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PERINORMAL POLYNOMIAL DOMAINS

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Abstract. Let A be a domain. We relate the perinormality (as defined by Epstein and Shapiro) of A and $A[X]$ for a narrow class of Noetherian domains.

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1. Introduction

In [1] and [2], Epstein and Shapiro studied the integral domains A with the property that every overring B of A which satisfies going down over A is A-flat (called by them *perinormal domains*). Krull domains are typical examples of perinormal domains. [1, Question 3] asks to relate the perinormality of A and $A[X]$. Using the pullback approach from [2], we provide an answer for a narrow class of Noetherian domains. Recall that a field K is *Hilbertian* if given $f_i(T_1, \ldots, T_n, X)$ irreducible polynomials in $K(T_1, \ldots, T_n)[X]$, $1 \le i \le k$, and $g \in K[T_1, \ldots, T_n] - \{0\}$, there exist $a_1, \ldots, a_n \in K$ such that each $f_i(a_1, \ldots, a_n, X)$ is defined and irreducible in $K[X]$ and $g(a_1, \ldots, a_n) \neq 0$, cf. [3, Chapter 11]. For an ideal I of a ring A, denote by $V_A(I)$ the Zariski closed set defined by I. We use standard terminology like in [4]. Our result is:

Theorem 1.1. *Let A be a Noetherian domain with the integral closure A'. Assume that the conductor $(A : A')$ has height at least two as an ideal of A' and $A/(A : A')$ is zero-dimensional and not local. Then the first three assertions below are equivalent and imply the fourth one.*

(a) *$A[X_1, \ldots, X_n]$ is perinormal for every $n \ge 0$.*
(b) *$A[X_1, \ldots, X_n]$ is perinormal for some $n \ge 1$.*
(c) *$A(X_1, \ldots, X_n)$ is perinormal for some $n \ge 1$.*
(d) *A is perinormal.*

Moreover, if A/M is a Hilbertian field for every $M \in V_A(A : A')$, then all four assertions are equivalent.

2. Lemmata and proof of Theorem 1.1

The proof is based on two lemmas. In [2, Definition 3.2], an integral extension of rings $A \subseteq B$ is called *apparently fragile* if for every ring C with $A \subset C \subseteq B$, there exists a minimal prime P of A which is not unibranched in B. The extension is called *fragile* if $A_P \subseteq B_{A-P}$ is apparently fragile for every prime ideal P of A. Due to Cohen-Seidenberg theorems, it can be seen that, when A is an integrally closed Noetherian domain and B is a finite reduced ring extension of A, then $A \subseteq B$ is apparently fragile if and only if there is no domain C, $A \subset C \subseteq B$.

Lemma 2.1. *Let $A \subseteq C$ and $B \subseteq D$ be integral ring extensions. Then $A \times B \subseteq C \times D$ is fragile if and only if $A \subseteq C$ and $B \subseteq D$ are fragile.*

Proof. The assertion follows combining the following simple facts. The extension $A \times B \subseteq C \times D$ is integral, *Spec*$(A \times B)$ is the disjoint union of *Spec*(A) and *Spec*(B), and, if $P \in$ *Spec*(A), then the extension $(A \times B)_{P \times B} \subseteq (C \times D)_{(A-P)\times B}$ is isomorphic to $A_P \subseteq C_{A-P}$. □

In the sequel, if A is a ring and $B_1, \ldots, B_k$ are ring extensions of A, we embed diagonally A in $\prod_{i=1}^k B_i$ and simply write $A \subseteq \prod_{i=1}^k B_i$.

Lemma 2.2. *Let K be a field and $L_1, \ldots, L_k$ finite field extensions of K. Then the first three assertions below are equivalent and imply the fourth one.*

(a) $K[X_1, \ldots, X_n] \subseteq \prod_{i=1}^k L_i[X_1, \ldots, X_n]$ *is fragile for every* $n \geq 1$.
(b) $K(X_1, \ldots, X_n) \subseteq \prod_{i=1}^k L_i(X_1, \ldots, X_n)$ *is fragile for every* $n \geq 1$.
(c) $K(X_1, \ldots, X_n) \subseteq \prod_{i=1}^k L_i(X_1, \ldots, X_n)$ *is fragile for some* $n \geq 1$.
(d) $K \subseteq \prod_{i=1}^k L_i$ *is fragile.*

Moreover, if K is a Hilbertian field, then all four assertions are equivalent.

Proof. Note that due to [2, Proposition 3.8], we may change everywhere fragile by apparently fragile. The case $k = 1$ is obvious, so we may suppose that $k \geq 2$. Set $A = K[X_1, \ldots, X_n]$, $S = A - \{0\}$, $B = \prod_{i=1}^k L_i[X_1, \ldots, X_n]$, $C = \prod_{i=1}^k L_i$ and observe that we have $B = C[X_1, \ldots, X_n]$, $A_S = K(X_1, \ldots, X_n)$ and $B_S = \prod_{i=1}^k L_i(X_1, \ldots, X_n)$, because $A \subseteq B$ is finite. $(a) \Rightarrow (b)$ Deny; so there exists a field E situated strictly between A_S and B_S. By (a), we get $E \cap B = A$. We obtain $E = (E \cap B)_S = A_S$, a contradiction. $(b) \Rightarrow (c)$ is trivial. $(c) \Rightarrow (d)$ Deny; hence there exists a field M situated strictly between K and C, so $M(X_1, \ldots, X_n)$ is situated strictly between A_S and B_S, a contradiction.

Now we prove $(d) \Rightarrow (b)$ for K being a Hilbertian field. Suppose that (b) fails. Then there exists a field E situated strictly between A_S and B_S. We may assume

$E = A_S(\alpha)$ for some $\alpha = (\alpha_1, \dots, \alpha_n) \in B_S - A_S$. Here $\alpha_i = \alpha_i(X_1, \dots, X_n) \in L_i(X_1, \dots, X_n)$ and let $p \in S$ such that $p\alpha_i \in B$ for each i. It follows that the minimal polynomial $f(X_1, \dots, X_n, Y) \in A_S[Y]$ of α over A_S equals the minimal polynomial of α_i over A_S for each i between 1 and k. As K is Hilbertian, there exist $a_1, \dots, a_n \in K$ such that $p(a_1, \dots, a_n) \neq 0$ and $g = f(a_1, \dots, a_n, Y)$ is defined and irreducible in $K[Y]$. Let $\beta_i = \alpha_i(a_1, \dots, a_n)$ and $\beta = (\beta_1, \dots, \beta_n) \in C$. As g is irreducible and $g(\beta) = 0$, it follows that g is the minimal polynomial of β over K, hence $K(\beta)$ is a field situated strictly between K and C. The implication $(c) \Rightarrow (b)$ follows from the fact that, for $n \geq 1$, $K(X_1, \dots, X_n)$ is Hilbertian [3, Theorem 12.10] and from implications $(c) \Rightarrow (d)$ and $(d) \Rightarrow (b)$ (for K Hilbertian) proved above. $(b) \Rightarrow (a)$ Let D be a domain situated between A and B. By (b), we get $A_S = D_S$, hence $D \subseteq A_S \cap B = A$, thus $D = A$. □

Proof. *Proof of Theorem 1.1.* $(a) \Rightarrow (b)$ and $(a) \Rightarrow (d)$ are trivial and $(b) \Rightarrow (c)$ follows from the fact that perinormality is a local property [1, Proposition 2.5]. $(c) \Rightarrow (a)$ Set $B = A(X_1, \dots, X_n)$, $B' = A'(X_1, \dots, X_n)$ and $I = (A : A')$. As $A \subseteq A'$ is finite, we have that $B' = B \otimes_A A'$ is the integral closure of B. Note that IB is also an ideal of B'. Since A/I is zero-dimensional, it follows that $B/IB = (A/I)(X_1, \dots, X_n) \subseteq B'/IB = (A'/I)(X_1, \dots, X_n)$ are zero-dimensional rings. Since B is perinormal, it follows that $B/IB \subseteq B'/IB$ is fragile [2, Theorem 3.13] and I is a radical ideal of A, cf. [2, Lemma 3.6]. Thus $A/I \subseteq A'/I$ is isomorphic to a direct product of finite extensions $K_i \subseteq \prod_{j=1}^{k_i} L_{ij}$, $1 \leq i \leq l$, where $V_A(I) = \{M_1, \dots, M_l\}$, $K_i = A/M_i$, $i = 1, \dots, l$ and $\{L_{i1}, \dots, L_{ik_i}\}$=$\{A'/N | \ N \in V_{A'}(I), N \cap A = M_i\}$. Then the fragile extension $B/IB \subseteq B'/IB$ is isomorphic to the direct product of extensions $K_i(X_1, \dots, X_n) \subseteq \prod_{j=1}^{k_i} L_{ij}(X_1, \dots, X_n)$, so all these extensions are fragile, cf. Lemma 2.1. Let $m \geq 0$ and set $C = A[X_1, \dots, X_m]$, $C' = A'[X_1, \dots, X_m]$. By Lemma 2.2, all extensions $K_i[X_1, \dots, X_m] \subseteq \prod_{j=1}^{k_i} L_{ij}[X_1, \dots, X_m]$ are fragile, hence so is their product which is isomorphic to $C/IC \subseteq C'/IC$. By [2, Theorem 3.5], C is perinormal. For the "moreover" part, if (d) holds and A/M is Hilbertian for all $M \in V_A(A : A')$, we can repeat the preceding proof to get that (a) holds. □

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