EXAMPLES OF (NON-)BRAIDED TENSOR CATEGORIES

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Abstract. Six examples of non-braidable tensor categories which are extensions of the category $\text{Comod}(H)$, for $H$ a supergroup algebra; and two examples of braided categories where the only possible braiding is the trivial braiding are introduced.

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1. Introduction

Braided categories were introduced by Joyal and Street [5]. They are related to knot invariants, topology and quantum groups, since they can express symmetries. Some examples of braided categories are:

- graded modules over a commutative ring,
- $(co)m$odules over a (co)quasi-triangular Hopf algebra,
- the Braid category, [5, Section 2.2],
- the center of a tensor category.

In the last example, we begin with a tensor category and construct a braided one. In a general scenario, a natural question is it is possible to construct braidings starting with tensor categories. In particular, if $G$ is a finite group, can a $G$-extension of a tensor category be braided? In this work we show that this can be done in very few cases. Then, an extension of a braided category is not necessarily braided, so it is really complicated to extend that property.

However, constructing examples of non-braided categories is also important. A big family of these come from the category of $(co)$modules of a Hopf algebra without a (co)quasi-triangular structure, see [9, T 10.4.2]. Masuoka in [6] and [7] constructs explicit examples of non-Quasi-triangular or non-CoQuasi-triangular Hopf algebras. In particular these Hopf algebras can not be obtained from any group algebra by twist (or cocycle) deformation. Other examples were constructed in [4].
In the literature there are a few explicit examples of tensor categories, for this reason we construct in [8] eight tensor categories, following the description introduced in [3] of Crossed Products. These categories extend the module category over certain quantum groups, called supergroup algebras. In a few words, a crossed product tensor category is, as Abelian category, the direct sum of copies of a fixed tensor category, and the tensor product comes from certain data. Then founding all possible data, we explicitly construct tensor categories.

In the same work [3], the author also describes all possible braidings over a crossed product. Following this, three conditions were introduced to decide if a $G$-crossed product is braidable:

1. the base category has to be braided,
2. $G$ has to be Abelian, and the biGalois objects associated to each crossed product have to be trivial,
3. the 3-cocycle associated to each crossed product over an specific supergroup algebra has to be trivial, if $G$ is the cyclic group of order 2.

The goal in the present paper is to obtain all possible braidings over the categories introduced in [8]. With this, only two categories of the eight found in [8] are braided with the trivial braiding only, and the other 6 are not braidable.

In [8, Theorem 6.3], using the Frobenius-Perron dimension, we proved that these eight categories are the module category of a quasi-Hopf algebra. Although we do not know how to explicitly compute these algebras, as a corollary of this work, we know that six of these algebras are non-Quasi-triangular and two are Quasi-triangular only. In particular, we are obtaining information about certain quasi-Hopf algebras without knowing them explicitly; showing how useful it is to work in the category world. In a future, when we can explicitly describe these quasi-Hopf algebras, we will already know how their Quasi-triangular structures are.

2. Preliminaries and notation

Throughout this paper we shall work over an algebraically closed field $\mathbb{k}$ of characteristic zero. For basic knowledge of Hopf algebras see [9]. Let $H$ be a finite-dimensional Hopf algebra and $A$ be a left $H$-comodule. Then $A$ is also a right $H$-comodule with right coaction $a \mapsto a_0 \otimes S(a_{-1})$, see [1, Proposition 2.2.1(iii)]. A left $H$-Galois extension of $A^{co(H)}$ is a left $H$-comodule algebra $(A, \rho)$ such that $A \otimes_{A^{co(H)}} A \to H \otimes A$, $a \otimes b \mapsto (1 \otimes a)\rho(b)$ is bijective. Similarly, we define right $H$-Galois extension.
Consider $L$ another finite-dimensional Hopf algebra. An $(H,L)$-biGalois object \cite{10} is an algebra $A$ that is a left $H$-Galois extension and a right $L$-Galois extension of the base field $k$ such that the two comodule structures make it an $(H,L)$-bicomodule. Two biGalois objects are isomorphic if there exists a bijective bicomodule morphism that is also an algebra map. For $A$ an $(H,L)$-biGalois object, define the tensor functor

$$F_A : \text{Comod}(L) \to \text{Comod}(H), \quad F_A = A \Box_L - .$$

By \cite{10}, every tensor functor between comodule categories is one of these, and $F_A \simeq F_B$ as tensor functors if and only if $A \simeq B$ as biGalois objects.

If $A = H$, then every natural monoidal equivalence $\beta : F_H \to F_H$ is given by $f \otimes \text{id}_X : H \Box_H X \to H \Box_H X$, $(X,\rho_X) \in \text{Comod}(H)$, where $f : H \to H$ is a bicomodule algebra isomorphism.

**Lemma 2.1.** Every natural monoidal equivalence $\text{id}_{\text{Comod}(H)} \to \text{id}_{\text{Comod}(H)}$ is given by $(\varepsilon f \otimes \text{id}_X)\rho_X$.

**Proof.** For $X \in \text{Comod}(H)$, the coaction induces an isomorphism $X \simeq H \Box_H X$ with inverse induced by $\varepsilon$, the counit. Then $\text{id}_{\text{Comod}(H)} \simeq F_H$ as tensor functors. Since all natural monoidal autoequivalences of $F_H$ are given by $f \otimes \text{id}_X$ then all natural monoidal autoequivalences of $\text{id}_{\text{Comod}(H)}$ are given by $(\varepsilon f \otimes \text{id}_X)\rho_X$. \hfill $\square$

**Definition 2.2.** \cite[Definition 10.1.5]{9} $(H,R)$ is a Quasi-triangular (or QT) Hopf algebra if $H$ is a Hopf algebra and there exists $R \in H \otimes H$, called the $R$-matrix, invertible such that

$$(\Delta \otimes \text{id})R = R^{13}R^{23}, \quad (\text{id} \otimes \Delta)R = R^{13}R^{12}, \quad \Delta^{\text{op}}(h) = R\Delta(h)R^{-1}, h \in H.$$

Dualizing we can define, $(H,r)$ is a CoQuasi-triangular (or CQT) Hopf algebra if $H$ is a Hopf algebra and $r : H \otimes H \to k$, called the $r$-form, is a linear functional which is invertible with respect to the convolution multiplication and satisfies for arbitrary $a,b,c \in H$

$$r(c \otimes ab) = r(c_1 \otimes b)r(c_2 \otimes a), \quad r(ab \otimes c) = r(a \otimes c_1)r(b \otimes c_2),$$

$$r(a_1 \otimes b_1)a_2b_2 = r(a_2 \otimes b_2)b_1a_1.$$

**Remark 2.3.** Drinfeld defined a quantum group as a non-commutative, non-cocommutative Hopf algebra. Examples of these are the QT Hopf algebras. The
importance of quantum groups lies in they allow to construct solutions for the quantum Yang-Baxter equation in statistical mechanics (the $R$-matrix is a solution of this equation). An example of quantum group are the supergroup algebras.

A *supergroup algebra* is a supercocommutative Hopf algebra of the form $\mathbb{k}[G] \ltimes \wedge V$, where $G$ is a finite group and $V$ is a finite-dimensional $G$-module. They appear and have an interesting role in the classification of triangular algebras, see [2, Theorem 4.3].

**Example 2.4.** Consider $H = \mathbb{k}C_2 \ltimes \mathbb{k}V$, for $V$ a 2-dimensional vector space and $C_2$ the 2-cyclic group generated by $u$ with $u \cdot v = -v$ for $v \in V$. As an algebra, it is generated by elements $v \in V, g \in C_2$ subject to relations $vw + wv = 0; gv = (g \cdot v)g$ for all $v, w \in V, g \in C_2$. The coproduct and antipode are determined by

$$\Delta(v) = v \otimes 1 + u \otimes v; \Delta(g) = g \otimes g; S(v) = -uv; S(g) = g^{-1}, \quad v \in V, g \in C_2.$$ 

Taking $R = \frac{1}{2}(1 \otimes 1 + 1 \otimes u + \otimes 1 - u \otimes u)$, $(H, R)$ is a QT-Hopf algebra. We can construct a CoQuasi-triangular structure taking $r = R^*$ since $H$ is auto-dual. Then $(H, R^*)$ is a CQT-Hopf algebra.

**Definition 2.5.** A *finite tensor category* is a locally finite, $\mathbb{T}$-linear, rigid, monoidal Abelian category $D$ with $\text{End}_D(1) \cong \mathbb{T}$. Given a finite group $\Gamma$, a (faithful) $\Gamma$-*grading* on a finite tensor category $D$ is a decomposition $D = \bigoplus_{g \in \Gamma} D_g$, where $D_g$ are full Abelian subcategories of $D$ such that

- $D_g \neq 0$;
- $\otimes : D_g \times D_h \rightarrow D_{gh}$ for all $g, h \in \Gamma$.

We have that $C := D_e$ is a tensor subcategory of $D$. The category $D$ is call a $\Gamma$-*extension* of $C$. Denote by $[V, g]$ the homogeneous elements in $D$, for $V \in D_g, g \in \Gamma$.

A *braided tensor category* is a tensor category $\mathcal{C}$ with natural isomorphisms $c_{X,Y} : X \otimes Y \rightarrow Y \otimes X$ such that

$$\alpha_{V,W,U}c_{U,V,W}\alpha_{U,V,W} = (\text{id} \otimes c_{U,W})\alpha_{V,U,W}(c_{U,V} \otimes \text{id}), \quad (1)$$

$$\alpha_{W,U,V}^{-1}c_{U,V,W}\alpha_{U,V,W}^{-1} = (c_{U,W} \otimes \text{id})\alpha_{W,U,V}^{-1}(\text{id} \otimes c_{V,W}). \quad (2)$$

If $(H, r)$ is a CQT-Hopf algebra then $\text{Comod}(H)$ is a braided tensor category with braiding given by $c_{V \otimes W}(x \otimes y) = r(y_{-1} \otimes x_{-1})y_0 \otimes x_0$, for all $V, W \in \text{Comod}(H)$.

The following theorem gives us the first condition to know when an extension can be braided.
Theorem 2.6. Let $\mathcal{D} = \bigoplus_{g \in G} \mathcal{D}_g$ be a $\Gamma$-extension of $\mathcal{C}$. If $\mathcal{D}$ is a braided tensor category then $\mathcal{C}$ is a braided tensor category.

Proof. Let $c$ be the braiding of $\mathcal{D}$, then $c_{[V,e],[W,e]} : [V \otimes W, e] \to [W \otimes V, e]$ and $c_{[V,e],[W,e]} = [\overline{c}_{V,W}, e]$ for some natural isomorphism $\overline{c}_{V,W} : V \otimes W \to W \otimes V$, for $V,W$ objects in $\mathcal{C}$. Since the associativity isomorphism satisfies $a_{[V,e],[W,e],[U,e]} = [a_{V,W,U}, e]$, where $a$ is the associativity morphism for $\mathcal{C}$; then $c$ is a braiding for $\mathcal{C}$. □

In [3], the author describes and classifies a family of such extensions and calls it crossed product tensor category. Fix $H$ a finite-dimensional Hopf algebra. In the case when $\mathcal{C} = \text{Comod}(H)$, in [8], we described crossed products in terms of Hopf-algebraic datum. A continuation they are introduced.

If $g \in G(H)$ and $L$ is a $(H,H)$-biGalois object then the cotensor product $L \square_H k_g$ is one-dimensional. Let $\phi(L, g) \in \Gamma$ be the group-like element such that $L \square_H k_g \simeq k_{\phi(L,g)}$ as left $H$-comodules. Assume that $A$ is an $H$-biGalois object with left $H$-comodule structure $\lambda : A \to H \otimes_k A$. If $g \in G(H)$ is a group-like element we can define a new $H$-biGalois object $A^g$ on the same underlying algebra $A$ with unchanged right comodule structure and a new left $H$-comodule structure given by $\lambda^g : A^g \to H \otimes_k A^g$, $\lambda^g(a) = g^{-1} a g \otimes a_0$ for all $a \in A$.

Theorem 2.7. [8, Lemma 5.7, Theorem 5.4] Let $\Upsilon = (L_a, (g(a,b), f^{a,b}), \gamma)_{a,b \in \Gamma}$ be a collection where

- $L_a$ is a $(H,H)$-biGalois object;
- $g(a,b) \in G(H)$;
- $f^{a,b} : (L_a \square_H L_b) \to L_{ab}$ are bicomodule algebra isomorphisms;
- $\gamma \in Z^3(G(H), k^\times)$ normalized,

such that for all $a,b,c \in \Gamma$:

$$L_e = H, \quad (g(e,a), f^{e,a}) = (e, \text{id}_{L_a}) = (g(a,e), f^{a,e});$$

$$\phi(L_a, g(b,c))g(a, bc) = g(a, b)g(ab, c);$$

$$f^{ab,c}(f^{a,b} \otimes \text{id}_{L_a}) = f^{a,b,c}(\text{id}_{L_a} \otimes f^{b,c}).$$

Then $\text{Comod}(H)(\Upsilon) := \bigoplus_{g \in \Gamma} \text{Comod}(H)$ as a structure of tensor category.
Proof. We give an sketch of the proof. Let $\Upsilon$ be a collection as in the Theorem. For $V, W \in \text{Comod}(H)$, $a, b \in \Gamma$, define

\[
[V, a] \otimes [W, b] := [V \otimes (L_a \Box_H W) \otimes k_{g(a,b)}, ab],
\]

\[
[V, 1]^* := [V^*, 1],
\]

\[
[k, a]^* := [k_{g(a,a^{-1})}, a^{-1}].
\]

Using [8, Eq (5.8)], we obtain the pentagon diagram and therefore $\text{Comod}(H)(\Upsilon)$ is a monoidal category. Since $\text{Comod}(H)$ is finite tensor category, then $\text{Comod}(H)(\Upsilon)$ is also finite tensor category. \qed

The following theorem gives us a second condition to decided if our extensions can be braided.

**Theorem 2.8.** If $\text{Comod}(H)(\Upsilon)$ is braided with braiding $c$ then the following conditions have to hold

1. $L_a \simeq H$ for all $a \in \Gamma$;
2. $\Gamma$ is Abelian;
3. $\Upsilon$ comes from a data $(g, f_{a,b}, \gamma)_{a,b \in \Gamma}$ with
   - $g \in Z^2(\Gamma, G(H))$ normalized,
   - $f_{a,b} : H^{g(a,b)} \rightarrow H$ a bicomodule algebra isomorphism with $f_{a,b,c} f_{a,b} = f_{a,bc}$,
   - $\gamma \in Z^3(G(H), k^*)$ normalized.

Proof. (1) Take, for any $V \in \text{Comod}(H)$, $c_{[V, e][1,a]} : [V, a] \rightarrow [L_a \Box_H V, a]$, this defines a natural isomorphism $\tau_a : \text{id}_C \rightarrow L_a \Box_H -$ which is monoidal since $c$ is a braiding. Then $L_a \simeq H$ as bicomodule algebras for all $a \in \Gamma$.

(2) Consider $c_{[1,a][1,b]} : [k_{g(a,b)}, ab] \rightarrow [k_{g(b,a)}, ba]$ then $ab = ba$ for all $a, b \in \Gamma$ and $\Gamma$ is Abelian.

(3) Since $L_a$ is trivial, then Equation (4) of Theorem 2.7 is equivalent to $g \in Z^2(\Gamma, G(H))$ and it is normalized by Equation (3) of Theorem 2.7. Moreover $f_{a,b} : H^{g(a,b)} \rightarrow H$ is a bicomodule algebra isomorphism that satisfies $f_{a,b,c} f_{a,b} = f_{a,bc}$ which is equivalent to Equation (5) of Theorem 2.7. \qed

**Remark 2.9.** By definition of bicomodule morphism, $f_{a,b} : H \rightarrow H$ has to be an algebra isomorphism such that $f_{a,b}(h_1) \otimes f_{a,b}(h_2) = g^{-1} h_1 g \otimes f_{a,b}(h_2)$ and $f_{a,b}(h_1) \otimes f_{a,b}(h_2) = f_{a,b}(h_1) \otimes h_2$, then $g^{-1} h_1 g \otimes f_{a,b}(h_2) = f_{a,b}(h_1) \otimes h_2$. 

\[
\]
In the case when $H = \wedge V \# kC_2$, as Example 2.4, using the previous Theorem we obtained eight tensor categories non-equivalent pairwise, [8, Section 6.3], named $\mathcal{C}_0(1, \text{id}, \pm 1), \mathcal{C}_0(\iota, \pm 1), \mathcal{D}(1, \text{id}, \pm 1), \mathcal{D}(\iota, \pm 1)$. 

In all cases, the underlying Abelian category is $\text{Comod}(\mathcal{H}) \oplus \text{Comod}(\mathcal{H})$ and for $V, W, Z \in \text{Comod}(\mathcal{H})$ they are defined in the following way:

- The tensor product, dual objects and associativity in $\mathcal{C}_0(1, \text{id}, \pm 1)$ are given by
  \[
  [V, e][W, g] = [V \otimes W, g], \quad [V, u][W, g] = [V \otimes U_0 \mathcal{H} W, ug],
  \]
  \[
  [V, e]^* = [V^*, e], \quad [1, u]^* = [k, u],
  \]

$\alpha_{[V, u],[W, u],[Z, u]}$ is not trivial, and $U_0$ is certain BiGalois object, see [8, Section 4].

- The tensor product, dual objects and associativity in $\mathcal{C}_0(\iota, \pm 1)$ are given by
  \[
  [V, e][W, e] = [V \otimes W, 1], \quad [V, u][W, u] = [V \otimes U_0 \mathcal{H} W \otimes k_u, e],
  \]
  \[
  [V, e][W, u] = [V \otimes W, u], \quad [V, u][W, e] = [V \otimes U_0 \mathcal{H} W, u],
  \]
  \[
  [V, e]^* = [V^*, e], \quad [1, u]^* = [k_u, u],
  \]

$\alpha_{[V, u],[W, u],[Z, u]}$ is not trivial.

- The tensor product, dual objects and associativity in $\mathcal{D}(1, \text{id}, \pm 1)$ are given by
  \[
  [V, e][W, g] = [V \otimes W, g], \quad [V, u][W, g] = [V \otimes W, ug],
  \]
  \[
  [V, e]^* = [V^*, e], \quad [1, u]^* = [k, u],
  \]

$\alpha_{[V, u],[W, u],[Z, u]} = [\pm \text{id}_V \otimes w \otimes z, u]$ and the others are trivial.

- The tensor product, dual objects and associativity in $\mathcal{D}(\iota, \pm 1)$ are given by
  \[
  [V, e][W, e] = [V \otimes W, e], \quad [V, u][W, u] = [V \otimes W \otimes k_u, e],
  \]
  \[
  [V, e][W, u] = [V \otimes W, u], \quad [V, u][W, e] = [V \otimes W, e],
  \]
  \[
  [V, e]^* = [V^*, e], \quad [1, u]^* = [k_u, u],
  \]

$\alpha_{[V, u],[W, u],[Z, u]} = [\pm \text{id}_V \otimes \tau(\epsilon \rho_Z \otimes \text{id}_Z \otimes k_u), u]$, where $\iota : H^u \rightarrow H$ is the unique bicomodule algebra isomorphism which satisfies $\iota(u) = -u$ and $\iota(x) = -x$ for $x \in V$; and $\tau : X \otimes Y \rightarrow Y \otimes X, \tau(z \otimes k) = k \otimes z$ for all $X, Y \in \text{Comod}(\mathcal{H})$, see [8, Remark 2.2].

**Remark 2.10.** By Lemma 2.8(1), we obtain that only the categories $\mathcal{D}(1, \text{id}, \pm 1)$ and $\mathcal{D}(\iota, \pm 1)$ could be braided, since the BiGalois objects have to be trivial.

By direct calculation on Equation (1), $\mathcal{D}(\iota, -1)$ is not braided with trivial braiding. So, in this case, we want to know if there exist another possible braidings.
3. Braided crossed product

Let $\Gamma$ be an Abelian group. In [8], following the ideas developed in [3], we described all $\Gamma$-crossed product tensor categories which are extensions of $\text{Comod}(H)$ for $H$ a Hopf algebra in terms of certain Hopf-algebraic datum. Fix $(H,r)$ a CQT-Hopf algebra. In the first Lemma of this Section, we do the same for the braiding associated tensor category. Consider a pair $(g,f_{a,b},\gamma)_{a,b\in \Gamma}$, as in Lemma 2.8, and let $\mathcal{C}$ be the associated tensor category. Consider a pair $(v^a,w^a)_{a\in \Gamma}$ where $v^a, w^a : H \to H$ are left $H$-comodule algebra isomorphisms. Let $W^a = \varepsilon w^a$, $V^a = \varepsilon v^a$ and $F^{a,b} = \varepsilon f^{a,b}$. If for all $a,b,c \in \Gamma$ and $X \in \text{Comod}(H)$ we have

\[ v^1 = w^1 = \text{id}_H, \quad (g(a,b), f^{a,b}) = (g(b,a), f^{b,a}), \quad (7) \]

\[ W^b(x_3)W^a(x_2)(W^{ab})^{-1}(x_1)x_0 = F^{a,b}(x_2)r(x_1 \otimes g(a,b))x_0, \quad x \in X, \quad (8) \]

\[ V^b(x_3)V^a(x_2)(V^{ab})^{-1}(x_1)x_0 = r(x_1 \otimes g(a,b))F^{a,b}(x_2)x_0, \quad x \in X, \quad (9) \]

\[ V^a(g(b,c)) = (\gamma_{a,b,c}\gamma_{b,c,a})^{-1}\gamma_{b,a,c}, \quad (10) \]

\[ W^b(g(c,a)) = \gamma_{c,a,b}\gamma_{b,c,a}\gamma_{c,b,a}, \quad (11) \]

then we obtain a braiding over $\mathcal{C}$ given by

\[ c_{[V,w],[W,b]} = c_{V,W}((V^a \otimes \text{id}_V)\rho_V \otimes (W^a \otimes \text{id}_W)\rho_W) \otimes \text{id}, \quad V,W \in \text{Comod}(H), a,b \in \Gamma. \]

All braidings over $\mathcal{C}$ come from a pair $(v^a,w^a)_{a \in \Gamma}$ which satisfies (7) to (11).

**Proof.** By [3, Definition 5.3], a datum $(g,f^{a,b},\gamma)_{a,b\in \Gamma}$ has associated a braiding if there exist a triple $(\theta^a,\tau^a,t_{a,b})_{a,b\in G}$ where

- $\theta^a, \tau^a : \text{id}_\mathcal{C} \to \text{id}_\mathcal{C}$ are monoidal natural isomorphisms,
- for all $a,b \in G$, $t_{a,b} : (U_{a,b},\sigma^{a,b}) \to (U_{b,a},\sigma^{b,a})$ are isomorphisms in $\mathcal{Z}(\mathcal{C})$, where $\sigma^{a,b}_X = \tau(\varepsilon f^{a,b} \otimes \text{id}_X)\rho_X$, for $(X,\rho_X) \in \text{Comod}(H)$, and $U_{a,b} = k_{g(a,b)}$,

such that for all $a,b,c \in \Gamma$ and $X \in \mathcal{C}$, the following conditions hold

\[ \quad \theta^1 = \tau^1 = \text{id}, \quad \theta^1_1 = \text{id}_1 = \tau^1_1, \quad t_{a,1} = t_{1,a} = \text{id}_1, \quad (12) \]

\[ c_{U_{a,b},X}^{a,b} = ((\tau^b_X)^{-1}\gamma^a_X) \otimes \text{id}_{U_{a,b}}, \quad (13) \]

where $\gamma^a_X = \gamma_{c,a,b}\gamma_{b,c,a}\gamma_{c,b,a}^{-1}$. 

\[ (\gamma^a_X)^{-1} = \gamma_{a,b,c}\gamma_{b,c,a}\gamma_{c,b,a}. \]

\[ \]
from a left H-comodule algebra isomorphism, then \( \theta_X^a := (\varepsilon u^a \otimes \text{id}) \rho_X \) and \( \tau_X^a := (\varepsilon w^a \otimes \text{id}) \rho_X \) for all \( X \in \text{Comod}(H) \). Since \( U_{g(a,b)} = k_{g(a,b)} \), we can take \( t_{a,b} = 1 \).

Each \( t_{a,b} \) is a left \( H \)-comodule isomorphism if and only if \( g(a,b) \otimes t_{a,b} \text{id}_k = g(b,a) \otimes t_{a,b} \text{id}_k \) which gives \( g(a,b) = g(b,a) \) for all \( a, b \in \Gamma \). Moreover, each \( t_{a,b} \) is a braided morphism if and only if \( \sigma_X^{a,b} t_{a,b} = \sigma_X^{b,a} t_{a,b} \) for \( a, b \in \Gamma \) and \( X \in \text{Comod}(H) \) if and only if \( \sigma^{a,b} = \sigma^{b,a} \). Then \( t_{a,b} \) is an isomorphism in \( Z(\text{Comod}(H)) \) if and only if Condition (7) holds.

Condition (12) is equivalent to \( v^1 = w^1 = \text{id}_H \) and \( t_{a,1} = t_{1,a} = 1 \), since \( \theta_X^a = \text{id}_k = \tau_X^a \) is always true. Condition (13) is equivalent to

\[
F^{a,b}(x_{-2}) r(x_{-1} \otimes g(a,b)) x_0 \otimes k = W^{b,h}(x_{-3}) W^{a}(x_{-2})(W^{ab})^{-1}(x_{-1}) x_0 \otimes k,
\]

for \( x \otimes k \in X \otimes k_{g(a,b)} \), which is equivalent to Condition (8). In the same way, Condition (14) is equivalent to Condition (9). Condition (15) is equivalent to \( \gamma_{a,b,c} V^a(g(b,c)) t_{b,c} \gamma_{b,c,a} = t_{b,c} \gamma_{b,a,c} \gamma_{c,a} \) but if we take \( c = 1 \) then

\[
1 = t_{b,a}, \quad \text{for } a, b \in \Gamma,
\]

so, this Condition is equivalent to Condition (10), and Condition (16) is equivalent to Condition (11).

By [3, Theorem 5.4], this pair produces a braiding over \( C \) given by

\[
c^{[V,a],[W,b]} = [c^{V,W}(\theta_Y^a \otimes \tau_W^a), a, b], \quad \text{for all } V, W \in \text{Comod}(H), a, b \in \Gamma,
\]

and all braidings come from such a pair.

Now, we focus our attention into the case \( \Gamma = C_2 \). By Lemma 2.8, a datum \( \Upsilon' = (g, f, \gamma) \) with \( g \in G(H) \) a group-like element, \( f : H^g \rightarrow H \) a bicomodule algebra isomorphism and \( \gamma \in k^\times \), \( \gamma^2 = 1 \); generates a tensor category \( C = \text{Comod}(H)(\Upsilon') \).

The following theorem gives us the third and last condition to decide if our categories are braidable.

**Theorem 3.3.** The category \( \text{Comod}(H)(\Upsilon') \) is a braided \( C_2 \)-extension if and only if, there exists a pair of isomorphisms of left \( H \)-comodule algebras \( v, w : H \rightarrow H \) such that for all \( X \in \text{Comod}(H) \) and \( x \in X \)

a. \( \varepsilon(w(x_{-2}) w^{-1}(x_{-1})) x_0 = x \),
b. $\varepsilon(w(x_2)w(x_1)x_0 = \varepsilon f(x_2)r(x_1 \otimes g)x_0$,

c. $\varepsilon(v(x_2)v^{-1}(x_1))x_0 = x$,

d. $\varepsilon(v(x_2)v(x_1))x_0 = r(x_2 \otimes g)\varepsilon f(x_1)x_0$,

e. $\varepsilon(v(g)) = \gamma^{-1}$,

f. $\varepsilon(w(g)) = \gamma$.

**Proof.** Condition (7) is always true. Condition (8) is equivalent to $r(x_1 \otimes 1)x_0 = x$, and items a,b. Condition (9) is equivalent to $r(x_1 \otimes 1)x_0 = x$, and items c,d. Condition (10) is equivalent to item e. Condition (11) is equivalent to item f.

Regarding condition $r(x_1 \otimes 1)x_0 = x$, it is always true over a CoQuasi-triangular Hopf algebra. \qed

If $H = \wedge V \# kC_2$, as Example 2.4, by [8, Proposition 4.10], the isomorphisms $v$ and $w$ are identities. Then if the extension is braided the only possible braiding is the trivial, see Equation (17), since the category Comod $H$ has a braiding giving by the r-form. With this information, Conditions a-f are equivalent to

\begin{align*}
a'. \quad &\varepsilon(x_2x_1)x_0 = x, \\
b'. \quad &\varepsilon(x_2x_1)x_0 = \varepsilon f(x_2)r(x_1 \otimes g)x_0, \\
c'. \quad &\varepsilon(x_2x_1)x_0 = r(x_2 \otimes g)\varepsilon f(x_1)x_0, \\
d'. \quad &\varepsilon(g) = \gamma^{-1}, \\
e'. \quad &\varepsilon(g) = \gamma.
\end{align*}

Since $g$ is a group-like element, d’ and e’ imply that $\gamma = 1$. Thus, the only categories that could be braided are $D(1, id, 1)$ and $D(u, i, 1)$.

**Corollary 3.4.** A $C_2$-extension over Comod $(\wedge V \# kC_2)$ is braided if and only if, for all comodule $X$, $r(f(x_1) \otimes g)x_0 = x$, for all $x \in X$.

**Proof.** Condition a’ is always true over comodules. Since $x_1y_1r(x_2 \otimes y_2) = r(x_1 \otimes y_1)y_2x_2$ for $x, y \in H$ we have

$$(x_1g)r(x_2 \otimes g) \otimes x_0 = r(x_1 \otimes g)g x_2 \otimes x_0.$$  

Applying $\varepsilon f \otimes id_X$, we obtain $r(x_2 \otimes g)\varepsilon f(x_1)x_0 = \varepsilon f(x_2)r(x_1 \otimes g)x_0$. This implies that Conditions b’ and c’ are equivalent. Since

$$r(f(x) \otimes g) = r(f(x_1) \otimes g)\varepsilon g(f(x_2)) = r(x_1 \otimes g)\varepsilon f(x_2))$$

we have $r(f(x_1) \otimes g)x_0 = \varepsilon f(x_2)r(x_1 \otimes g)x_0$, then Condition b’ is equivalent to

$$r(f(x_1) \otimes g)x_0 = x.$$  

We are ready for our main result.
Theorem 3.5. The categories $D(1, \text{id}, 1)$ and $D(u, \iota, 1)$ are braided tensor categories. The remaining 6 categories found in [8] are non-braidable.

Proof. By [3, Theorem 5.4], the only possible option for $v$ and $w$ is for there to be the identity. Then the categories $D(1, \text{id}, 1)$ and $D(u, \iota, 1)$ have associated at most a single pair $(\text{id}, \text{id})$, which would give it a braided structure. For the remaining six categories, we already know that they are non-braidable.

Since $D(1, \text{id}, 1)$ has trivial associativity and $	ext{Comod}(H)$ is braided then the braiding for $D(1, \text{id}, 1)$ is

$$c_{[V,a],[W,b]} = [c_{V,W}, ab], \quad \text{for all } V, W \in \text{Comod}(H), a, b \in C_2.$$  (18)

Over $D(u, \iota, 1)$ it is enough to check Equations (1) and (2) where the associativity is not trivial. Since $(f \otimes \text{id})(\text{id} \otimes g) = (\text{id} \otimes g)(f \otimes \text{id})$ for any $f, g$ morphisms in the category, also the braiding given in (18) also satisfies the desired Equations. □

Corollary 3.6. For $X \in \text{Comod}(\land V \# kC_2)$, $r(\iota(x_{-1}) \otimes g)x_0 = x$, for all $x \in X$.

Remark 3.7. Since $	ext{Comod}(H)$ is not symmetric, then these two categories are not symmetric either.

References


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