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ON ALMOST SUBNORMAL SUBGROUPS AND MAXIMAL SUBGROUPS IN SKEW LINEAR GROUPS

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Abstract. In this paper, we study almost subnormal subgroups and maximal subgroups in skew linear groups satisfying a generalized Laurent polynomial identity.

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1. Introduction and preliminaries

Let D be a division ring with center F. Recently, some skew linear groups satisfying an identity was investigated [4,10,11,12,13]. For example, in [4] it was shown that every subnormal subgroup N of $\mathrm{GL}_n(D)$ satisfying a generalized group identity over $\mathrm{GL}_n(D)$ is central, i.e. $N \subseteq F$, provided F is infinite. Later, in [12] this result was extended for almost subnormal subgroups of $\mathrm{GL}_n(D)$. Additionally, L. Makar-Limanov proved that if D is infinite dimensional over its infinite center F, then any subnormal subgroup of D^* satisfying a generalized Laurent polynomial identity over D is central [11].

Our first aim in this paper is to generalize the above results for almost subnormal subgroups of $\mathrm{GL}_n(D)$ satisfying a generalized Laurent polynomial identity in the case when D is algebraic over its uncountable center F and $[D : F] = \infty$. In fact, we prove that if N is an almost subnormal subgroup of $\mathrm{GL}_n(D)$ satisfying a generalized Laurent polynomial identity, then N is central (Theorem 2.5). Secondly, we focus on maximal subgroups of $\mathrm{GL}_n(D)$ satisfying a Laurent polynomial identity by proving that if M is a maximal subgroup of $\mathrm{GL}_n(D)$ such that D is infinite dimensional over its infinite center F and $F[M]$ is algebraic over F, then M is absolutely irreducible (Theorem 2.6). This result generalizes partially [10, Theorem 4.1]. In the case when $n = 1$, we investigate maximal subgroups of an almost subnormal subgroup of D^*. In [13], maximal subgroups of subnormal subgroups of D^* was studied and it was shown that every nilpotent maximal subgroup of a subnormal subgroup of

D^* is abelian [13, Theorem 2.3]. We extend this result for any maximal subgroup M of a non-central almost subnormal subgroup of D^* in the case when D is infinite dimensional over its infinite center F and $C_D(M) \setminus F$ contains an algebraic element over F. Namely, we show that if M satisfies a Laurent polynomial identity, then M is abelian (Theorem 2.10).

Now, we recall some notation we use in this paper. Let D be a division ring with center F and G be the free group generated by m non-commuting indeterminates $x_1, x_2, \ldots, x_m$. Denote by $\mathrm{M}_n(D) *_F FG$ the free product of the matrix ring $\mathrm{M}_n(D)$ and the group algebra FG over F. An element $f(x_1, x_2, \cdots, x_m) \in \mathrm{M}_n(D) *_F FG$ is called a *generalized Laurent polynomial* over $\mathrm{M}_n(D)$ (see [5] for the definition of generalized Laurent polynomials over an arbitary algebra). In particular, if $f \in FG$, then f is called a *Laurent polynomial* over F.

Assume that $f(x_1, x_2, \ldots, x_m)$ is non-zero and N is a subset of the general skew linear group $\mathrm{GL}_n(D)$. If $f(c_1, c_2, \ldots, c_m) = 0$ for every $(c_1, c_2, \cdots, c_m) \in N^m$, then we say that N *satisfies the generalized Laurent polynomial identity* (briefly, GLPI) $f = 0$. In this case, $f = 0$ is called a *generalized Laurent polynomial identity* of N. Additionally, if f is a Laurent polynomial, then we simply say that $f = 0$ is a *Laurent polynomial identity* of N or N *satisfies the Laurent polynomial identity* (shortly, LPI) $f = 0$.

Let K be a group. Following Hartley [8], H is an *almost subnormal subgroup* of K if there is a family of subgroups $H = H_r \leq H_{r-1} \leq \cdots \leq H_1 = K$ of K such that for each $1 < i \leq r$, either H_i is normal in H_{i-1} or H_i has finite index in H_{i-1}. We call such a series of subgroups an *almost normal series* of H in K. It was noted in [12] that there is a division ring whose multiplicative group contains some almost subnormal subgroup that is not subnormal.

2. Results

Let us denote by $\mathrm{M}_n(D)[t_1, t_2, \ldots, t_m]$ the polynomial ring in the determinates $t_1, t_2, \ldots, t_m$ over $\mathrm{M}_n(D)$. The following lemma can be obtained by applying the Vandermonde argument [14, Proposition 2.3.26 and 2.3.27].

Lemma 2.1. *Let $f(t_1, t_2, \ldots, t_m) \in \mathrm{M}_n(D)[t_1, t_2, \ldots, t_m]$. If there exist infinitely many elements $\alpha_1, \alpha_2, \ldots, \alpha_m$ in the center F of D such that $f(\alpha_1, \alpha_2, \ldots, \alpha_m) = 0$, then f is identically zero.*

Lemma 2.2. *Let D be a division ring with infinite center F and $\mathrm{M}_n(D)$ be the matrix ring over D. If $a \in \mathrm{M}_n(D)$ is algebraic over F, then there exist infinitely many elements $\alpha \in F$ such that $1 + \alpha a \in \mathrm{GL}_n(D)$ and*

$$(1+\alpha a)^{-1} = -\frac{(1+\alpha a)^{k-1} + v_{k-1}(\alpha)(1+\alpha a)^{k-2} + \cdots + v_1(\alpha)}{v_0(\alpha)}, \tag{1}$$

where $v_i(t) \in F[t]$.

Proof. Since $a \in \mathrm{M}_n(D)$ is algebraic over F, there exists a polynomial

$$u(x) = x^k + b_{k-1}x^{k-1} + \cdots + b_1 x + b_0 \in F[x]$$

such that $u(a) = 0$. Let t be a central indeterminate. Put $h_t(x) = u\left(\dfrac{x-1}{t}\right)$. Then, $h_t(x) = \dfrac{1}{t^k}(x-1)^k + \dfrac{b_{k-1}}{t^{k-1}}(x-1)^{k-1} + \cdots + \dfrac{b_1}{t}(x-1) + b_0$. Hence

$$\begin{aligned} t^k h_t(x) &= (x-1)^k + tb_{k-1}(x-1)^{k-1} + \cdots + t^{k-1}b_1(x-1) + t^k b_0 \\ &= x^k + v_{k-1}(t)x^{k-1} + \cdots + v_1(t)x + v_0(t), \end{aligned}$$

where $v_i(t) \in F[t]$. It is clear that $h_t(1+ta) = 0$, so

$$(1+ta)^k + v_{k-1}(t)(1+ta)^{k-1} + \cdots + v_1(t)(1+ta) + v_0(t) = 0.$$

Since $v_0(t)$ has finitely many roots in F, there exist infinitely many elements $\alpha \in F$ such that

$$(1+\alpha a)^{-1} = -\frac{(1+\alpha a)^{k-1} + v_{k-1}(\alpha)(1+\alpha a)^{k-2} + \cdots + v_1(\alpha)}{v_0(\alpha)}.$$

This completes the proof. $\square$

Let $F\langle y_1, y_2, \ldots, y_m\rangle$ be the free algebra in $y_1, y_2, \ldots, y_m$ over F and

$$\mathrm{M}_n(D)\langle y_1, y_2, \ldots, y_m\rangle = \mathrm{M}_n(D) *_F F\langle y_1, y_2, \ldots, y_m\rangle$$

be the free product of $\mathrm{M}_n(D)$ and $F\langle y_1, y_2, \ldots, y_m\rangle$ over F. Denote by

$$\mathrm{M}_n(D)\langle y_1, y_2, \ldots, y_m\rangle[[t_1, t_2, \ldots, t_m]]$$

the ring of formal power series in the indeterminates $t_1, t_2, \ldots, t_m$ with coefficients in $\mathrm{M}_n(D)\langle y_1, y_2, \ldots, y_m\rangle$.

Lemma 2.3. *If* $f(x_1, x_2, \ldots, x_m)$ *is a non-zero element in* $\mathrm{M}_n(D) *_F FG$*, then*

$$f(1+t_1y_1, 1+t_2y_2, \ldots, 1+t_my_m)$$

is a non-zero element in $\mathrm{M}_n(D)\langle y_1, y_2, \ldots, y_m\rangle[[t_1, t_2, \ldots, t_m]]$.

Proof. If

$$f(1+t_1y_1, 1+t_2y_2, \ldots, 1+t_my_m) \equiv 0$$

in $\mathrm{M}_n(D)\langle y_1, y_2, \ldots, y_m\rangle[[t_1, t_2, \ldots, t_m]]$, then

$$f\left(1+t_1\frac{x_1-1}{t_1}, 1+t_2\frac{x_2-1}{t_2}, \ldots, 1+t_m\frac{x_m-1}{t_m}\right) = 0.$$

This means that $f(x_1, x_2, \ldots, x_m) = 0$, a contradiction. The proof is complete. □

Recall that a *generalized polynomial identity* is a generalized Laurent polynomial identity in which all powers of indeterminates are non-negative.

Lemma 2.4. *If* $\mathrm{M}_n(D)$ *satisfies a generalized polynomial identity, then* D *is centrally finite, i.e.* D *is a finite dimensional vector space over* F.

Proof. This lemma is followed from [2, Theorem 6.1.9]. □

Now, we are ready to prove the main result of this work.

Theorem 2.5. *Let* D *be an algebraic division ring with uncountable center* F *and* $[D:F]=\infty$. *If* N *is an almost subnormal subgroup of* $\mathrm{GL}_n(D)$ *satisfying a* GLPI $f(x_1, x_2, \ldots, x_m) = 0$, *then* N *is central.*

Proof. We first claim that if $\mathrm{GL}_n(D)$ satisfies a GLPI $g(x_1, x_2, \ldots, x_m) = 0$ then D is centrally finite. In fact, by Lemma 2.3,

$$g(1+t_1y_1, 1+t_2y_2, \ldots, 1+t_my_m) \neq 0.$$

Moreover,

$$g(1+t_1y_1, 1+t_2y_2, \ldots, 1+t_my_m) = \sum_{j_1, j_2, \ldots, j_m \geq 0} t_1^{j_1}t_2^{j_2}\cdots t_m^{j_m} p_{j_1j_2\cdots j_m}(y_1, y_2, \ldots, y_m),$$

where $p_{j_1j_2\cdots j_m}$ are generalized polynomials over $\mathrm{M}_n(D)$ and

$$p_{00\ldots0} = g(1, 1, \ldots, 1) = 0.$$

Thus, there exist $j_1^*, j_2^*, \ldots, j_m^* > 0$ such that $p_{j_1^*j_2^*\cdots j_m^*}(y_1, y_2, \ldots, y_m) \neq 0$. Now, since D is algebraic over its uncountable center F, by [1, Theorem 2.10], $\mathrm{M}_n(D)$ is algebraic over F. Let $c_1, c_2, \ldots c_m$ be arbitary elements in $\mathrm{M}_n(D)$, by Lemma 2.2, we have $1+\alpha_1c_1, 1+\alpha_2c_2, \ldots, 1+\alpha_mc_m \in \mathrm{GL}_n(D)$, for infinitely many elements $\alpha_1, \alpha_2, \ldots, \alpha_m$ in F. Hence

$$g(1+\alpha_1c_1, 1+\alpha_2c_2, \ldots, 1+\alpha_mc_m) = 0,$$

for infinitely many elements $\alpha_1, \alpha_2, \dots, \alpha_m$ in F. Due to Equation (1) in Lemma 2.2, we can write $g(1+\alpha_1 c_1, 1+\alpha_2 c_2, \dots, 1+\alpha_m c_m) = \dfrac{h(\alpha_1, \alpha_2, \dots, \alpha_m)}{k(\alpha_1, \alpha_2, \dots, \alpha_m)}$, where

$$h(t_1, t_2, \dots, t_m) \in \mathrm{M}_n(D)[t_1, t_2, \dots, t_m], k(t_1, t_2, \dots, t_m) \in F[t_1, t_2, \dots, t_m].$$

Since $g(1+\alpha_1 c_1, 1+\alpha_2 c_2, \dots, 1+\alpha_m c_m) = 0$, for infinitely many elements $\alpha_1, \alpha_2, \dots, \alpha_m$ in F, it follows that $h(\alpha_1, \alpha_2, \dots, \alpha_m) = 0$ for infinitely many elements α_1, α_2, $\dots$, α_m in F. By Lemma 2.1, $h(t_1, t_2, \dots, t_m)$ is identically zero, so is $g(1+t_1c_1, 1+t_2c_2, \dots, 1+t_mc_m)$. Observe that

$$g(1+t_1c_1, 1+t_2c_2, \dots, 1+t_mc_m) = \sum_{j_1, j_2, \dots, j_m \geq 0} t_1^{j_1} t_2^{j_2} \cdots t_m^{j_m} p_{j_1 j_2 \dots j_m}(c_1, c_2, \dots, c_m),$$

so $p_{j_1^* j_2^* \dots j_m^*}(c_1, c_2, \dots, c_m) = 0$. Thus, $p_{j_1^* j_2^* \dots j_m^*}(y_1, y_2, \dots, y_m)$ is a generalized polynomial identity of $\mathrm{M}_n(D)$. By Lemma 2.4, D is centrally finite. The claim is proved.

Now suppose that N is non-central. We consider the following two cases.

Case 1. In the case when $n \geq 2$, by [12, Theorem 3.3], N is a normal subgroup of $\mathrm{GL}_n(D)$. Fix an element $k \in N \setminus F$, and put

$$g(y_1, \dots, y_m) = f(y_1 k y_1^{-1}, \dots, y_m k y_m^{-1}).$$

Then, since $aka^{-1} \in N$ for any $a \in \mathrm{GL}_n(D)$, $g = 0$ is a GLPI of $\mathrm{GL}_n(D)$. Hence, D is centrally finite, a contradiction.

Case 2. In the case when $n = 1$, suppose $N = N_r \leq N_{r-1} \leq \cdots \leq N_1 = D^*$ is an almost normal series in D^*. We claim that if N satisfies a GLPI, then so does D^*. It suffices to prove that N_{r-1} satisfies a GLPI. In fact, if N_r is normal in N_{r-1}, then by the same argument in Case 1 we get the claim. If $[N_{r-1} : N_r] = \ell < \infty$, then $a_1^{\ell!}, a_2^{\ell!}, \dots, a_m^{\ell!} \in N_r$ for any $a_1, a_2, \dots, a_m \in N_{r-1}$. Hence, $f(a_1^{\ell!}, a_2^{\ell!}, \dots, a_m^{\ell!}) = 0$. Thus $g(x_1, x_2, \dots, x_m) = f(x_1^{\ell!}, x_2^{\ell!}, \dots, x_m^{\ell!})$ is a GLPI of N_{r-1}. The claim is proved. Therefore, D is centrally finite, a contradiction.

Thus, the proof is now complete. $\square$

Next, we prove some results for maximal subgroups in $\mathrm{GL}_n(D)$.

Theorem 2.6. *Let D be a division ring with infinite center F and $[D : F] = \infty$. Assume that M is a maximal subgroup of* $\mathrm{GL}_n(D)$ *such that $F[M]$ is algebraic over F, where $F[M]$ is the F-subalgebra of* $\mathrm{M}_n(D)$ *generated by M over F. If M satisfies an* LPI*, then M is absolutely irreducible.*

Proof. By the maximality of M, either $F[M]^* = M$ or $F[M]^* = \mathrm{GL}_n(D)$. We show that the first case can not occur. Indeed, by hypothesis M satisfies an LPI, so $F[M]^*$ satisfies an LPI. Using the technique in the first part of the proof of

Theorem 2.5, we can prove that $F[M]$ satisfies a polynomial identity. Therefore, by [10, Theorem 3.5], D is centrally finite, a contradiction.

Thus, we conclude $F[M]^* = \mathrm{GL}_n(D)$ and $F[M] = \mathrm{M}_n(D)$. Hence M is absolutely irreducible. $\square$

Corollary 2.7. *Let D be a division ring with infinite center F. Assume that M is a maximal subgroup of $\mathrm{GL}_n(D)$ such that $F[M]$ is algebraic over F. If M satisfies a group identity, then D is centrally finite or M is absolutely irreducible.*

Lemma 2.8. *Let G be a group and N be an almost subnormal subgroup of G. For any subgroup H of G, the subgroup $H \cap N$ is an almost subnormal subgroup of H.*

Proof. The proof is elementary. $\square$

Lemma 2.9. *Let D be a division ring with infinite center F. If D^* contains a non-central almost subnormal subgroup which satisfies an* LPI *$f = 0$, then D is centrally finite.*

Proof. This lemma is from [7, Theorem 1.1]. $\square$

Theorem 2.10. *Let D be a division ring with infinite center F and $[D : F] = \infty$. Let N be a non-central almost subnormal subgroup of D^*. Suppose that M is a maximal subgroup of N such that $\mathrm{C}_D(M) \setminus F$ contains an algebraic element over F. If M satisfies an* LPI*, then M is abelian.*

Proof. This proof is a slight modification of the one of [9, Theorem 2]. Suppose that $\alpha \in \mathrm{C}_D(M) \setminus F$ is algebraic over F. Put $L := F(\alpha)$ and $B := \mathrm{C}_D(L)$. Then, $[L : F] < \infty$. By the Double Centralizer Theorem, B is a division ring with center L. Since $\alpha \in \mathrm{C}_D(M)$, we have $M \leq B^*$. Therefore, $M \leq N \cap B^* \leq N$. By the maximality of M in N, we have $N \cap B^* = N$ or $N \cap B^* = M$. The first case can not occur. To prove this, we claim that $N \not\subseteq B^*$. Suppose that $N \subseteq B^*$. Then, $F(N) \subseteq B$. Since N normalizes $F(N)$, by [3, Theorem 1], we have $F(N) = D$ and consequently $B = D$. This contradicts the fact that α is not in F. Hence $N \not\subseteq B^*$ and $N \cap B^* = M$. By Lemma 2.8, we have $B^* \cap N$ is an almost subnormal subgroup of B^*. Thus, M is an almost subnormal subgroup of $B^* = \mathrm{C}_D(F(\alpha))^*$.

Now, suppose that M is non-abelian. By [12, Corollary 2.3], M is a non-central almost subnormal subgroup of B^*. Since M satisfies an LPI, by Lemma 2.9, we have $[B : L] < \infty$. Recall that $[L : F] = r < \infty$, so $[B : F] < \infty$. By part (ii) of [6, Centralizer Theorem, p. 42], we have

$$D \otimes_F L \cong \mathrm{M}_r(B \otimes_L \mathrm{C}_D(B)) \cong \mathrm{M}_r(B \otimes_L \mathrm{C}_D(\mathrm{C}_D(L))) \cong \mathrm{M}_r(B \otimes_L L) \cong \mathrm{M}_r(B).$$

Since $\mathrm{M}_r(B)$ is a finite dimensional vector space over F, we conclude that $[D:F]<\infty$, a contradiction. The proof is complete. $\square$

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