UNITS OF THE GROUP ALGEBRA OF THE GROUP $C_n \times D_6$
OVER ANY FINITE FIELD OF CHARACTERISTIC 3

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Dedicated to the memory of Professor John Clark

Abstract. In this paper, we establish the structure of the unit group of the group algebra $F_3(C_n \times D_6)$ for $n \geq 1$. 

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1. Introduction

Let $KG$ denote the group algebra of the group $G$ over the field $K$. Let $U(KG)$ be the set of invertible elements of $KG$. The homomorphism $\varepsilon : KG \rightarrow K$ given by $\varepsilon \left( \sum_{g \in G} a_g g \right) = \sum_{g \in G} a_g$ is called the augmentation mapping of $KG$. It is a well known fact that $U(KG) \cong U(K) \times V(KG)$ where $V(KG) = \{ u \in U(KG) | \varepsilon(u) = 1 \}$.

Let $G$ be a finite $p$-group and $K$ a field of characteristic $p$, it is well known that $|V(KG)| = |K|^{|G|^{-1}}$. Sandling in [8], provides a basis for $V(F_pG)$ where $G$ is an abelian $p$-group and $F_p$ is the Galois field of $p$ elements. In [10], it is shown that $Z(V_1)$ and $V_1/Z(V_1)$ are elementary abelian 3-groups where $V_1 = 1 + J(F_3D_6)$, $J(F_3D_6)$ is the Jacobson radical of $F_3D_6$ and $Z(V_1)$ is the center of $V_1$. The structure of $U(F_3D_6)$ was determined in terms of split extensions of elementary abelian groups in [4]. The structure of $FA_4$ and $FS_4$ were established in [7,9] where $F$ is any finite field, $A_4$ is the alternating group of degree 4 and $S_4$ is the symmetric group of degree 4. Additionally, the structure of $U(F_3(C_3 \times D_6))$ and $U(F_3D_{12})$ was established in [5,6] respectively. Consult [1] for an overview of modular group algebras.
The map $*: KG \rightarrow KG$ defined by $(\sum_{g \in G} a_g) * = \sum_{g \in G} a_g g^{-1}$ is an antimorphism of $KG$ of order 2. An element $v$ of $V(KG)$ satisfying $v^{-1} = v^*$ is called unitary. We denote by $V_u(KG)$ the subgroup of $V(KG)$ formed by the unitary elements of $KG$. In [3] a basis for $V_u(KG)$ is constructed for any field of characteristic $p > 2$ and any finite abelian $p$-group. Additionally the order of $V_u(F_{2^k}G)$ is determined for special cases of $G$ in [2]. Let $\hat{g} = \sum_{h \in (g)} h \in RG$. Our main results are:

**Theorem 1.1.**

$$\mathcal{U}(F_{3^l}(C_n \times D_6)) \cong (C_3^{3n+1} \times C_3^{2n+1}) \times \mathcal{U}(F_{3^l}(C_n \times C_2)).$$

**Corollary 1.2.**

$$\mathcal{U}(F_{3^l}(C_n \times D_6)) \cong \begin{cases} (C_3^{3n+1} \times C_3^{2n+1}) \times (C_3^{2} \times C_3^{3}\times \cdots \times C_3^{3m}(V) \times C_3^2) & \text{if } n \mid (3^l - 1) \\ (C_3^{3n+1} \times C_3^{2n+1}) \times (C_3^{2} \times C_3^{3}\times \cdots \times C_3^{3m+1}(V) \times C_3^2) & \text{if } n = 3^m \end{cases}$$

where $f_i(V) = t((|C_{3m}^{3^{i-1}}| - 2|C_{3m}^{3^i}| + |C_{3m}^{3^i+1}|)$.

2. The structure of $\mathcal{U}(F_{3^l}(C_n \times D_6))$

Let $G = C_n \times D_6 = \langle x, y, z \mid x^3 = y^2 = z^n = 1, xy = x^{-1}, xz = zx, yz = zy \rangle$ where $n \geq 1$. The natural group homomorphism $G \rightarrow G/\langle x \rangle$ extends linearly to the algebra homomorphism $\theta : F_{3^l}(C_n \times D_6) \rightarrow F_{3^l}(C_n \times C_2)$ where

$$\sum_{i=1}^{3} x^{i-1}(\alpha_i + \alpha_{i+3}z + \cdots + \alpha_{i+3n}z^{n-1} + \alpha_{i+3n+3}y + \alpha_{i+3n+6}yz + \cdots + \alpha_{i+6n}yz^{n-1}) \mapsto$$

$$\sum_{i=1}^{3} (\alpha_i + \alpha_{i+3}b + \cdots + \alpha_{i+3n}b^{n-1} + \alpha_{i+3n+3}a + \alpha_{i+3n+6}ab + \cdots + \alpha_{i+6n}ab^{n-1})$$

and $C_n \times C_2 = \langle a, b \mid a^2 = b^n = 1, ab = ba \rangle$. If we restrict $\theta$ to $\mathcal{U}(F_{3^l}(C_n \times D_6))$, we can construct the group epimorphism $\theta' : \mathcal{U}(F_{3^l}(C_n \times D_6)) \rightarrow \mathcal{U}(F_{3^l}(C_n \times C_2))$. Consider the group homomorphism $\psi : \mathcal{U}(F_{3^l}(C_n \times C_2)) \rightarrow \mathcal{U}(F_{3^l}(C_n \times C_2))$ by

$$\gamma_1 + \gamma_2 + \cdots + \gamma_n b^{n-1} + \delta_1 a + \delta_2 ab + \cdots + \delta_n ab^{n-1} \mapsto$$

$$\gamma_1 + \gamma_2 z + \cdots + \gamma_n z^{n-1} + \delta_1 y + \delta_2 yz + \cdots + \delta_n yz^{n-1}$$

where $\gamma_i, \delta_j \in F_{3^l}$. Clearly $\theta' \circ \psi$ is the identity map of $\mathcal{U}(F_{3^l}(C_n \times C_2))$. Therefore $\mathcal{U}(F_{3^l}(C_n \times D_6))$ is a split extension of $\mathcal{U}(F_{3^l}(C_n \times C_2))$ by $ker(\theta')$ and $\mathcal{U}(F_{3^l}(C_n \times D_6)) \cong H \times \mathcal{U}(F_{3^l}(C_n \times C_2))$ where $H \cong ker(\theta')$. Now, $\theta : F_{3^l}(C_n \times D_6) \rightarrow F_{3^l}(C_n \times D_6)/(\langle x \rangle) \cong F_{3^l}(C_n \times D_6)/J(\langle x \rangle)$ where $J(\langle x \rangle)$ is the ideal of $F_{3^l}(C_n \times D_6)$ generated by all $x - 1$ where $x \in \langle x \rangle$. Additionally, $\theta' : \mathcal{U}(F_{3^l}(C_n \times D_6)) \rightarrow$
\[ \mathcal{U}(\mathbb{F}_3((C_n \times D_6)/(x))) \cong \mathcal{U}(\mathbb{F}_3(C_n \times D_6))/1 + \mathcal{J}(\langle x \rangle). \]

As the characteristic of \( \mathbb{F}_3 \) is 3 and \( x \) is of order 3, \( \mathcal{J}(\langle x \rangle) \) is nilpotent of index 3. Therefore \( H \) has exponent 3.

**Lemma 2.1.** \( C_H(x) \cong C_3^{3nt} \) where \( C_H(x) = \{ h \in H \mid xh = hx \} \).

**Proof.** Let \( h = 1 + \sum_{j=1}^{n} \mathcal{A}_j + \sum_{k=1}^{n} \mathcal{B}_k y \in H \) where

\[
\mathcal{A}_j = \sum_{i=1}^{2} \alpha_{i+2(j-1)z}z^{-1}(x^i - 1) \quad \text{and} \quad \mathcal{B}_k = \sum_{i=1}^{2} \alpha_{i+2(k+n-1)x}x^{k-1}(x^i - 1)
\]

and \( \alpha_j \in \mathbb{F}_3 \). Now

\[
xh - hx = x \left( 1 + \sum_{j=1}^{n} \mathcal{A}_j + \sum_{k=1}^{n} \mathcal{B}_k y \right) - \left( 1 + \sum_{j=1}^{n} \mathcal{A}_j + \sum_{k=1}^{n} \mathcal{B}_k y \right) x
\]

\[
= x \left( \sum_{k=1}^{n} \mathcal{B}_k y \right) - \left( \sum_{k=1}^{n} \mathcal{B}_k y \right) x.
\]

Now,

\[
x \mathcal{B}_k y - \mathcal{B}_k y x = z^{k-1}[(\alpha_{2k+2n-1}(x^2 - x) + \alpha_{2k+2n}(1 - x)) - (\alpha_{2k+2n-1}(1 - x^2) + \alpha_{2k+2n}(x - x^2))]y
\]

\[
= 3 \hat{x} z^{k-1}(\alpha_{2k+2n} - \alpha_{2k+2n-1}).
\]

Therefore, every element of \( C_H(x) \) takes the form

\[
1 + \sum_{j=1}^{n} \mathcal{A}_j + \sum_{i=1}^{n} \alpha_{i+2n} \hat{x} y z^{l-1}
\]

where \( \mathcal{A}_j = \sum_{i=1}^{2} \alpha_{i+2(j-1)z}z^{-1}(x^i - 1) \) and \( \alpha_i \in \mathbb{F}_3 \). Clearly \( (\hat{x})^2 = 3 \hat{x} = 0 \) and \( \hat{x} \mathcal{A}_j = \mathcal{A}_j \hat{x} \). Therefore \( C_H(x) \) is an abelian group of order \( 3^{2nt} \cdot 3^{nt} = 3^{3nt} \). \( \square \)

Next, consider a subset \( S \) of \( H \) where the elements of \( S \) take the form:

\[
1 + \sum_{j=1}^{n} \mathcal{R}_j
\]

where \( \mathcal{R}_j = \sum_{i=1}^{2} i r_j x^i (1 + y) z^{j-1} \) and \( r_i \in \mathbb{F}_3 \).

**Lemma 2.2.** \( S \cong C_3^{nt} \).
Proof. Let $s_1 = 1 + \sum_{j=1}^{n} \mathcal{R}_j \in S$ and $s_2 = 1 + \sum_{j=1}^{n} \mathcal{X}_j \in S$ where

$$\mathcal{R}_j = \sum_{i=1}^{2} ir_j x^i(1 + y)z^{j-1}, \quad \mathcal{X}_j = \sum_{i=1}^{2} it_j x^i(1 + y)z^{j-1}$$

and $r_i, t_j \in \mathbb{F}_3$. Now

$$s_1 s_2 = \left( 1 + \sum_{j=1}^{n} \mathcal{R}_j \right) \left( 1 + \sum_{j=1}^{n} \mathcal{X}_j \right)$$

and

$$\mathcal{R}_j \mathcal{X}_k = \left( \sum_{i=1}^{2} ir_j x^i(1 + y)z^{j-1} \right) \left( \sum_{i=1}^{2} it_k x^i(1 + y)z^{k-1} \right)$$

$= (r_j x + r_j xy + 2r_j x^2 + 2r_j x^2 y)(t_k x + t_k xy + 2t_k x^2 + 2t_k x^2 y)z^{j+k-2}$

$= \sum_{i=1}^{3} (12 - 3i)r_j t_k x^{i-1}(1 + y)z^{j+k-2}$

$= 0.$

Clearly $s_1 s_2 \in S$ and $S$ is abelian, therefore $S \cong C_3^{nt}$. $\square$

Theorem 2.3.

$$\mathcal{U}(\mathbb{F}_3(C_n \times D_6)) \cong (C_3^{nt} \times C_3^{nt}) \times \mathcal{U}(\mathbb{F}_3(C_n \times C_2)).$$

Proof. Let $c = 1 + \sum_{j=1}^{n} \mathfrak{A}_j + \sum_{l=1}^{n} \alpha_{l+2n} \hat{x}yz^{l-1} \in C_H(x)$ and $s = 1 + \sum_{j=1}^{n} \mathcal{R}_j \in S$

where $\mathfrak{A}_j = \sum_{i=1}^{2} \alpha_{i+2(j-1)} z^{j-1}(x^i - 1)$, $\mathcal{R}_j = \sum_{i=1}^{2} ir_j x^i(1 + y)z^{j-1}$ and $\alpha_i, r_j \in \mathbb{F}_3$.

Now

$$c^s = s^2 cs$$

$$= \left( 1 + \sum_{j=1}^{n} \mathcal{R}_j \right) \left( 1 + \sum_{j=1}^{n} \mathfrak{A}_j + \sum_{l=1}^{n} \alpha_{l+2n} \hat{x}yz^{l-1} \right) \left( 1 + \sum_{j=1}^{n} \mathcal{R}_j \right)$$

$$= \left( 1 + \sum_{j=1}^{n} \mathcal{R}_j \right) \left( 1 + \sum_{j=1}^{n} \mathfrak{A}_j + \sum_{l=1}^{n} \alpha_{l+2n} \hat{x}yz^{l-1} \right) \left( 1 + \sum_{j=1}^{n} \mathcal{R}_j \right).$$
Additionally, \( R_j^2 = 0 \) and \( xR_j = 3xR_j(1+y)z^{j-1} = 0 = R_jx \), therefore

\[
e^a = 1 + \sum_{j=1}^{n} R_j + \sum_{l=1}^{n} \alpha_{l+2n} \hat{x} y z^{l-1} + 2 \left( \sum_{j=1}^{n} R_j \right) \left( \sum_{j=1}^{n} \hat{A}_j \right) + \left( \sum_{j=1}^{n} \hat{A}_j \right) \left( \sum_{j=1}^{n} R_j \right).
\]

Now, \( R_j \hat{A}_k = r_j(\alpha_{2k} - \alpha_{2k-1}) \hat{x}(1-y)z^{j+k-2}, \hat{A}_k R_j = r_j(\alpha_{2k} - \alpha_{2k-1}) \hat{x}(1+y)z^{j+k-2} \) and

\[
2R_j \hat{A}_k + \hat{A}_k R_j = r_j(\alpha_{2k} - \alpha_{2k-1}) \hat{x}[2(1-y) + (1+y)]z^{j+k-2}
\]

\[
= r_j(\alpha_{2k-1} - \alpha_{2k}) \hat{x} y z^{j+k-2}.
\]

Additionally, \( R_j \hat{A}_k R_j = 0 \) since \( \hat{x} R_j = 0 \). Therefore \( e^a \in C_H(x) \) and consequently \( C_H(x) \) is a normal subgroup of \( H \). Note that \( |H| = 3^{4nt} \) and that \( C_H(x) \cap S = \{1\} \).

By the Second Isomorphism Theorem, \( H = C_H(x).S \). Thus, \( H \cong C_H(x) \times S \cong C_{3^{nt}} \times C_{3^t} \).

\[\square\]

Corollary 2.4.

\[
U(H^{3t} (C_n \times D_6)) \cong \begin{cases} (C_{3^t} \cong C_{3^t}) \times C_{3^t} \times \cdots \times C_{3^t} & \text{if } n|(3^t - 1) \\ (C_{3^t} \cong C_{3^t}) \times C_{3^t} \times \cdots \times C_{3^t} \times C_{3^t} \times C_{3^t-1} & \text{if } n = 3^m \end{cases}
\]

where \( f_i(V) = t((C_{3^m}^{3^{i-1}}) - 2|C_{3^m}^{3^i}| + |C_{3^m}^{3^{i+1}}|) \).

**Proof.** It is well known that \( F_{3^t}(C_2 \times C_n) \cong (F_{3^t}C_2)C_n \cong (F_{3^t} \oplus F_{3^t})C_n \cong F_{3^t}C_n \oplus F_{3^t}C_n \). It is well known that if \( n|(3^t - 1) \), then \( F_{3^t}C_n \cong \oplus_{i=1}^{n-1} F_{3^t} \). Therefore

\[
U(F_{3^t}(C_2 \times C_n)) \cong C_{3^t-1} \text{ when } n|(3^t - 1).
\]

When \( n = 3^m \), the number of cyclic groups \( f_i(V) \) of order \( 3^i \) in the direct product of \( V(F_{3^t}G) \) is \( f_i(V) = t((C_{3^m}^{3^{i-1}}) - 2|C_{3^m}^{3^i}| + |C_{3^m}^{3^{i+1}}|) \) \( (8) \).

\[\square\]

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**References**


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