NILPOTENT AND LINEAR COMBINATION OF IDEMPOTENT MATRICES

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Abstract. A ring \(R\) is Zhou nil-clean if every element in \(R\) is the sum of a nilpotent and two tripotents. Let \(R\) be a Zhou nil-clean ring. If \(R\) is of bounded index or 2-primal, we prove that every square matrix over \(R\) is the sum of a nilpotent and a linear combination of two idempotents. This provides a large class of rings over which every square matrix has such decompositions by nilpotent and linear combination of idempotent matrices.

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1. Introduction

Throughout, all rings are associative with an identity. Very recently, Zhou investigated a class of rings in which elements are the sum of a nilpotent and two tripotents that commute (see [7]). We call such ring a Zhou nil-clean ring. Many elementary properties of such rings are investigated in [4].

Decomposition of a matrix into the sum of simple matrices is of interest. In this paper, we consider a linear combination of the form

\[ P = N + c_1P_1 + c_2P_2, \]

where \(N\) is a nilpotent matrix and \(P_1, P_2\) are idempotent matrices and \(c_1\) and \(c_2\) are scalars. Such decomposition of matrices over Zhou nil-clean rings is thereby determined in this way. A ring \(R\) is of bounded index if there exists \(m \in \mathbb{N}\) such that \(x^m = 0\) for all nilpotent \(x \in R\). A ring \(R\) is 2-primal if its primal radical coincides with the set of nilpotents in \(R\) [3]. For instance, every commutative (reduced) ring is 2-primal. Let \(R\) be Zhou nil-clean. If \(R\) is of bounded index or 2-primal, we prove that every square matrix over \(R\) is the sum of a nilpotent and linear combination of two idempotent matrices. This provides a large class of rings over which every square matrix has such decompositions by nilpotent and linear combination of idempotent matrices.
We use $N(R)$ to denote the set of all nilpotent elements in $R$. $N$ stands for the set of all natural numbers.

2. Zhou nil-clean rings

**Definition 2.1.** A ring $R$ is a Zhou ring if every element in $R$ is the sum of two tripotents that commute.

The structure of Zhou rings was studied in [6]. We now investigate matrices over Zhou rings. We begin with

**Lemma 2.2.** Every square matrix over $\mathbb{Z}_3$ is the sum of two idempotents and a nilpotent.

**Proof.** See [5, Lemma 2.1].

**Lemma 2.3.** Every square matrix over $\mathbb{Z}_5$ is the sum of a nilpotent and a linear combination of two idempotent matrices.

**Proof.** As every matrix over $\mathbb{Z}_5$ is similar to a companion matrix, we may assume

$$A = \begin{pmatrix}
0 & c_0 \\
1 & 0 & c_1 \\
1 & 0 & c_2 \\
\vdots & \ddots & \ddots \\
\vdots & \ddots & \ddots & \ddots \\
0 & c_{n-2} & \ddots & \ddots & 0 \\
1 & 0 & \cdots & \cdots & c_{n-1}
\end{pmatrix}.
$$

Case I. $c_{n-1} = 0$. Choose

$$W = \begin{pmatrix}
0 & 0 \\
1 & 0 & 0 \\
1 & 0 & 0 \\
\vdots & \ddots & \ddots \\
\vdots & \ddots & \ddots & \ddots \\
0 & 0 & \cdots & \cdots & 0 \\
1 & 0 & \cdots & \cdots & 0 & 1
\end{pmatrix},
$$

$$E_1 = \begin{pmatrix}
0 & c_0 \\
0 & 0 & c_1 \\
0 & 0 & c_2 \\
\vdots & \ddots & \ddots \\
\vdots & \ddots & \ddots & \ddots & \ddots \\
0 & c_{n-2} & \ddots & \cdots & 0 \\
0 & 1 & \cdots & \cdots & 1
\end{pmatrix},
$$

$$E_2 = \begin{pmatrix}
0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0 \\
\vdots & \ddots & \ddots \\
\vdots & \ddots & \ddots & \ddots & \ddots \\
0 & 0 & \cdots & \cdots & 0 \\
0 & 1 & \cdots & \cdots & 1
\end{pmatrix}.$$
Then $E_1^2 = E_1$ and $E_2^2 = E_2$, and so $A = E_1 + (-1)E_2 + W$.  

Case II. $c_{n-1} = 1$. Choose 

$$W = \begin{pmatrix} 0 & 0 \\ 1 & 0 & 0 \\ \vdots & \vdots \\ \cdots & 0 & 0 \\ 1 & 0 \end{pmatrix}, E = \begin{pmatrix} 0 & c_0 \\ 0 & 0 & c_1 \\ 0 & 0 & c_2 \\ \vdots & \vdots \\ \cdots & 0 & c_{n-2} \\ 0 & 1 \end{pmatrix}.$$ 

Then $E_2^2 = E_1$, and so $A = E_1 + 0 + W$.  

Case III. $c_{n-1} = -1$. Choose 

$$W = \begin{pmatrix} 0 & 0 \\ 1 & 0 & 0 \\ \vdots & \vdots \\ \cdots & 0 & 0 \\ 1 & 0 \end{pmatrix}, E = \begin{pmatrix} 0 & c_0 \\ 0 & 0 & c_1 \\ 0 & 0 & c_2 \\ \vdots & \vdots \\ \cdots & 0 & c_{n-2} \\ 0 & 1 \end{pmatrix}.$$ 

Then $E_2^2 = E_1$, and so $A = (-1)E_1 + 0 + W$.  

Case IV. $c_{n-1} = 2$. Choose 

$$W = \begin{pmatrix} 0 & 0 \\ 1 & 0 & 0 \\ \vdots & \vdots \\ \cdots & 0 & 0 \\ 1 & 0 \end{pmatrix}, E_1 = \begin{pmatrix} 0 & c_0 \\ 0 & 0 & c_1 \\ 0 & 0 & c_2 \\ \vdots & \vdots \\ \cdots & 0 & c_{n-2} \\ 0 & 1 \end{pmatrix},$$ 

$$E_2 = \begin{pmatrix} 0 \\ 0 & 0 \\ 0 & 0 & 0 \\ \vdots & \vdots \\ \cdots & 0 & 0 \\ 0 & 1 \end{pmatrix}.$$ 

Then $E_1^2 = E_1$ and $E_2^2 = E_2$, and so $A = E_1 + E_2 + W$.  


Case IV. $c_{n-1} = -2$. Choose

$$W = \begin{pmatrix}
0 & 0 \\
1 & 0 & 0 \\
1 & 0 & 0 \\
\vdots & \vdots & \vdots \\
\vdots & 0 & 0 \\
1 & 0 
\end{pmatrix},
E_1 = \begin{pmatrix}
0 & 0 & c_0 \\
0 & 0 & c_1 \\
0 & 0 & c_2 \\
\vdots & \vdots & \vdots \\
\vdots & 0 & c_{n-2} \\
0 & 1 
\end{pmatrix},
E_2 = \begin{pmatrix}
0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0 \\
\vdots & \vdots & \vdots \\
\vdots & 0 & 0 \\
0 & 1 
\end{pmatrix}.$$ 

Then $E_1^2 = E_1$ and $E_2^2 = E_2$, and so $A = (-1)E_1 + (-1)E_2 + W$.

Therefore we complete the proof. \hfill \Box

Recall that a ring $R$ is a Yaqub ring if it is the subdirect product of $\mathbb{Z}_3$’s. A ring $R$ is a Bell ring if it is the subdirect product of $\mathbb{Z}_5$’s. We have

**Lemma 2.4.** Every Zhou ring is isomorphic to a strongly nil-clean ring of bounded index, a Yaqub ring, a Bell ring or products of such rings.

**Proof.** See [5, Lemma 2.3]. \hfill \Box

**Lemma 2.5.** (See [2, Lemma 6.6]) Let $R$ be of bounded index. If $J(R)$ is nil, then $J(M_n(R))$ is nil for all $n \in \mathbb{N}$.

We are ready to prove the following.

**Theorem 2.6.** Let $R$ be a Zhou nil-clean ring of bounded index. Then every square matrix over $R$ is the sum of a nilpotent and linear combination of two idempotent matrices.

**Proof.** In view of Lemma 2.3, $R$ is isomorphic to $R_1, R_2, R_3$ or the products of these rings, where $R_1$ is a strongly nil-clean ring of bounded index, $R_2$ is a Yaqub ring and $R_3$ is a Bell ring.

Step 1. Let $A \in M_n(R_1)$. In view of [2, Corollary 6.8], there exist an idempotent $E \in M_n(R_1)$ and $W \in N(M_n(R_1))$ such that $A = E + W$. 

Then $E_1^2 = E_1$ and $E_2^2 = E_2$, and so $A = (-1)E_1 + (-1)E_2 + W$.

Therefore we complete the proof. \hfill \Box
Step 2. Let $A \in M_n(R_2)$, and let $S$ be the subring of $R_2$ generated by the entries of $A$. That is, $S$ is formed by finite sums of monomials of the form: $a_1a_2 \cdots a_m$, where $a_1, \ldots, a_m$ are entries of $A$. Since $R_2$ is a commutative ring in which $3 = 0$, $S$ is a finite ring in which $x = x^3$ for all $x \in S$. Thus, $S$ is isomorphic to finite direct product of $\mathbb{Z}_3$. As $A \in M_n(S)$, it follows by Lemma 2.1 that $A$ is the sum of two idempotents and a nilpotent matrix over $S$.

Step 3. Let $A \in M_n(R_3)$, and let $S$ be the subring of $R_3$ generated by the entries of $A$. Analogously, $S$ is isomorphic to finite direct product of $\mathbb{Z}_5$. As $A \in M_n(S)$, it follows by Lemma 2.2 that $A$ is the sum of a linear combination of two idempotents and a nilpotent matrix over $S$.

Let $A \in M_n(R)$. We may write $A = (A_1, A_2, A_3)$ in $M_n(R_1) \times M_n(R_2) \times M_n(R_3)$, where $A_1 \in M_n(R_1), A_2 \in M_n(R_2), A_3 \in M_n(R_3)$. According to the preceding discussion, we obtain the result. \hfill \Box

**Example 2.7.** Let $n \geq 2$ be an integer, if $n = 2^k3^l5^m$, then every square matrix over $R = \mathbb{Z}_n$ is a linear combination of two idempotents and a nilpotent.

**Proof.** It is obvious by [5, Example 3.5] that $R$ is a Zhou nil-clean ring, also it is clear that $R$ is of bounded index. Then the result follows from Theorem 2.5. \hfill \Box

3. **2-Primal rings**

An element $w$ in a ring $R$ is called strongly nilpotent if any chain $x_1 = x, x_2, x_3, \ldots$ with $x_{n+1} \in x_nRx_n$ forces $x_m = 0$ for some $m \in \mathbb{N}$. Let $P(R)$ be the primal radical of $R$, i.e., the intersection of all prime ideals of $R$. Then $P(R)$ is exactly the set of all strongly nilpotents in $R$ [1, Remark 2.8]. We derive

**Theorem 3.1.** Let $R$ be a ring. Then the following are equivalent:

1. $R$ is 2-primal and Zhou nil-clean.
2. $a - a^5 \in R$ is strongly nilpotent for all $a \in R$.
3. $R/P(R)$ has the identity $x = x^5$.
4. Every element in $R$ is the sum of two tripotents and a strongly nilpotent that commute.

**Proof.** (1) $\Rightarrow$ (2) This is obvious, as every nilpotent in $R$ is strongly nilpotent.

(2) $\Rightarrow$ (3) Since every strongly nilpotent in $R$ is contained in $P(R)$, we are through.

(3) $\Rightarrow$ (4) Let $a \in R$. Then $a = a^5$; hence, $a - a^5 \in P(R)$ is nilpotent. Thus, $R$ is Zhou nil-clean. In view of [7, Theorem 2.11], every element in $R$ is the sum of
two tripotents $u, v$ and a nilpotent $w$ that commute. Write $w^n = 0 (n \in \mathbb{N})$. Then $\overline{w} = \overline{w^5} \in R/P(R)$. Hence, $w \in P(R)$, i.e., $w$ is strongly nilpotent, as desired.

(4) $\Rightarrow$ (1) As every strongly nilpotent in $R$ is nilpotent, $R$ is Zhou nil-clean, by [7, Theorem 2.11]. In view of [7, Theorem 2.11], $2 \times 3 \times 5 \in N(R)$. Write $2^6 \times 3^7 \times 5^9 = 0(n \in \mathbb{N})$. Since $(2, 3, 5) = 1$, by the Chinese Remainder Theorem, $R \cong R_1 \times R_2 \times R_3$, where $R_1 = R/2^6R$, $R_2 = R/3^7R$ and $R_3 = R/5^9R$. Step 1. Let $a \in N(R_1)$. Then $a = e + w$ with $e^3 = e, w \in P(R)$ and $ae = ea$. As $2 \in N(R_1)$, we see that $2 \in P(R_1)$, as it is central. Hence, $a^2 - a^4 \in P(R)$, and so $a(a - a^3) \in P(R)$. As $P(R)$ is an ideal, we see that $(a - a^3)^2 \in P(R)$. Hence, $(a^3 - a^5)^2 \in P(R)$. It follows that $(a - a^5)^2 \in P(R)$. This implies that $a^2 \in P(R)$. This implies that $e^2 \in P(R)$, and so $e = e^3 \in P(R)$. Therefore $a \in P(R)$. Thus, $N(R) \subseteq P(R)$; hence, $R_1$ is 2-primal.

Step 2. Let $a \in N(R_2)$. Then $a = e + w$ with $e^3 = e, w \in P(R)$ and $ae = ea$. As $3 \in N(R_1)$, we see that $3 \in P(R_1)$, as it is central. Hence, $a - a^3 \in P(R)$. Hence, $a^3 - a^5 = a^2(a - a^3) \in P(R)$. It follows that $a - a^5 = (a - a^3) + (a^3 - a^5) \in P(R)$. This implies that $a \in P(R)$, and so $N(R) \subseteq P(R)$; hence, $R_2$ is 2-primal.

Step 3. Let $a \in N(R_3)$. Then there exist two tripotents $e, f \in R$ and a strongly nilpotent $w \in R$ that commute such that $a = e + f + w$. As $5 \in N(R_3)$, we easily see that $5 \in P(R_3)$, as it is central. Hence, $a^5 \equiv e^5 + f^5 (\text{mod } P(R))$. Hence, $a^5 \equiv e + f = a$, and so $a \in P(R)$. This shows that $R_3$ is 2-primal.

Therefore $R$ is 2-primal, as asserted. $\square$

Corollary 3.2. Let $R$ be a ring. Then the following are equivalent:

(1) $R$ is 2-primal and Zhou nil-clean.

(2) Every element in $R$ is the sum of four idempotents and a strongly nilpotent that commute.

Proof. (1) $\Rightarrow$ (2) This is obvious, by [4, Theorem 2.5]. (2) $\Rightarrow$ (1) Let $a \in R$. Then there exist idempotents $e, f, g, h \in R$ and a strongly nilpotent $w \in R$ that commute such that $2 - a = e + f + g + h + w$. Hence, $a = (1 - e) - f + (1 - g) - h - w$. Obviously, $(1 - e) - f, (1 - g) - h \in R$ are both tripotents. Therefore $a$ is the sum of two tripotents and a strongly nilpotent that commute. According to Theorem 3.1, $R$ is 2-primal and Zhou nil-clean. $\square$

Theorem 3.3. Every subring of 2-primal Zhou nil-clean rings is 2-primal Zhou nil-clean.

Proof. Let $S$ be a subring of a 2-primal Zhou nil-clean $R$. For any $a \in S$, we have $a \in R$. By virtue of Theorem 3.1, $a - a^5 \in P(R)$. 


Given any chain \( x_1 = a - a^5, x_2, x_3, \ldots \) in \( S \) with \( x_{n+1} \in x_n S x_n \), we see that this chain is a chain in \( R \) with \( x_{n+1} \in x_n R x_n \). Thus, we can find some \( m \in \mathbb{N} \) such that \( x_m = 0 \). This implies that \( a - a^5 \) is strongly nilpotent. Hence, \( a - a^5 \in P(S) \).

By using Theorem 3.1 again, \( S \) is a 2-primal Zhou nil-clean ring.

Consequently the center of a 2-primal Zhou nil-clean ring is 2-primal Zhou nil-clean. Every corner of 2-primal Zhou nil-clean rings is 2-primal Zhou nil-clean.

**Corollary 3.4.** Every finite subdirect product of 2-primal Zhou nil-clean rings is 2-primal Zhou nil-clean ring.

**Proof.** Let \( R \) be the subdirect product of 2-primal Zhou nil-clean rings \( R_1, \ldots, R_n \). Then \( R \) is isomorphic to the subring of \( R_1 \times \cdots \times R_n \). In view of Theorem 5.3, \( R \) is a 2-primal Zhou nil-clean ring.

**Example 3.5.** Let \( R \) be a ring. Set \( S = \{(x, y) \in R \times R \mid x - y \in J(R)\} \), which is a subring of \( R \times R \). Then \( R \) is 2-primal Zhou nil-clean if and only if \( S \) is 2-primal Zhou nil-clean.

**Proof.** \( \Rightarrow \) Clearly, \( S \) is a subring of \( R \times R \). Thus, \( S \) is 2-primal Zhou nil-clean.

\( \Leftarrow \) Since \( R \) is a homomorphic image of \( S \), we easily obtain the result.

**Example 3.6.** Let \( V \) be a countably-infinite-dimensional vector space over \( \mathbb{Z}_5 \), with \( \{v_1, v_2, \ldots\} \) a basis, let

\[ A = \{f \in \text{End}(V) \mid \text{rank}(f) < \infty, f(v_i) \in \sum_{k=1}^{i} v_k \mathbb{Z}_5 \text{ for all } i \in \mathbb{N}\}; \]

and let \( R \) be the \( \mathbb{Z}_5 \)-algebra of \( \text{End}(V) \) generated by \( A \) and the identity endomorphism. Then \( R \) is 2-primal Zhou nil-clean.

**Proof.** In view of [3, Example 4.2.20],

\[ P(R) = \{f \in A \mid f(v_i) \in \sum_{k=1}^{i-1} v_k \mathbb{Z}_5 \text{ for all } i \in \mathbb{N}\}, \]

and then \( R/P(R) \) is isomorphic to the ring of all eventually-constant sequences in the direct product of \( \mathbb{Z}_5 \)'s; hence, \( R/P(R) \) has the identity \( x = x^5 \). Therefore \( a - a^5 \in P(R) \) for all \( a \in R \). By using Theorem 3.1, \( R \) is a 2-primal Zhou nil-clean ring, as asserted.

**Proposition 3.7.** Let \( R \) be a ring. Then the following are equivalent:

1. \( R \) is 2-primal Zhou nil-clean.
(2) $T_n(R)$ is 2-primal Zhou nil-clean for some $n \in \mathbb{N}$.
(3) $T_n(R)$ is 2-primal Zhou nil-clean for all $n \in \mathbb{N}$.

**Proof.** (1) $\Rightarrow$ (3) Let $I = \{ \begin{pmatrix} 0 & a_{12} & \cdots & a_{1n} \\ 0 & 0 & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{pmatrix} \in T_n(R) \mid$ each $a_{ii} = 0 \}$. Then $I$ is a nilpotent ideal of $T_n(R)$. Since $T_n(R)/I \cong \bigoplus_{i=1}^{n} R_i$ with each $R_i = R$, the finite direct product $\bigoplus_{i=1}^{n} R_i$ is Zhou nil-clean. It is obvious that $T_n(R)$ is Zhou nil-clean.

Let $x \in N(T_n(R))$. Then $\overline{x} \in N(T_n(R)/I)$. Given any chain $x_1 = x, x_2, x_3, \ldots$ in $T_n(R)$ with $x_{m+1} \in x_m T_m(R) x_m$, we get a chain $\overline{x_1} = \overline{x}, \overline{x_2}, \overline{x_3}, \ldots$ in $T_m(R)/I$ with $\overline{x_{m+1}} \in \overline{x_m}(T_m(R)/I)\overline{x_m}$. As $\overline{x} \in T_n(R)/I$ is strongly nilpotent, we see that $\overline{x_k} = 0$ for some $k \in \mathbb{N}$, i.e., $x_k \in I$. Since $I^n = 0$, we see that $x_{k+n} \in I^n = 0$, and so $x \in T_n(R)$ is strongly nilpotent. Hence, $T_n(R)$ is 2-primal, as asserted.

(3) $\Rightarrow$ (2) This is obvious.

(2) $\Rightarrow$ (1) Clearly, $R$ is isomorphic to a subring of $T_n(R)$, thus we obtain the result by Theorem 3.3. $\square$

**Theorem 3.8.** Let $R$ be a 2-primal Zhou nil-clean ring. Then every square matrix over $R$ is the sum of a nilpotent and linear combination of two idempotent matrices.

**Proof.** Since $R$ is a Zhou nil-clean ring, it follows by [7, Theorem 2.11] that $J(R)$ is nil and $R/J(R)$ has the identity $x = x^5$. Hence, $R/J(R)$ is Zhou nil-clean of bounded index 5. By virtue of Theorem 2.5, every matrix in $M_n(R/J(R))$ is the sum of a nilpotent and linear combination of two idempotent matrices. Clearly, $J(R) \subseteq N(R) = P(R) \subseteq J(R)$, we have $J(R) = P(R)$. Therefore $M_n(J(R)) = M_n(P(R)) = P(M_n(R))$ is nil. It follows from $M_n(R/J(R)) \cong M_n(R)/M_n(J(R))$ that every matrix in $M_n(R)$ is the sum of a nilpotent and linear combination of two idempotent matrices. $\square$

**Corollary 3.9.** Let $R$ be a commutative Zhou nil-clean ring. Then every square matrix over $R$ is the sum of a nilpotent and linear combination of two idempotent matrices.

**Proof.** Since every commutative ring is 2-primal, we obtain the result by Theorem 3.8. $\square$

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