

A NOTE ON AUTOMORPHISM LIFTABLE MODULES

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ABSTRACT. A module M is said to be an *automorphism liftable module* if for each submodule N of M , every automorphism of the quotient M/N can be lifted to an endomorphism of M . In this work, some properties of automorphism liftable modules are investigated. Also, characterization for some special rings such as perfect, semiperfect and uniserial are given by using automorphism liftable modules.

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1. Introduction and basic definitions

Throughout this paper, R denotes an associative ring with identity and all modules are assumed to be unital right R -modules unless otherwise stated. For any module M , $J(M)$ and $E(M)$ denote the Jacobson radical and injective hull of M .

Singh and Srivastava [13], introduced a new class of modules namely dual automorphism invariant modules, which is the dual notion of automorphism invariant modules introduced by Lee and Zhou [9]. Further study of such modules was carried out by various authors in various articles [1,8,11].

A module M is called an *automorphism-invariant module* [9] if M is invariant under any automorphism of its injective hull, i. e., for any automorphism σ of $E(M)$, $\sigma(M) \subseteq M$. In other-hand, M is called an automorphism invariant module if every isomorphism between two essential submodules of M extends to an automorphism of M . Clearly, any quasi injective or pseudo injective module is an automorphism-invariant module.

A submodule N of a module M is said to be *small* in M if there is no proper submodule K of M such that $N + K = M$. The notation $N \ll M$ means that N is small in M . A *projective cover* of a module M is an epimorphism $f : P \rightarrow M$ with $Ker(f) \ll P$ and P is projective.

A module M is said to be *quasi projective* if for any module A , epimorphism $q : M \rightarrow A$ and homomorphism $f : M \rightarrow A$, there is an endomorphism f' of M such that the diagram

$$\begin{array}{ccc} & & M \\ & \swarrow f' & \downarrow f \\ M & \xrightarrow{q} & A \longrightarrow 0 \end{array}$$

is commutative, that is, $qf' = f$.

A module M is called a *dual automorphism-invariant module* if whenever K_1 and K_2 are small submodules of M , then any epimorphism $\eta : M/K_1 \rightarrow M/K_2$ with small kernel lifts to an endomorphism ϕ of M .

$$\begin{array}{ccc} M & \xrightarrow{\phi} & M \\ \downarrow & & \downarrow \\ M/K_1 & \xrightarrow{\eta} & M/K_2 \end{array}$$

In fact, the lifting ϕ is an isomorphism [13]. Every semisimple module is a dual automorphism-invariant module. Quasi projective and pseudo projective modules are dual automorphism-invariant modules. Also, the Prüfer group $\mathbb{Z}(p^\infty)$ is not a dual automorphism invariant module over the ring of integers [13].

In [14], Tuganbaev introduced a new notion namely automorphism-extendable module. A right module M is said to be an *automorphism-extendable module* if for each submodule N of M , every automorphism of the module N can be extended to an endomorphism of M . In [14], he proved that every automorphism-invariant module is an automorphism-extendable module. But the converse need not be true. For example, \mathbb{Z} is an automorphism-extendable \mathbb{Z} -module but it is not an automorphism-invariant \mathbb{Z} -module [14].

The dual notion of automorphism-extendable module was introduced by Selvaraj and Santhakumar in [12]. Such modules were called as automorphism liftable modules. Also, they characterized some rings by the properties of automorphism liftable modules and they studied automorphism liftable modules with the summand sum and summand intersection properties. At the same time Abyzov and Truong [2] introduced the same concept in the name of dual automorphism-extendable module. They proved that dual automorphism-extendable module satisfies the D3 condition. Also, in [1] Abyzov et al. studied the notion of dual strictly automorphism-extendable modules which can be viewed as a strongest notion of automorphism liftable modules. They proved that over a perfect ring a module is dual strictly automorphism-extendable if and only if it is dual automorphism-invariant.

A module M is a *Hopf module* provided that every surjective endomorphism of M is an automorphism. A submodule K of M is said to be *weak supplement* of M if there is a submodule N of M such that $N + K = M$ and $N \cap K \ll M$. M is a *weakly supplemented module* if every submodule of M has a weak supplement.

In this paper, we discuss the question “When a direct sum of automorphism liftable modules will become automorphism liftable?”. Also we introduce automorphism liftable cover to characterize some rings.

2. Automorphism liftable modules

Selvaraj and Santhakumar introduced the notion automorphism liftable module in [12], and discussed some basic properties of the same. Here we discuss some more basic properties of automorphism liftable modules.

The definition of automorphism liftable module as defined in [12] is

Definition 2.1. [12] A module M is said to be an *automorphism liftable module* if for each submodule N of M , every automorphism of the quotient M/N can be lifted to an endomorphism of M .

Example 2.2.

- (1) It is very clear that every quasi (pseudo) projective module is automorphism liftable.
- (2) Let $R = \begin{pmatrix} \mathbb{F}_2 & \mathbb{F}_2 \\ 0 & \mathbb{F}_2 \end{pmatrix}$ be a ring and $M = \begin{pmatrix} \mathbb{F}_2 & \mathbb{F}_2 \\ 0 & 0 \end{pmatrix}$ be a right R -module. Then the only proper subset of M is $N = \left\{ \begin{pmatrix} a & a \\ 0 & 0 \end{pmatrix} \mid a \in \mathbb{F}_2 \right\}$. Then M/N isomorphic to \mathbb{F}_2 . Hence the only automorphism of M/N is the identity. Therefore M is an automorphism liftable module over the ring R .
- (3) A weakly supplemented dual automorphism invariant module is an automorphism liftable module [1,12].
- (4) Over a perfect ring every flat module is an automorphism liftable module [12].
- (5) Over a quasi-Frobenius ring every injective module is an automorphism liftable module [12].

We know that if any two modules M and N with projective covers are isomorphic, then their projective covers are isomorphic. The converse part need not be true for all modules. However, here we try to solve the converse with certain condition by using automorphism liftable modules.

Proposition 2.3. *Suppose $P_1 \xrightarrow{\phi_1} M_1 \rightarrow 0$ and $P_2 \xrightarrow{\phi_2} M_2 \rightarrow 0$ are projective covers of modules M_1 and M_2 , respectively and $P_1 \cong P_2$ with $M_1 \oplus M_2$ is an automorphism liftable module. Then $M_1 \cong M_2$.*

Proof. Since M_1 and M_2 have projective covers, $M_1 \oplus M_2$ has a projective cover $P_1 \oplus P_2 \xrightarrow{\phi_1 \oplus \phi_2} M_1 \oplus M_2 \rightarrow 0$. Let $\rho_1 : P_1 \rightarrow P_2$ be an automorphism and $\rho_2 : P_2 \rightarrow P_1$ be the inverse of ρ_1 . Consider a homomorphism $\rho = \rho_1 \oplus \rho_2 : P_1 \oplus P_2 \rightarrow P_1 \oplus P_2$, which is an automorphism of $P_1 \oplus P_2$. Since $J(P_1 \oplus P_2) = J(P_1) \oplus J(P_2)$ is invariant under ρ , there exists an automorphism $\hat{\rho} : (P_1 \oplus P_2)/J(P_1 \oplus P_2) \rightarrow (P_1 \oplus P_2)/J(P_1 \oplus P_2)$. Also, $(P_1 \oplus P_2)/J(P_1 \oplus P_2) \cong (P_1/J(P_1)) \oplus (P_2/J(P_2))$. Take $J_1 = \phi_1(J(P_1))$ and $J_2 = \phi_2(J(P_2))$. Since $J(P_1)$ and $J(P_2)$ are small submodules of P_1 and P_2 respectively, clearly, J_1 and J_2 are small submodules of M_1 and M_2 respectively. Also, $(P_1/J(P_1)) \oplus (P_2/J(P_2)) \cong (M_1/J_1) \oplus (M_2/J_2) \cong (M_1 \oplus M_2)/(J_1 \oplus J_2)$. Then the automorphism $\rho = \rho_1 \oplus \rho_2$ yields an automorphism $\delta = \delta_1 \oplus \delta_2$, $\delta_1 : M_1/J_1 \rightarrow M_2/J_2$ and $\delta_2 : M_2/J_2 \rightarrow M_1/J_1$ are isomorphism yielded by ρ_1 and ρ_2 respectively.

Since $M_1 \oplus M_2$ is automorphism liftable, $\exists \sigma : M_1 \oplus M_2 \rightarrow M_1 \oplus M_2$ is a lifting of δ . Consider the map $f = I_1 \circ \sigma \circ n_2 : M_1 \rightarrow M_2$, where $I_1 : M_1 \rightarrow M_1 \oplus M_2$ is natural inclusion map and $n_2 : M_1 \oplus M_2 \rightarrow M_2$ is natural projection. Then we have $f(M_1) + J_2 = M_2$ and $J(M_2)$ is small in M_2 , $f(M_1) = M_2$, i.e. f is epic. By [12, Corollary 2.10], f is a splitting epimorphism. This implies that f is an isomorphism. This completes the proof. \square

The direct sum of two automorphism liftable modules need not be automorphism liftable. For example [12], \mathbb{Z}_2 and \mathbb{Z}_4 are automorphism liftable \mathbb{Z} -modules. But, $\mathbb{Z}_2 \oplus \mathbb{Z}_4$ is not an automorphism liftable \mathbb{Z} -module. This motivates us to rise the question ‘‘When a direct sum of automorphism liftable modules will become automorphism liftable?’’. Through following results we partially answer the above question.

Theorem 2.4. *Let P be a projective simple module, and M be any automorphism liftable module such that $\text{Hom}(M/K, P) = 0$ for any submodule K of M . Then $P \oplus M$ is automorphism liftable.*

Proof. Set $N = P \oplus M$. Let K be a submodule of N , then either $P \subseteq K$ or $P \cap K = 0$.

Case 1: Suppose $P \subseteq K$, $N/K = M/K_1$ for some submodule $K_1 \subseteq M$. Let $f : N/K \rightarrow N/K$ be an automorphism, then we can inherit an automorphism $f_1 : M/K_1 \rightarrow M/K_1$ from f . Since M is automorphism liftable, $\exists g_1 : M \rightarrow M$ is

liftable of f_1 . Let $g = g_1 \oplus I_P$, I_P is an identity map of P . Then $g : N \rightarrow N$ is a lifting of f .

Case 2: Suppose $P \cap K = 0$, $N/K \cong P \oplus M/K_1$ for some submodule $K_1 \subseteq M$. Let $f : N/K \rightarrow N/K$ be an automorphism. Now f can be viewed as $f = \begin{pmatrix} f_{11} & f_{12} \\ f_{21} & f_{22} \end{pmatrix}$, where $f_{11} : P \rightarrow P$, $f_{12} : P \rightarrow M/K$, $f_{21} : M/K \rightarrow P$ and $f_{22} : M/K \rightarrow M/K$. By hypothesis $f_{21} = 0$. Then $f = \begin{pmatrix} f_{11} & f_{12} \\ 0 & f_{22} \end{pmatrix}$. Since f is an automorphism, $\exists g = \begin{pmatrix} g_{11} & g_{12} \\ 0 & g_{22} \end{pmatrix}$ such that $fg = I = gf$.

$$\begin{aligned} fg &= I \\ \begin{pmatrix} f_{11} & f_{12} \\ 0 & f_{22} \end{pmatrix} \begin{pmatrix} g_{11} & g_{12} \\ 0 & g_{22} \end{pmatrix} &= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} . \\ \begin{pmatrix} f_{11}g_{11} & f_{11}g_{12} + f_{12}g_{22} \\ 0 & f_{22}g_{22} \end{pmatrix} &= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \end{aligned}$$

This implies that $f_{11}g_{11} = I$ and $f_{22}g_{22} = I$. Similarly, we have $g_{11}f_{11} = I$ and $g_{22}f_{22} = I$. Hence f_{22} is an isomorphism of M/K . Since M is automorphism liftable, $\exists h_{22} : M \rightarrow M$ is a lifting of f_{22} . By the projectivity of P , $\exists h_{12} : P \rightarrow M$ is a lifting of f_{12} . Take $h_{11} = f_{11}$ and $h_{21} = 0$.

Let $h = \begin{pmatrix} h_{11} & h_{12} \\ 0 & h_{22} \end{pmatrix}$. It is easy to check that h is a lifting of f . Hence $P \oplus M$ is an automorphism liftable module. \square

Theorem 2.5. *If $M_i (i \in I)$ are automorphism liftable modules such that for every submodule N of the direct sum $M = \oplus M_i$, $N_i = \oplus (N \cap M_i)$ holds, then M is again an automorphism liftable module.*

Proof. Since every submodule N of the direct sum $M = \oplus M_i$, holds $N_i = \oplus (N \cap M_i)$, every quotient module M/N of M is of the form $\oplus (M/N_i)$ with $N_i \subseteq M_i$. By the proof of [4, Lemma 3], every homomorphism $M_i \rightarrow M_j/N_j$ is trivial. Let $f : M_i/N_i \rightarrow M_j/N_j$ be a homomorphism, $g = f \circ n$, where $n : M_i \rightarrow M_i/N_i$ is natural projection, is a homomorphism from $M_i \rightarrow M_j/N_j$. Hence $g = f \circ n = 0$ implies $f = 0$. Therefore any automorphism $\sigma : M/N \rightarrow M/N$ is a coordinate wise automorphism. This implies that M is an automorphism liftable module. \square

Proposition 2.6. *If $M_i (i \in I)$ are quasi projective modules and $M = \oplus M_i$ is an automorphism liftable module, then M is quasi projective.*

Proof. The proof is followed by [12, Theorem 2.7]. \square

3. Automorphism liftable cover

In this section, we introduce an analog notion of dual automorphism invariant cover, namely automorphism liftable cover via AL-stable submodule. We can view it as a generalization of the quasi projective cover.

Definition 3.1. A submodule Q of M is said to be *AL-stable* if for any submodule N containing Q and automorphism $g : M/N \rightarrow M/N$ having lifting $f \in \text{End}M$ of g , $f(Q) \subseteq Q$.

Clearly, every invariant submodule of M is an AL-stable submodule of M . A zero module is trivially an AL-stable submodule of any module.

The following lemma shows the purpose of defining the notion AL-stable submodule.

Lemma 3.2. *If Q is an AL-stable submodule of an automorphism liftable module M , then the quotient module M/Q is an automorphism liftable module.*

Proof. Let $f : (M/Q)/(N/Q) \rightarrow (M/Q)/(N/Q)$ be an automorphism, where N is a submodule of M . Equivalently, $f : M/N \rightarrow M/N$ is an automorphism. Then it can be extended to $g : M \rightarrow M$, and also Q is an AL-stable submodule. Hence $g_1 : M/Q \rightarrow M/Q$ defined by $g_1(m + Q) = g(m) + Q$ is a lifting of f . \square

Lemma 3.3. *Let Q be an AL-stable submodule of an automorphism liftable module M , and N be a submodule of M such that $Q \subseteq N$. If N is not an AL-stable submodule of M , then N/Q is not an AL-stable submodule of M/Q .*

Proof. If N is not an AL-stable submodule of M , then there exist submodules N_1 containing N , isomorphism $f : M/N_1 \rightarrow M/N_1$ having lift $f_1 : M \rightarrow M$ such that $f_1(n_0) \notin N$ for some $n_0 \in N$. Define $g : M/Q \rightarrow M/Q$ by $g(m + Q) = f_1(m) + Q$. Then $g(n_0 + Q) = f_1(n_0) + Q \notin N/Q$. Hence N/Q is not an AL-stable submodule of M/Q . \square

Definition 3.4. An automorphism liftable module L is said to be an *automorphism liftable cover* of a module M if

- (1) there exists a minimal epimorphism $\psi : L \rightarrow M$, i.e. $\text{Ker}(\psi)$ is small.
- (2) $\text{Ker}(\psi)$ does not contain any non-zero AL-stable submodule of L .

The following theorem gives an existence of an automorphism liftable cover.

Theorem 3.5. *If a module M has a quasi-projective cover, then it has an automorphism liftable cover.*

Proof. Let $f : Q \rightarrow M$ be a quasi projective cover of M . Let A be a maximal AL-stable submodule of Q contained in $\text{Ker}(f)$. Then by Lemma 3.2, Q/A is an automorphism liftable module. Since $A \subseteq \text{Ker}(f)$, there exists $g : Q/A \rightarrow M$ such that $f = g \circ h$ where $h : Q \rightarrow Q/A$ is a natural homomorphism. Since h is onto, f is onto and $\text{Ker}(g) = \text{Ker}(f)/A$.

Suppose $(B/A) + \text{Ker}(g) = Q/T$. Then $B + \text{Ker}(f) = Q$ and so $B = Q$. Therefore $B/A = Q/A$. Hence $\text{Ker}(g) \subseteq Q/A$ is a small submodule. Let, if possible, $C/A \subseteq \text{Ker}(g)$ be a non zero AL-stable submodule of Q/A . Then $\text{Ker}(f) \supset C \supset A$. Since A is a maximal AL-stable submodule of Q in $\text{Ker}(f)$, C is not an AL-stable submodule of Q . Then by Lemma 3.3, C/A is not an AL-stable submodule of Q/A . Therefore $g : Q/A \rightarrow M$ is an automorphism liftable cover of M . \square

From [15, Proposition 2.6], we have that any module has a quasi projective cover whenever, it has a projective cover. Then the following corollary follows from the above theorem.

Corollary 3.6. *If a module M has a projective cover, then it has an automorphism liftable cover.*

In general, the converse of above corollary need not be true. With some certain condition as given in Theorem 3.7, it will be true.

Theorem 3.7. *Let $\lambda : P \rightarrow M$ be an epimorphism from a projective module P onto a module M . Then M has a projective cover if and only if $P \oplus M$ has an automorphism liftable cover.*

Proof. Suppose M has a projective cover $p : P_1 \rightarrow M$. Then $I_P \oplus p : P \oplus P_1 \rightarrow P \oplus M$ is a projective cover of $P \oplus M$. By Corollary 3.6, $P \oplus M$ has an automorphism liftable cover.

Conversely, assume that $P \oplus M$ has an automorphism liftable cover $\phi : L \rightarrow P \oplus M$. Then $\phi \circ n : L \rightarrow P$ is an epimorphism, where n is a natural projection of $P \oplus M$ to P . By the projectivity of P , we can view P as a direct summand of L , i.e., $L = P \oplus L_1$ for some automorphism liftable module L_1 . Then ϕ can be splitted as $\phi = I_P \oplus \phi'$, where ϕ' is the restriction of ϕ to L_1 . Since $\text{Ker}(\phi')$ is the homomorphic image of $\text{Ker}(\phi)$, $\text{Ker}(\phi')$ is small in L_1 . Hence $\phi' : L_1 \rightarrow M$ is an epimorphism with small kernel.

By the projectivity of P , there exists a homomorphism $\lambda' : P \rightarrow L_1$ such that $\phi' \circ \lambda' = \lambda$. Since $P \oplus L_1$ is automorphism liftable module, P and L_1 are relatively projective to each other. Therefore the map $\lambda' : P \rightarrow L_1$ splits, i.e., L_1 is a direct summand of P . Hence $\phi' : L_1 \rightarrow M$ is a projective cover of M . \square

Recall that a ring R is said to be a (semi) perfect ring if every (finitely generated) module has a projective cover. In [5], Golan proved that a ring is perfect if and only if every module has a quasi projective cover. In [11], Selvaraj and Santhakumar characterized perfect ring by using dual automorphism invariant cover with certain condition. Also in [12], they proved that a ring is perfect if and only if every flat module is automorphism liftable. Here we characterized the same by using automorphism liftable cover.

Corollary 3.8. *A ring R is (semi) perfect if and only if every (finitely generated) module has an automorphism liftable cover.*

Proof. Suppose every (finitely generated) module has an automorphism liftable cover. Since every (finitely generated) module can be written as an epimorphic image of a (finitely generated) projective module, by Theorem 3.7, R is a (semi) perfect ring.

Conversely, suppose R is a (semi) perfect ring. Then by Corollary 3.6, every (finitely generated) module M has an automorphism liftable cover. \square

In [6], Golan prove that a ring R is semiperfect if and only if for all $n \geq 1$, every cyclic R_n -module has a quasi projective cover if and only if there exists an $n > 1$ such that every cyclic R_n -module has a quasi projective cover. Inspired by this, we characterize the semiperfect ring by automorphism liftable covers.

Theorem 3.9. *For any ring R , the following are equivalent:*

- (1) R is semiperfect;
- (2) For all $n \geq 1$, every cyclic R_n -module has an automorphism liftable cover;
- (3) There exists an $n > 1$ such that every cyclic R_n -module has an automorphism liftable cover.

Proof. (1) \Rightarrow (2) It follows from the fact that if R is semiperfect so is R_n for all $n \geq 1$, by [7, Theorem 3].

(3) \Rightarrow (1) Let $n > 1$ be satisfy the condition that every cyclic R_n -module has an automorphism liftable cover. Let S be a right ideal of R , S_n the right ideal of R_n consisting of all matrices with entries from S . Let $e_{ij} \in R_n$ be the matrix with 1_R in the (i, j) th position and zero else where. Then $R_n/e_{11}S_n$ is isomorphic to

$P \oplus M$, where $M = e_{11}R_n/e_{11}S_n$ and $P = \sum_{i=2}^n e_{ii}R_n$. P is clearly R_n -projective and the map $\lambda : P \rightarrow M$ which sends $[a_{ij}]$ to $e_{21}[a_{ij}] + e_{11}S_n$ is an R_n -epimorphism. Since $P \oplus M$ has an automorphism liftable cover over R_n , by Theorem 3.7, M has a projective cover $\phi : Q \rightarrow M$ over R_n . Then $\phi(Qe_{11}) = (\phi Q)e_{11} = Me_{11}$ which is isomorphic, as an R -module, to R/S . Therefore Q is R_n -projective and so Qe_{11} is R -projective [7]. The induced R -homomorphism $\phi' : Qe_{11} \rightarrow R/S$ is then a projective cover, proving (1).

(2) \Rightarrow (3) is trivial. \square

In [10], Nguyen et al. proved that a ring R is semisimple Artinian if and only if every cyclic right module over $M_n(R)$, $n \geq 2$ is a $D3$ -module. This result is also true when we replace $D3$ by automorphism liftable.

Theorem 3.10. *A ring R is semisimple Artinian if and only if every cyclic right module over $M_n(R)$, $n \geq 2$ is an automorphism liftable module.*

Proof. From the proof of [10, Lemma 2.6.], we have a Morita equivalence between R to $M_n(R)$ such that collection of all n -generated R -modules mapped onto collection of all cyclic $M_n(R)$ -modules. Then by [12, Theorem 2.12], every n -generated R -module is automorphism liftable if and only if every cyclic module over $M_n(R)$, $n \geq 2$ is automorphism liftable. Hence by [12, Proposition 3.1], R is semisimple Artinian if and only if every cyclic right module over $M_n(R)$, $n \geq 2$ is an automorphism liftable module. \square

In [3], Byrd proved that a ring R is uniserial if and only if every quasi projective module is quasi injective; equivalently if every quasi-injective module is quasi-projective. Here we characterize uniserial rings by using automorphism liftable and automorphism extendable modules.

Theorem 3.11. *For any ring R , the following are equivalent:*

- (1) R is uniserial;
- (2) Every quasi projective module is automorphism extendable;
- (3) Every quasi injective module is automorphism liftable;
- (4) R is quasi-Frobenius and every finitely generated quasi projective module is automorphism extendable;
- (5) R is quasi-Frobenius and every finitely generated quasi injective module is automorphism liftable.

Proof. (1) \Rightarrow (2), (1) \Rightarrow (3), (1) \Rightarrow (4) and (1) \Rightarrow (5) are follows from [3, Theorem 2.8].

(2) \Rightarrow (1) and (3) \Rightarrow (1): By [3, Theorem 2.8], it is enough to prove that every quasi (injective) projective module is quasi (projective) injective. Let M be a quasi (injective) projective module. Then $M \oplus M$ is a quasi (injective) projective module. By hypothesis $M \oplus M$ is automorphism (liftable) extendable. Then by ([14, Lemma 7]) [12, Theorem 2.7], M is quasi (projective) injective.

(4) \Rightarrow (1) and (5) \Rightarrow (1) are similar as above. □

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