A NOTE ON THE DEPTH OF A SOURCE ALGEBRA OVER ITS DEFECT GROUP

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Abstract. By results of Boltje and Külshammer, if a source algebra $A$ of a principal $p$-block of a finite group with a defect group $P$ with inertial quotient $E$ is a depth two extension of the group algebra of $P$, then $A$ is isomorphic to a twisted group algebra of the group $P \rtimes E$. We show in this note that this is true for arbitrary blocks. We observe further that the results of Boltje and Külshammer imply that $A$ is a depth two extension of its hyperfocal subalgebra, with a criterion for when this is a depth one extension. By a result of Watanabe, this criterion is satisfied if the defect groups are abelian.

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Let $p$ be a prime and $\mathcal{O}$ a complete local principal ideal domain with an algebraically closed residue field $k$ of characteristic $p$, allowing the case $\mathcal{O} = k$. We will make without further comment use of the fact that by [9, II, Prop. 8], the canonical group homomorphism $\mathcal{O}^\times \to k^\times$ splits canonically, and hence group cohomology with coefficients in $k^\times$ can be viewed as cohomology with coefficients in $\mathcal{O}^\times$. Following terminology in [4], a ring extension $B \to A$ is called of depth one if $A$ is isomorphic, as a $B$-$B$-bimodule, to a direct summand of $B^n$ for some positive integer $n$, and a ring extension $B \to A$ is called of depth two if $A \otimes_B A$ is isomorphic, as an $A$-$B$-bimodule, to a direct summand of $A^n$, for some positive integer $n$. Tensoring by $A \otimes_B -$ shows that a ring extension of depth one is also an extension of depth two.

Let $A$ be a source algebra of a block algebra over $\mathcal{O}$ of a finite group, with a defect group $P$. Boltje and Külshammer showed in [2, 2.4] that if $A$ is isomorphic to a twisted group algebra of the form $\mathcal{O}_\alpha(P \rtimes E)$ for some $p'$-subgroup $E$ of $\text{Aut}(P)$ and some $\alpha \in H^2(E; k^\times)$, inflated trivially to $P \rtimes E$, then the canonical map $\mathcal{O}P \to A$ is an extension of depth two. Moreover, they showed that the converse holds for principal blocks. The following result shows that this converse holds for arbitrary
blocks. See for instance [10, §11, §38] and [5, §6, §7] for background material on the Brauer homomorphism $\text{Br}_P$ and fusion in source algebras.

**Theorem 1.** Let $G$ be a finite group, $b$ a block of $OG$, $P$ a defect group of $b$ and $A = iO Gi$ a source algebra of $b$, where $i$ is a primitive idempotent in the $P$-fixed point algebra $(OGb)^P$ such that $\text{Br}_P(i) \neq 0$. The following are equivalent:

(i) The ring extension $O P \to A$ induced by the canonical map $P \to A$ is of depth two.

(ii) The ring extension $kP \to k \otimes_O A$ induced by the canonical map $P \to A$ is of depth two.

(iii) There is an isomorphism of interior $P$-algebras $A \cong O_\alpha(P \rtimes E)$ for some $p'$-subgroup $E$ of $\text{Aut}(P)$ and some $\alpha \in H^2(E; k^\times)$ inflated trivially to $P \rtimes E$.

(iv) There is an isomorphism of interior $P$-algebras $k \otimes_O A \cong k_\alpha(P \rtimes E)$ for some $p'$-subgroup $E$ of $\text{Aut}(P)$ and some $\alpha \in H^2(E; k^\times)$ inflated trivially to $P \rtimes E$.

**Proof.** The equivalence of (iii) and (iv) is an immediate consequence of results of Puig (either apply [7, 14.6] over both $O$ and $k$, or use the lifting property [6, 7.8] for source algebras). Statement (iv) implies (i) and (ii) by Boltje and Kulshammer [2, 2.4]. The implication (i) ⇒ (ii) is trivial. It suffices to show that (ii) implies (iv). We may therefore assume that $O = k$. Suppose that (ii) holds but that (iv) does not hold. As an $A$-$kP$-bimodule, $A$ is indecomposable since $1_A = i$ is primitive in $A^P$. Thus, if (ii) holds, then the Krull-Schmidt theorem implies that any indecomposable direct summand of $A \otimes_{k P} A$ as an $A$-$kP$-bimodule is isomorphic to $A$ as an $A$-$kP$-bimodule. Now if (iv) does not hold, then by [7, 14.6], there is a proper subgroup $Q$ of $P$ and an injective group homomorphism $\varphi$ from $Q$ to $P$ such that the indecomposable $kP$-$kP$-bimodule $kP \otimes_{kQ} (\varphi kP)$ is isomorphic to a direct summand of $A$ as a $kP$-$kP$-bimodule. Thus $A \otimes_{kQ} (\varphi kP)$ is isomorphic to a direct summand of $A \otimes_{kP} A$ as an $A$-$kP$-bimodule, and hence so is $Aj \otimes_{kQ} (\varphi kP)$, where $j$ is a primitive idempotent in $A^Q$. Since $Aj$ is indecomposable as an $A$-$kQ$-bimodule, so is the $k(G \times Q)$-module $kGj$. Green’s indecomposability theorem implies that the $k(G \times P)$-module $kGj \otimes_{kQ} (\varphi kP)$ is indecomposable. Using that multiplication by $i$ yields a Morita equivalence between $kGb$ and $A$ it follows that the $A$-$kP$-bimodule $Aj \otimes_{kQ} (\varphi kP)$ is also indecomposable, hence isomorphic to $A$ as an $A$-$kP$-bimodule, by the above. Since $\text{Br}_P(i) \neq 0$ this is, however, only possible if $Q = P$, a contradiction. □
For the sake of completeness, we mention that the depth of an extension $D \to A$, where $D$ is a hyperfocal subalgebra (cf. [8]) in a source algebra $A$ of a block of a finite group, can be determined essentially as an application of the methods from [1] and [2]. The first statement of the following proposition is a special case of [1, 1.5].

**Proposition 2.** Let $A$ be a source algebra of a block of a finite group algebra over $O$ with defect group $P$, and let $D$ be a hyperfocal subalgebra of $A$. The following hold.

(i) The extension $D \to A$ is of depth two.

(ii) The extension $D \to A$ is of depth one if and only if $P$ acts by inner automorphisms on $D$.

**Proof.** As mentioned above, statement (i) is a special case of [1, 1.5], as $A$ is $P/Q$-graded, with $D$ as 1-component. Since the argument is short and some parts of the notation will be useful in the proof of (ii), we sketch this briefly. We identify $P$ with its canonical image in $A$. The following definitions and facts on the hyperfocal subalgebra $D$ of $A$ are from [8]. The subalgebra $D$ is $P$-stable, and the group $Q = P \cap D^\times$ is the $F$-hyperfocal subgroup of $P$, where $F$ is the fusion system of $A$ on $P$. An immediate consequence of these properties is that $D$ is indecomposable as an $O$-algebra. Indeed, we have $D^P \subseteq A^P$, which is local, and hence $P$ permutes the blocks of $D$ transitively. But we also have $\text{Br}_P(1_A) \neq 0$, and hence $D$ has a unique block. By [8, Theorem 1.8] we have $A = \bigoplus_{u \in [P/Q]} Du$, where $[P/Q]$ is a set of representatives in $P$ of $P/Q$. Since $D$ is $P$-stable, this is a decomposition of $A$ as a $D$-$D$-bimodule. Thus $A \otimes_D A = \bigoplus_{u \in [P/Q]} A \otimes_D Du$ is a decomposition of $A \otimes_D A$ as an $A$-$D$-bimodule. For $u \in P$, a trivial verification shows that the $A$-$D$-bimodule $A \otimes_D Du$ is isomorphic to $A$ via the map sending $a \otimes du$ to $adu$, where $a \in A$ and $d \in D$. Thus any indecomposable direct summand of the $A$-$D$-bimodule $A \otimes_D A$ isomorphic to $A$ via the map sending $a \otimes du$ to $adu$, where $a \in A$ and $d \in D$. This proves (i).

The summands $Du$ in the $D$-$D$-bimodule decomposition $A = \bigoplus_{u \in [P/Q]} Du$ are all indecomposable as $D$-$D$-bimodules. Indeed, $D$ is indecomposable by the above, and $Du$ is isomorphic to the image of $D$ under the Morita equivalence on $\text{mod}(D \otimes_O D^{pp})$ obtained from twisting the right $D$-module structure by the automorphism induced by conjugation with $u$. Thus the extension $D \to A$ is of depth one if and only if $Du \cong D$ as $D$-$D$-bimodules, for all $u \in [P/Q]$, hence for all $u \in P$. By standard facts on automorphisms (cf. [3, §55A]) this is equivalent to the condition that $u$ induces an inner automorphism of $D$, for all $u \in P$. This proves (ii).  

In conjunction with a result of Watanabe [11], this yields the following consequence.

**Corollary 3.** With the notation of Proposition 2, if $P$ is abelian, then the extension $D \to A$ is of depth one.

**Proof.** By [11, Theorem 2], if $P$ is abelian, then $P$ acts as inner automorphisms on $D$. Thus the result follows from Proposition 2 (ii). □

**Remark 4.** What we have called depth two in this note is called right D2 in [4, 3.1], with left D2 being the obvious analogue, requiring $A \otimes_B A$ to be a direct summand, as a $B$-$A$-bimodule, of $A^n$ for some positive integer $n$. It is easy to see directly that left and right D2 are equivalent conditions for the extensions $OP \to A$ and $D \to A$ considered in the results above; this follows also from a more general result in [4, 6.4]. See [2, §2.3] for a related discussion.

**References**


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