of M. Here we introduce a generalization of total graphs, denoted by $\Gamma_{N^{\star}}(M)$, as the (undirected) simple graph with vertices $\{x \in M - N \,|\, x + x' \in N^{\star}$ for some $x \neq x' \in M - N\}$, where distinct vertices x and y are adjacent if and only if $x + y \in N^{\star} = N - \{0\}$.

Let G be a simple graph. If there is a path from any vertex to any other vertex of graph G, then G is said to be *connected*, and G is said to be *totally disconnected*

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if there is no path connecting any pair of vertices. For vertices x_1 and x_2 of G, we define $d(x_1, x_2)$ to be the length of a shortest path between x_1 and x_2 ($d(x, x) = 0$ and $d(x_1, x_2) = \infty$ if there is no such path). The *diameter* of G is $\operatorname{diam}(G) = \sup\{d(x_1, x_2) | \, x_1$ and x_2 are vertices of $G\}$. The *girth* of G, denoted by $\operatorname{gr}(G)$, is the length of its shortest cycle; $\operatorname{gr}(G) = \infty$ if G contains no cycles, in this case, G is called an *acyclic graph*. A *complete graph* is one which every two vertices are adjacent. A complete graph with n vertices is denoted by K^n. A *bipartite graph* G is a graph whose vertex set $V(G)$ can be partitioned into disjoint subsets U_1 and U_2 in such a way that each edge of G has one end vertex in U_1 and the other in U_2. In particular, if E consists of all possible such edges, then G is called a *complete bipartite graph* and is denoted by $K^{m,n}$ when $|U_1| = m$ and $|U_2| = n$. For a vertex v of G, $deg(v)$ denotes the degree of v and we set $\delta(G) := \min\{deg(v)$: v is a vertex of $G\}$. A graph G is called *k-regular* if every vertex has degree k. A subgraph of G is the graph formed by a subset of the vertices and edges of G. Two subgraphs G_1 and G_2 of G are said to be *disjoint* if G_1 and G_2 have no common vertices and no vertex of G_1 (resp., G_2) is adjacent (in G) to any vertex not in G_1 (resp., G_2). The *union* of two graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ is the graph $G_1 \cup G_2$ whose vertex set is $V_1 \cup V_2$ and whose edge set is $E_1 \cup E_2$. A complete subgraph of G is called a *clique*. The *clique number*, $\omega(G)$, is the greatest integer $n \geq 1$ such that K^n is a subgraph of G, and $\omega(G) = \infty$ if $K^n \subseteq G$ for all $n \geq 1$. A *matching* in a graph G is a set of edges such that no two have a vertex in common. A spanning matching of a graph is said to be a *perfect matching*. A star graph S_k is the complete bipartite graph $K^{1,k}$. A *Hamiltonian cycle* is a cycle that visits each vertex exactly once. A graph that contains a Hamiltonian cycle is called a *Hamiltonian graph*. A *walk* is an alternating sequence of vertices and edges which are incident, that begins and ends with a vertex. A *tour* is a closed walk that traverses each edge at least once. An *Eulerian tour* in an undirected graph is a tour that traverses each edge exactly once. If such a tour exists, the graph is called *Eulerian*. A connected *component* (or just component) of an undirected graph is a maximal connected induced subgraph. An *independent set* is a set of vertices in a graph, no two of which are adjacent. That is, it is a set S of vertices such that for every two vertices in S, there is no edge connecting the two. The *vertex independence number* of G, often called the independence number, is the cardinality of a largest independent vertex set, i.e., the maximum size among all independent vertex sets of G. The *independence number* is denoted by $\alpha(G)$. A *vertex cover* of G is a set of vertices such that each edge of G is incident to at least one vertex of the set. The *vertex cover number* is the minimum size among all vertex covers in the graph, denoted by $\beta(G)$. A *coloring* of a graph is a proper (vertex) coloring with colors such that no two vertices sharing the same edge have the same color. A coloring using k colors is called a (proper) k-*coloring*. The smallest number of colors needed to color the vertices of G is called its *chromatic number* and is denoted by $\chi(G)$.

The main objective of this paper is to study some properties of the graph $\Gamma_{N^\star}(M)$. We also investigate the independence number and chromatic number of the graph $\Gamma_{N^\star}(M)$.

2. Properties of $\Gamma_{N^\star}(M)$

In this section, we investigate some properties of $\Gamma_{N^\star}(M)$. Throughout, N is a non-zero proper submodule of a non-zero R-module M, where R is commutative ring.

Definition 2.1. Let R be a commutative ring, M be an R-module, N be a submodule of M, and let $N^\star = N - \{0\}$. We define an undirected simple graph $\Gamma_{N^\star}(M)$ with vertices $\{x \in M - N \,|\, x + x' \in N^\star \text{ for some } x \neq x' \in M - N\}$, where distinct vertices x and y are adjacent if and only if $x + y \in N^\star$.

Remark 2.2.
1. *$x \in V(\Gamma_{N^\star}(M))$ if and only if $N(x) \neq \emptyset$, where $N(x) = \{x' \in V(\Gamma_{N^\star}(M)) \,|\, x' \neq x, x + x' \in N^\star\}$. So, there is no isolated vertex in $\Gamma_{N^\star}(M)$. In particular, $\Gamma_{N^\star}(M)$ is not totally disconnected.*
2. *Let $x, y \in V(\Gamma_{N^\star}(M))$ be adjacent with $x - y \in N^\star$, then $x + y - x + y \in N$ and $x + y + x - y \in N$; so $2x, 2y \in N$.*
3. *Let $x, y \in V(\Gamma_{N^\star}(M))$ with $x \neq y$ and $N(x) \cap N(y) \neq \emptyset$. Then $x - y \in N^\star$.*
4. *$\Gamma_{N^\star}(M)$ is a perfect matching if and only if for all $x, y \in V(\Gamma_{N^\star}(M))$ with $x \neq y$, one has $N(x) \cap N(y) = \emptyset$ or $|N(x)| = |N(y)| = 1$.*

Example 2.3. *Let $M = \mathbb{Z}_{12}$ and $N = \{0, 4, 8\}$. Then $\Gamma_{N^\star}(M)$ has the following form:*

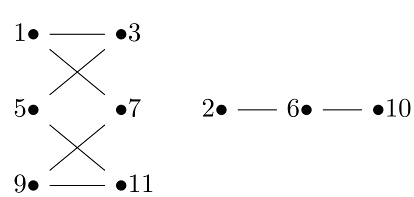

Figure 1.

Theorem 2.4. *If $x, y \in V(\Gamma_{N^\star}(M))$ are distinct vertices connected by a path of length 3 with $x + y \neq 0$, then x, y are adjacent.*

Proof. Let x, m_1, m_2, y be distinct vertices of $\Gamma_{N^\star}(M)$ with a path x — m_1—m_2— y. Since $x+m_1$, m_1+m_2 and $m_2+y \in N^\star$, we have $x+y = (x+m_1)+(y+m_2)-(m_1+m_2) \in N$. This yields $x+y \in N^*$, since $x+y \neq 0$; so x and y are adjacent. □

Corollary 2.5. *Let $e = xx'$ and $f = yy'$ be edges of $\Gamma_{N^\star}(M)$, where the sum of each end point of e and each end point of f does not equal zero. If two end points of e and f are adjacent, then so are the other two.*

Proof. Without loss of generality, we may assume that x' and y are adjacent; so there is a path x — x'— y— y' in $\Gamma_{N^\star}(M)$. Therefore, x and y' are adjacent, by Theorem 2.4. □

Remark 2.6. *In Corollary 2.5, the condition "does not equal zero" is necessary. For instance, in Example 2.3, set $x = 5, y = 9, x' = 3, y' = 11$. Then x and y' are adjacent, but x' and y are not.*

Theorem 2.7. *If $x, y \in V(\Gamma_{N^\star}(M))$ are distinct vertices connected by a path of length 4, then there exists a path of length 2 between them. In particular, there is $t \in V(\Gamma_{N^\star}(M))$ with $t \in N(x) \cap N(y)$.*

Proof. Let x, m_1, m_2, m_3, y be distinct vertices of $\Gamma_{N^\star}(M)$ with a path x — m_1— m_2— m_3— y. If $x+m_3 \neq 0$ or $m_1+y \neq 0$, then x and m_3, or y and m_1, are adjacent by Theorem 2.4, as we desired. So let $x = -m_3$ and $y = -m_1$. Then we have a path $x(= -m_3)$ — m_1— m_2— m_3— $y(= -m_1)$. Thus, $x(= -m_3)$ — $(-m_2)$— $y(= -m_1)$ is a path of length 2. □

Theorem 2.8. *Let $\Gamma_{N^\star}(M)$ be connected. If $t_1 - t_2 \in N^\star$ for all adjacent vertices t_1 and t_2 of $\Gamma_{N^\star}(M)$, then $diam(\Gamma_{N^\star}(M)) \in \{1, 2\}$.*

Proof. For every path of length 3 such as x — m_1— m_2— y in $\Gamma_{N^\star}(M)$, if $x+y \neq 0$, then x and y are adjacent by Theorem 2.4 and $\text{diam}(\Gamma_{N^\star}(M)) \leq 2$. Let $x = -y$. Then there is a path $x(= -y)$ — m_1— m_2— y. Our hypothesis yields x— m_2, and we are done. □

Theorem 2.9. *$diam(\Gamma_{N^\star}(M)) \in \{1, 2, 3, \infty\}$. In particular, if $\Gamma_{N^\star}(M)$ is connected, then $diam(\Gamma_{N^\star}(M)) \leq 3$.*

Proof. By Theorem 2.7, we can reduce every path of length greater than 3 to a path of length at most 3. □

Example 2.10. *Let $M = \mathbb{Z}_8$ and $N = \{0, 2, 4, 6\}$. Then $\Gamma_{N^\star}(M)$ has the following form:*

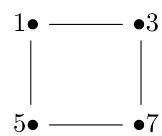

Figure 2.

Theorem 2.11. *Let $\Gamma_{N^\star}(M)$ be connected. Then it is complete if and only if $2t = 0$ for every $t \in V(\Gamma_{N^\star}(M))$.*

Proof. Suppose that $\Gamma_{N^\star}(M)$ is complete and $2t \neq 0$ for some $t \in V(\Gamma_{N^\star}(M))$. Then $-t$ is a vertex and $0 = t + (-t) \in N^\star$, which is a contradiction. Suppose $2t = 0$ for all vertices t. Then $\operatorname{diam}(\Gamma_{N^\star}(M)) \leq 3$ by Theorem 2.9. Let x — t— y be a path in $\Gamma_{N^\star}(M)$. Then by part (3) of Remark 2.2, $x + y \in N^\star$ (since by assumption $2y = 0$ implies that $y = -y$). Let $d(x, y) = 3$. So there is a path x — t_1— t_2— y in $\Gamma_{N^\star}(M)$. If $x + y = 0$, then $x = -y = y$ (since $2y = 0$); this contradicts our assumption, so $x + y \neq 0$. Hence, x and y are adjacent by Theorem 2.4. So, $\Gamma_{N^\star}(M)$ is complete. □

Theorem 2.12. *If $2x \neq 0$ for every $x \in V(\Gamma_{N^\star}(M))$, then $gr(\Gamma_{N^\star}(M)) \in \{3, 4, 6, \infty\}$.*

Proof. (1) It is clear that $\Gamma_{N^\star}(M)$ has more than two vertices. For all $x \in V(\Gamma_{N^\star}(M))$, let $|N(x)| = 1$. Then $\operatorname{gr}(\Gamma_{N^\star}(M)) = \infty$, since in this case $\Gamma_{N^\star}(M)$ is just a perfect matching.
(2) Suppose there is $t \in V(\Gamma_{N^\star}(M))$ such that $|N(t)| \geq 2$.

($1'$) If for all vertices t with $|N(t)| \geq 2$, we have $|N(x)| = 1$ for every $x \in N(t)$, then there are not any cycles in $\Gamma_{N^\star}(M)$.

($2'$) Suppose there exists $y \in N(t)$ such that $|N(y)| \geq 2$ and this condition is satisfied just for y. There is an $x \in N(t)$ such that $|N(x)| = 1$, since $|N(t)| \geq 2$. If $x \neq -y$, then by part 3 of Remark 2.2, one has $(-y)$ — x— t— y and if $x = -y$, then $(-y)$ — t— y— m for some $m \in N(y)$. This implies that $(-m)$ — $(-y)$— t— y— m, which contradicts $|N(x)| = 1$. So, we should have at least two vertices $x, y \in N(t)$ such that $|N(x)|, |N(y)| \geq 2$. If x and y are adjacent, then x — t— y— x and $\operatorname{gr}(\Gamma_{N^\star}(M)) = 3$. We assume that x and y are not adjacent for every $x, y \in N(t)$.

(a) Let $|N(x) \cap N(y)| = 1$. If $2t \in N^\star$, then $x + t, y + t \in N^\star$; so $x + y + 2t \in N$. This yields $x + y \in N$ and so $x + y = 0$ (since x and y are not adjacent). Therefore, $x = -y$ and there exists a path $(-y)$ — t— y— $(-t)$. This implies that $-y$ and

$-t$ are adjacent. So, $-t \in N(x) \cap N(y)$ where contradicts our assumption, since $|N(x) \cap N(y)| = 1$.

Now, assume that $2t \notin N^\star$ and $|N(t)| = 2$. There is a path $(-y)$— x— t— y — $(-x)$ in $\Gamma_{N^\star}(M)$ by part 3 of Remark 2.2; so $(-t)$— $(-y)$— x—t— y— $(-x)$ and $-t$ and $-x$ are adjacent. Hence $\mathrm{gr}(\Gamma_{N^\star}(M)) \leq 6$. If $|N(t)| \geq 3$, there is a path m — t— y— $(-x)$ for some vertex $m \neq x$. Since $m - x \neq 0$, one has $\mathrm{gr}(\Gamma_{N^\star}(M)) \leq 4$, by Theorem 2.4. If $-x, t$ are adjacent then there is a path $(-x)$ — t— y— $(-x)$ and $\mathrm{gr}(\Gamma_{N^\star}(M)) = 3$.

(b) Let $|N(x) \cap N(y)| \geq 2$. There is a path m — x— t— y— m. Hence $\Gamma_{N^\star}(M)$ contains a 4-cycle and $\mathrm{gr}(\Gamma_{N^\star}(M)) \leq 4$. □

Corollary 2.13. $\Gamma_{N^\star}(M)$ *is an acyclic graph if and only if it is a disjoint union of some star components.*

Proof. Suppose that graph $\Gamma_{N^\star}(M)$ is an acyclic graph. If $\Gamma_{N^\star}(M)$ has a non-star component, then there exists at least one path of length 3 as x — t_1— t_2— y in $\Gamma_{N^\star}(M)$. We assumed that $\Gamma_{N^\star}(M)$ is an acyclic graph, so $x + y = 0$, by Theorem 2.4. Hence, we have a path $(-t_2)$ — x— t_1— t_2— y. Theorem 2.7 yields there is a cycle in $\Gamma_{N^\star}(M)$, which contradicts our assumption. Hence all paths are of lengths 1 or 2. This implies that all components are in the form of stars. □

Theorem 2.14. *The following statements hold for the clique number of* $\Gamma_{N^\star}(M)$.

(1) $\omega(\Gamma_{N^\star}(M) = 2$ *if* $N(x) \cap N(y) = \emptyset$ *for every distinct* $x, y \in V(\Gamma_{N^\star}(M))$.
(2) *If there exist adjacent vertices* x *and* y *in* $\Gamma_{N^\star}(M)$ *such that* $N(x) \cap N(y) \neq \emptyset$, *then* $\omega(\Gamma_{N^\star}(M)) \geq 3$.
(3) *If* $2t = 0$ *for all* $t \in V(\Gamma_{N^\star}(M))$ *and there are adjacent vertices* x *and* y *in* $\Gamma_{N^\star}(M)$ *such that* $x' + y' \neq 0$ *for some* $x' \in N(x)$ *and* $y' \in N(y)$, *then* $\omega(\Gamma_{N^\star}(M)) \geq 4$.

Proof. (1) It is clear, since $\Gamma_{N^\star}(M)$ is a perfect matching.
(2) It is clear, since there is a triangular cycle.
(3) There is a path x' — x— y— y' in $\Gamma_{N^\star}(M)$. In view of Theorem 2.4, x' and y' are adjacent. So, x', y and x, y' are adjacent by part 3 of Remark 2.2. Hence $\omega(\Gamma_{N^\star}(M) \geq 4$. □

Definition 2.15. (See [5, Definition 2.9]) Let $m \in M - N$. We call the subset $m + N^\star$ a *column* of $\Gamma_{N^\star}(M)$. If $2m \in N^\star$ for every $m \in M - N$, then we call $m + N^\star$ a *connected column* of $\Gamma_{N^\star}(M)$.

Theorem 2.16. *Suppose* $\Gamma_{N^\star}(M)$ *contains at least one connected column and* $|N^\star| \geq 4$ *with* $2m \neq 0$ *for every* $m \in N^\star$. *Then gr*$(\Gamma_{N^\star}(M) = 3$.

Proof. Let $x + N^\star$ be a connected column in $\Gamma_{N^\star}(M)$. Then $2x \in N^\star$. Let $n \neq 2x, -2x$ in such a way that $n \in N^\star$. Then x — $(x+n)$— $(x-n)$— x is a cycle of length 3 in $\Gamma_{N^\star}(M)$. □

Recall that a vertex x of a connected graph G is called a *cut-point* of G if there are vertices u, w of G such that x lies on every path from u to w (with $x \neq u$, $x \neq w$). Equivalently, for a connected graph G, x is called a *cut-point* of G if $G - \{x\}$ is not connected.

Theorem 2.17. *Let $\Gamma_{N^\star}(M)$ be connected with $2x \neq 0$ for all $x \in V(\Gamma_{N^\star}(M))$. Then $\Gamma_{N^\star}(M)$ has no cut-points.*

Proof. Assume the vertex x of $\Gamma_{N^\star}(M)$ is a cut-point. Then there exist vertices u, w of $\Gamma_{N^\star}(M)$ such that x lies on every path from u to w (therefore, $x \neq u, w$). By Theorem 2.9, the shortest path from u to w is of length 2 or 3.
Case 1. Suppose u — x— w is a path of shortest length from u to w. There is a path $(-w)$— u— x— w— $(-u)$ in $\Gamma_{N^\star}(M)$. So there exists a path u— $(-w)$— $(-x)$ — $(-u)$—w by part 3 of Remark 2.2, which contradicts our assumption.
Case 2. Suppose (without loss of generality) that u— x— y—w is a path of shortest length from u to w in $\Gamma_{N^\star}(M)$. Therefore, $N(u) \cap N(w) = \emptyset$. Since u and w are not adjacent, by Theorem 2.4, we have $u + w = 0$ and $(-y)$ — $u(= -w)$— x— y—w. So, there exists a path u— $(-y)$— $(-x)$— w, which contradicts our assumption. □

Remark 2.18. *Suppose there is a path as u— t— w in $\Gamma_{N^\star}(M)$ such that $|N(u)| = |N(w)| = 1$. Then $\Gamma_{N^\star}(M)$ has a cut-point.*

Theorem 2.19. *The degree of every vertex x of $\Gamma_{N^\star}(M)$ is either $|N^\star|$ or $|N^\star|-1$. In particular, if $2m \in N^\star$ for every vertex m of $\Gamma_{N^\star}(M)$, then $\Gamma_{N^\star}(M)$ is a $|N^\star|-1$-regular graph.*

Proof. Let $x \in V(\Gamma_{N^\star}(M))$. If x is adjacent to y, then $x + y = a \in N^\star$ and hence, $y = a - x$ for some $a \in N^\star$. There are two cases:
Case 1. Suppose that $2x \in N^\star$. Then x is adjacent to $a - x$ for every $a \in N^\star - \{2x\}$. Thus the degree of x is $|N^\star| - 1$. In particular, if $2m \in N^\star$ for every $m \in V(\Gamma_{N^\star}(M))$, then $\Gamma_{N^\star}(M)$ is a $|N^\star| - 1$-regular graph.
Case 2. Suppose that $2x \notin N^\star$. Then x is adjacent to $a - x$ for all $a \in N^\star$. Thus the degree of x is $|N^\star|$. □

In general, it is not easy to determine when the graph $\Gamma_{N^\star}(M)$ is Eulerian or Hamiltonian. Here we consider $M = \mathbb{Z}_n$, for some positive integer n, and investigate being Eulerian or Hamiltonian (or both) for $\Gamma_{N^\star}(M)$.

Lemma 2.20. *The followings hold.*

(1) [4, Theorem 3.4] *If G is a simple graph with $\nu \geq 3$ and $\delta \geq \nu/2$, where $\nu = |V(G)|$, then G is Hamiltonian.*

(2) [4, Theorem 1.4] *A connected graph G is Eulerian if and only if it contains no vertices of odd degree.*

Example 2.21. *Let $M = \mathbb{Z}_n$ and $N = 2\mathbb{Z}_n$ with $n \geq 8$. Considering Theorem 2.19, $\Gamma_{N^\star}(M)$ is $|N^\star| - 1$-regular; so $\delta = |N^\star| - 1 = |N| - 2 \geq N/2$, where $|N|(= n/2) \geq 4$. Hence by Lemma 2.20, $\Gamma_{N^\star}(M)$ is Hamiltonian.*

Remark 2.22. *If $\Gamma_{N^\star}(M)$ is connected, $2x \notin N^\star$ for every $x \in V(\Gamma_{N^\star}(M))$, and if $|N^\star|$ is an even integer, then $\Gamma_{N^\star}(M)$ is Eulerian.*

Remark 2.23. *Let $M = \mathbb{Z}_n$ and $N = k\mathbb{Z}_n$ (so, $N = d\mathbb{Z}_n$ for $d = (k, n)$), and let $\Gamma_{N^\star}(M)$ be connected.*

(1) *If n is an odd integer, then $|N^\star|$ is even. Let $2x \in N^\star$ for some $x \in V(\Gamma_{N^\star}(M))$. Then $2x = td$ for some $t \in \mathbb{Z}$. Hence $x \in N$ is not a vertex. So $2x \notin N^\star$ for every $x \in V(\Gamma_{N^\star}(M))$. Hence, by Lemma 2.20 and Theorem 2.19, $\Gamma_{N^\star}(M)$ is Eulerian.*

(2) *Assume that n and k are even integers; then d is an even integer. By Theorem 2.19, the degree of every vertex x is $|N^\star|$ or $|N^\star| - 1$.*

(*i*) *Let $n = 2^l$ for some $l \in \mathbb{N}$. If $d > 2$, then there exists at least one vertex x such that $2x \notin N^\star$. So, the degree of the vertex x is $|N^\star|$. Note that $|N^\star|$ is an odd integer. Hence by Lemma 2.20, $\Gamma_{N^\star}(M)$ is not Eulerian. If $d = 2$, then by Theorem 2.19, $\Gamma_{N^\star}(M)$ is a $|N^\star| - 1$-regular graph, so it is Eulerian.*

(*ii*) *Let $n = 2^l m$ for some $l, m \in \mathbb{N}$ such that $(2, m) = 1$. Since $d = 2m' \in N^\star$ for some $m' \in V(\Gamma_{N^\star}(M))$, the degree of vertex m' is $|N^\star| - 1$. Note that $n = td$ for some $t \in \mathbb{Z}$. If t is an odd integer, then $|N^\star|$ is even. So the degree of m' is odd and $\Gamma_{N^\star}(M)$ is not Eulerian. If t is an even integer, then $|N^\star|$ is odd. We have two cases.*

(*i'*) *If $d > 2$, there exists at least one vertex l such that $2l \notin N^\star$. Therefore, by Theorem 2.19, $deg(l) = |N^\star|$, hence $\Gamma_{N^\star}(M)$ is not Eulerian.*

(*ii'*) *If $d = 2$, by Theorem 2.19, $\Gamma_{N^\star}(M)$ is a $|N^\star| - 1$-regular graph and so it is Eulerian.*

(3) *Let n be an even integer and k be an odd integer. Since $|N^\star|$ is an odd integer and by part (1), $2x \notin N^\star$ for every $x \in V(\Gamma_{N^\star}(M))$, $\Gamma_{N^\star}(M)$ is not Eulerian.*

3. Independence number and chromatic number of $\Gamma_{N^\star}(M)$

One of the interesting computing problems in graph theory is determining the independence number of a graph. Here we obtain the independence number of $\Gamma_{N^\star}(M)$ with some special conditions. It is well-known that $\alpha(K^n) = 1$.

Lemma 3.1. [4, Theorem 1.7]

(1) *A set is independent if and only if its complement is a vertex cover.*
(2) *The sum of the size of the largest independent set $\alpha(G)$ and the size of a minimum vertex cover $\beta(G)$ is equal to the number of vertices in the graph.*

Theorem 3.2. *Let $\Gamma_{N^\star}(M)$ be connected and let $\nu = |V(G)|$.*

(1) *If diam$(\Gamma_{N^\star}(M)) = 3$ and $d(m, -m) = 3$ for every $m \in V(\Gamma_{N^\star}(M))$, then $\alpha(\Gamma_{N^\star}(M)) = \beta(\Gamma_{N^\star}(M)) = \nu/2$.*
(2) *If diam$(\Gamma_{N^\star}(M)) = 2$ and $2m \neq 0$ for every $m \in V(\Gamma_{N^\star}(M))$, then $\alpha(\Gamma_{N^\star}(M)) = 2$ and $\beta(\Gamma_{N^\star}(M)) = \nu - 2$.*

Proof. (1) Choose $x \in V(\Gamma_{N^\star}(M))$. Put $A_x = \{-y \in V(\Gamma_{N^\star}(M)) \,|\, y \text{ is adjacent to } x\}$, $A'_x = \{y \in V(\Gamma_{N^\star}(M)) \,|\, y \neq -x \text{ and } y \text{ is not adjacent to } x\}$ and $P_x = A_x \cup A'_x$. For every $n \in V(\Gamma_{N^\star}(M)) - \{x, -x\}$, $n \in P_x$ or $-n \in P_x$.
Claim: P_x is an independent set in $\Gamma_{N^\star}(M)$.
By way of contradiction, suppose there exist $n_1, n_2 \in P_x$ that they are adjacent. Since $n_1, n_2 \in P_x$, so n_1, n_2 are not adjacent to x. We claim that for every vertex t other than x and $-x$, t is adjacent to either $-x$ or x (but not to both of them, otherwise, $x — t — (-x)$, this yields $d(x, -x) = 2$). Hence, n_1, n_2 are not adjacent to x, which implies that n_1, n_2 are adjacent to $-x$.
Suppose there exists $t(\neq x, -x) \in V(\Gamma_{N^\star}(M))$ such that t is not adjacent to x and $-x$. Since $d(x, -x) = 3$, there exists a path $x — m_1 — m_2 — (-x)$ in $\Gamma_{N^\star}(M)$. It is clear that $d(t, x) = d(t, -x) = 2$; otherwise, t is adjacent to x or $-x$. So, there exists $l \in V(\Gamma_{N^\star}(M))$ such that $t — l — x$. There is a path $t — l — x — m_1 — m_2 — (-x)$ such that t is adjacent to m_1 (since m_1 and x are adjacent and $d(t, -x) = 2$, so $t \neq -m_1$). Hence $t — m_1 — m_2 — (-x)$ implies that t and $-x$ are adjacent (since $t \neq x$) which is a contradiction. (Therefore, for every vertex t, all other vertices except $-t$ are adjacent to t or $-t$.) Since n_1, n_2 are not adjacent to x, so n_1, n_2 are adjacent to $-x$ and $n_1 — (-x) — n_2 — (-n_1)$ which by assumption n_1 and n_2 are adjacent. So, $d(n_1, -n_1) = 2$, a contradiction. This shows that P_x is independent. On the other hand, for every $x \neq y \in V(\Gamma_{N^\star}(M))$, one has

$|P_x| = |P_y|$. We have to show that P_x is the largest independent set in $\Gamma_{N^\star}(M)$. Suppose there exists an independent set U in $\Gamma_{N^\star}(M)$ such that $|U| > |P_x| = |\nu|/2$, where $\nu = |V(G)|$. So, there exists $g \in V(\Gamma_{N^\star}(M))$ such that $g, -g \in U$. This implies that U is not independent. Hence P_x is the largest independent set in $\Gamma_{N^\star}(M)$ and $\alpha(\Gamma_{N^\star}(M)) = \beta(\Gamma_{N^\star}(M)) = \nu/2$ by Lemma 3.1.

(2) Let $\Gamma_{N^\star}(M)$ be connected with $diam(\Gamma_{N^\star}(M)) = 2$ and let $2m \neq 0$ for every $m \in V(\Gamma_{N^\star}(M))$. Put $G_x = \{x, -x\}$ for some $x \in V(\Gamma_{N^\star}(M))$. We show that G_x is the largest independent set in $\Gamma_{N^\star}(M)$.

Claim: Every vertex x is adjacent to every other vertex except $-x$.

By way of contradiction, assume that there is $m \in V(\Gamma_{N^\star}(M))$ such that $d(m, x) = 2$ for some $x (\neq -m) \in V(\Gamma_{N^\star}(M))$. So there is a path m— t— x in $\Gamma_{N^\star}(M)$ for some $t \in V(\Gamma_{N^\star}(M))$; therefore, $(-x)$— m— t— x such that x and $-m$ are adjacent. Moreover, $d(x, -x) = 2$. Hence, there is a path x— l— $(-x)$ in $\Gamma_{N^\star}(M)$ for some $l \in V(\Gamma_{N^\star}(M))$. Now, the path $(-m)$— x— l— $(-x)$ implies that $-m$ is adjacent to $-x$ by Theorem 2.4. So m is adjacent to x, a contradiction.

Therefore, G_x is the largest independent set in $\Gamma_{N^\star}(M)$. In this case $\alpha(\Gamma_{N^\star}(M)) = 2$ and $\beta(\Gamma_{N^\star}(M)) = \nu - 2$ where $\nu = |V(G)|$ by Lemma 3.1. □

One of the important aims in graph theory is determining the chromatic number of a given graph. Here we investigate the chromatic number of $\Gamma_{N^\star}(M)$ in some special cases. It is well-known that $\chi(K^n) = n$ and $\chi(G) \geq \omega(G)$.

Remark 3.3. *For all distinct $x, y \in V(\Gamma_{N^\star}(M))$, if $N(x) \cap N(y) = \emptyset$, then it is obvious that $\Gamma_{N^\star}(M)$ is a perfect matching and $\chi(\Gamma_{N^\star}(M)) = 2$.*

Theorem 3.4. *Let $\Gamma_{N^\star}(M)$ be connected and non-complete. If there exist two adjacent vertices x and y of $\Gamma_{N^\star}(M)$ with $N(x) \cap N(y) \neq \emptyset$ or for every two non-adjacent vertices x and y of $\Gamma_{N^\star}(M)$, $N(x) \cap N(y) \neq \emptyset$, then $\chi(\Gamma_{N^\star}(M)) \geq 3$.*

Proof. Since $\Gamma_{N^\star}(M)$ is not complete, there exist non-adjacent vertices x and y in $\Gamma_{N^\star}(M)$. By assumption, $N(x) \cap N(y) \neq \emptyset$; so $(-y)$— x— t— y and l— $(-y)$— x— t— y— l for some $l \in N(y) \cap N(-y)$. Thus $\chi(\Gamma_{N^\star}(M)) \geq 3$.
(It should be noted that if $y = -y$, then $\chi(\Gamma_{N^\star}(M)) \geq 3$.) □

Theorem 3.5. (1) *Let $\Gamma_{N^\star}(M)$ be connected with $diam(\Gamma_{N^\star}(M)) = 3$ and $d(m, -m) = 3$ for every $m \in V(\Gamma_{N^\star}(M))$. Then $\chi(\Gamma_{N^\star}(M)) = 2$.*

(2) *Let $\Gamma_{N^\star}(M)$ be connected with $diam(\Gamma_{N^\star}(M)) = 2$ and let $2m \neq 0$ for every $m \in V(\Gamma_{N^\star}(M))$. Then $\chi(\Gamma_{N^\star}(M)) = \nu/2$, where $\nu = |V(G)|$.*

Proof. (1) Let $x \in V(\Gamma_{N^\star}(M))$. Considering our hypothesis and by the proof of part 1 of Theorem 3.2, for every vertex t other than x and $-x$, t is adjacent to either $-x$ or x (but not to both of them, otherwise, x— t— $(-x)$, this implies that $d(x,-x)=2$). If t is adjacent to x, then t is not adjacent to $-x$; so $t \in P_{-x}$, otherwise, $t \in P_x$. Hence $P_x \cup P_{-x} = V(\Gamma_{N^\star}(M))$. Now we assign color a to elements of P_x and color b to elements of P_{-x}. Therefore, $\chi(\Gamma_{N^\star}(M)) = 2$.

(2) Let $l_1 \in V(\Gamma_{N^\star}(M))$. Here, by the proof of part 2 of Theorem 3.2, every vertex m is adjacent to all other vertices except $-m$. At first we assign color 1 to l_1 and $-l_1$. Choose $l_i (\neq l_1, -l_1) \in V(\Gamma_{N^\star}(M))$ where $i > 1$. Since l_i is adjacent to l_1 and $-l_1$ and all of other vertices except $-l_i$, we assign color i to l_i and $-l_i$. Continuing in this manner for remaining vertices of $\Gamma_{N^\star}(M)$, one has $\chi(\Gamma_{N^\star}(M)) = \nu/2$. □

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References

[1] A. Abbasi and S. Habibi, *The total graph of a commutative ring with respect to proper ideals*, J. Korean Math. Soc., 49(1) (2012), 85-98.

[2] A. Abbasi and S. Habibi, *The total graph of a module over a commutative ring with respect to proper submodules*, J. Algebra Appl., 11(3) (2012), 1250048 (13 pp).

[3] D. F. Anderson and A. Badawi, *The total graph of a commutative ring*, J. Algebra, 320(7) (2008), 2706-2719.

[4] J. A. Bondy and U. S. R. Murty, Graph Theory with Applications, American Elsevier Publishing Co., Inc., New York, 1976.

[5] S. P. Redmond, *An ideal-based zero-divisor graph of a commutative ring*, Comm. Algebra, 31(9) (2003), 4425-4443.

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