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ON THE DOT PRODUCT GRAPH OF A COMMUTATIVE RING II

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Abstract. In 2015, the second-named author introduced the *dot product graph* associated to a commutative ring A. Let A be a commutative ring with nonzero identity, $1 \leq n < \infty$ be an integer, and $R = A \times A \times \cdots \times A$ (n times). We recall that the *total dot product graph* of R is the (undirected) graph $TD(R)$ with vertices $R^* = R \setminus \{(0, 0, \ldots, 0)\}$, and two distinct vertices x and y are adjacent if and only if $x \cdot y = 0 \in A$ (where $x \cdot y$ denotes the normal dot product of x and y). Let $Z(R)$ denote the set of all zero-divisors of R. Then the *zero-divisor dot product graph* of R is the induced subgraph $ZD(R)$ of $TD(R)$ with vertices $Z(R)^* = Z(R) \setminus \{(0, 0, \ldots, 0)\}$. Let $U(R)$ denote the set of all units of R. Then the *unit dot product graph* of R is the induced subgraph $UD(R)$ of $TD(R)$ with vertices $U(R)$. In this paper, we study the structure of $TD(R)$, $UD(R)$, and $ZD(R)$ when $A = Z_n$ or $A = GF(p^n)$, the finite field with p^n elements, where $n \geq 2$ and p is a prime positive integer.

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1. Introduction

Let R be a commutative ring with $1 \neq 0$. Then $Z(R)$ denote the set of zero-divisors of R and the group of units of R will be denoted by $U(R)$. As usual, Z_n denotes the ring of integers modulo n. The nonzero elements of $S \subseteq R$ will be denoted by S^*. Over the past several years, there has been considerable attention in the literature to associating graphs with commutative rings (and other algebraic structures) and studying the interplay between ring-theoretic and graph-theoretic properties (for example, see [1]–[20] and [22]–[26]). In particular, as in [9], the *zero-divisor graph* of R is the (simple) graph $\Gamma(R)$ with vertices $Z(R) \setminus \{0\}$, and distinct vertices x and y are adjacent if and only if $xy = 0$. This concept is due to Beck [16], who let all the elements of R be vertices and was mainly interested in coloring. The zero-divisor graph of a ring R has been studied extensively by many authors.

In 2015, Badawi [15] introduced the *dot product graph* associated to a commutative ring A. Let A be a commutative ring with nonzero identity, $1 \leq n < \infty$ be an integer, and $R = A \times A \times \cdots \times A$ (n times). We recall from [1] that the *total dot product graph* of R is the (undirected) graph $TD(R)$ with vertices $R^* = R \setminus \{(0, 0, \ldots, 0)\}$, and two distinct vertices x and y are adjacent if and only if $x \cdot y = 0 \in A$ (where $x \cdot y$ denotes the normal dot product of x and y). Let $Z(R)$ denote the set of all zero-divisors of R. Then the *zero-divisor dot product graph* of R is the induced subgraph $ZD(R)$ of $TD(R)$ with vertices $Z(R)^* = Z(R) \setminus \{(0, 0, \ldots, 0)\}$. Let $U(R)$ denote the set of all units of R. Then the *unit dot product graph* of R is the induced subgraph $UD(R)$ of $TD(R)$ with vertices $U(R)$. Let $p \geq 2$ be a prime integer, $n \geq 1$, $A = GF(p^n)$ be the finite field with p^n elements, and $R = A \times A$. In Section 2 of this paper, we study the structure of $ZD(R)$, $UD(R)$, and $TD(R)$. Let $n \geq 2$, $A = Z_n$, and $R = A \times A$. In Section 3 of this paper, we study the structure of $UD(R)$. In Section 4, we study some induced subgraphs of $ZD(Z_n \times Z_n)$, where $n \geq 2$. In Section 5, we introduce the *equivalence unit dot product* of R, $EUD(R)$ and we show that $UD(R)$, can be recovered from $EUD(R)$.

Let G be a graph. Two vertices v_1, v_2 of G are said to be *adjacent* in G if v_1, v_2 are connected by an edge (line segment) of G and we write $v_1 - v_2$. A finite sequence of edges from a vertex v_1 of G to a vertex v_2 of G is called a *path* of G and we write $v_1 - a_1 - a_2 - \cdots - a_k - v_2$, where $k < \infty$ and the a_i, $1 \leq i \leq k$, are some distinct vertices of G. Hence it is clear that every edge of G is a path of G, but not every path of G is an edge of G. We say that G is *connected* if there is a path between any two distinct vertices of G. At the other extreme, we say that G is *totally disconnected* if no two vertices of G are adjacent. We denote the complete graph on n vertices by K_n, recall that a graph G is called complete if every two vertices of G are adjacent and the complete bipartite graph on m and n vertices by $K_{m,n}$ (recall that $K_{m,n}$ is the graph with two sets of vertices, say V_1, V_2, such that $|V_1| = n, |V_2| = m$, $V_1 \cap V_2 = \emptyset$, every two vertices in V_1 are not adjacent, every two vertices in V_2 are not adjacent, and every vertex in V_1 is adjacent to every vertex in V_2). We will sometimes call a $K_{1,n}$ a *star graph*. We say that two (induced) subgraphs G_1 and G_2 of G are *disjoint* if G_1 and G_2 have no common vertices and no vertex of G_1 (resp., G_2) is adjacent (in G) to any vertex not in G_1 (resp., G_2). Assume that a graph $G = G_1 \cup G_2 \cup \cdots \cup G_n$, where each vertex of G_i

is not connected to a vertex of G_j for every $1 \leq i, j \leq n$ with $i \neq j$. Then we say that G is the *disjoint union* of $G_1, \ldots, G_n$.

2. The structure of $UD(R = A \times A)$ when A is a field

Let p be a positive prime number, $n \geq 2$. Then $A = GF(p^n)$ denotes the finite field with p^n elements. Let $R = A \times A$. Then $TD(R)$ is not connected by [15, Theorem 2.1]. The first two results give a complete description of the structure of $UD(R)$ and $TD(R)$.

Theorem 2.1. *Let $n \geq 1$, $m = 2^n - 1$ and $R = GF(2^n) \times GF(2^n)$. Then*

(1) $ZD(R) = \Gamma(R) = K_{m,m}$.

(2) *$UD(R)$ is the disjoint union of one K_m and $(2^{(n-1)} - 1)$ $K_{m,m}$'s.*

(3) *$TD(R)$ is the disjoint union of one K_m and $2^{(n-1)}$ $K_{m,m}$'s.*

Proof. (1) The result is clear by [15, Theorem 2.1], [9, Theorem 2.1], and [10, Theorem 2.2].

(2) Let $A = GF(2^n)$. Then $R = A \times A$. Let $v_1, v_2 \in U(R)$. Since R is a vector space over A, $v_1 = u(1, a) \in R$ and $v_2 = v(1, b) \in R$ for some $u, v, a, b \in A^*$. Hence v_1 is adjacent to v_2 if and only if $v_1 \cdot v_2 = uv + uvab = 0$ in A if and only if $b = -a^{-1} = a^{-1}$ in A . Thus for each $a \in U(A) = A^*$, let $X_a = \{u(1, a) \mid u \in A^*\}$ and $Y_a = \{u(1, a^{-1}) \mid u \in A^*\}$. It is clear that $|X_a| = |Y_a| = 2^n - 1$. Let $a = 1$. Since $char(A) = char(R) = 2$, $X_a = Y_a$ and the dot product of every two distinct vertices in X_a is zero. Hence every two distinct vertices in X_a are adjacent. Thus the vertices in X_a form the graph K_m that is a complete subgraph of $TD(R)$. Let $a \in U(A)$ such that $a \neq 1$. Since $a^2 \neq 1$ for each $a \in U(A) \setminus \{1\}$, we have $X_a \cap Y_a = \emptyset$, every two distinct vertices in X_a are not adjacent, and every two distinct vertices in Y_a are not adjacent. Since $char(A) = char(R) = 2$, it is clear that every vertex in X_a is adjacent to every vertex in Y_a. Thus the vertices in $X_a \cup Y_a$ form the graph $K_{m,m}$ that is a complete bi-partite subgraph of $TD(R)$. By construction, there are exactly $(2^n - 2)/2 = 2^{n-1} - 1$ disjoint complete bi-partite $K_{m,m}$ subgraphs of $TD(R)$. Hence $UD(R)$ is the disjoint union of one complete subgraph K_m and $(2^{n-1} - 1)$ complete bi-partite $K_{m,m}$ subgraphs.

(3) The claim follows from (1) and (2). $\square$

Theorem 2.2. *Let $p \geq 3$ be a positive prime integer, $n \geq 1$, $m = p^n - 1$, and let $R = GF(p^n) \times GF(p^n)$. Then*

(1) $ZD(R) = \Gamma(R) = K_{m,m}$.

(2) *If $4 \nmid m$, then $UD(R)$ is the disjoint union of $m/2$ $K_{m,m}$'s.*

(3) *If* $4 \mid m$*, then* $UD(R)$ *is the disjoint union of two* K_m*'s and* $(m-2)/2$ $K_{m,m}$*'s.*

(4) *If* $4 \nmid m$*, then* $TD(R)$ *is the disjoint union of* $(m+2)/2$ $K_{m,m}$*'s.*

(5) *If* $4 \mid m$*, then* $TD(R)$ *is the disjoint union of two* K_m*'s and* $m/2$ $K_{m,m}$*'s.*

Proof. (1) The result is clear by [15, Theorem 2.1], [9, Theorem 2.1], and [10, Theorem 2.2].

(2) Let $A = GF(p^n)$. Then $R = A \times A$. Let $v_1, v_2 \in U(R)$. Since R is a vector space over A, $v_1 = u(1,a) \in R$ and $v_2 = v(1,b) \in R$ for some $u, v, a, b \in A^*$. Hence v_1 is adjacent to v_2 if and only if $v_1 \cdot v_2 = uv + uvab = 0$ in A if and only if $b = -a^{-1}$ in A. Since R is a vector space over A, for each $a \in U(A) = A^*$, let $X_a = \{u(1,a) \mid u \in A^*\}$ and $Y_a = \{u(1,-a^{-1}) \mid u \in A^*\}$. It is clear that $|X_a| = |Y_a| = m = p^n - 1$. Since $4 \nmid m$, $U(A) = A^*$ has no elements of order 4. Thus $a^2 \neq -1$ for each $a \in U(A)$. Hence $X_a \cap Y_a = \emptyset$; so every two distinct vertices in X_a are not adjacent, and every two distinct vertices in Y_a are not adjacent. By construction of X_a and Y_a, it is clear that every vertex in X_a is adjacent to every vertex in Y_a. Thus the vertices in $X_a \cup Y_a$ form the graph $K_{m,m}$ that is a complete bi-partite subgraph of $TD(R)$. By construction, there are exactly $m/2$ disjoint complete bi-partite $K_{m,m}$ subgraphs of $TD(R)$. Hence $UD(R)$ is the disjoint union of $m/2$ $K_{m,m}$'s.

(3) Note that $|U(A)| = m$. Since $U(A) = A^*$ is cyclic and $4 \mid m$, $U(A)$ has exactly one subgroup of order 4. Thus $U(A)$ has exactly two elements of order 4, say b, c. Since $a \in U(A)$ is of order 4 if and only if $a^2 = -1$, it is clear that $x^2 = -1$ for some $x \in U(A)$ if and only if $x = b, c$. Let $X_b = \{u(1,b) \mid u \in U(A)\}$ and let $X_c = \{u(1,c) \mid u \in U(A)\}$. It is clear that $|X_b| = |X_c| = m$. Let $H = \{b, c\}$. Then the dot product of every two distinct vertices in X_h is zero for each $h \in H$. Thus every two distinct vertices in X_h are adjacent for every $h \in H$. Thus for each $h \in H$, the vertices in X_h form the graph K_m that is a complete subgraph of $TD(R)$. Let $a \in U(A) \setminus H$, $X_a = \{u(1,a) \mid u \in A^*\}$, and $Y_a = \{u(1,-a^{-1}) \mid u \in A^*\}$. It is clear that $|X_a| = |Ya| = m$. Since $a \notin H$, we have $X_a \cap Y_a = \emptyset$, every two distinct vertices in X_a are not adjacent, and every two distinct vertices in Y_a are not adjacent. By construction, it is clear that every vertex in X_a is adjacent to every vertex in Y_a. Thus the vertices in $X_a \cup Y_a$ form the graph $K_{m,m}$ that is a complete bi-partite subgraph of $TD(R)$. By construction, there are $(m-2)/2$ disjoint $K_{m,m}$ subgraphs. Hence $UD(R)$ is the disjoint union of two K_m's and $(m-2)/2$ $K_{m,m}$'s.

(4) The claim follows from (1) and (2).

(5) The claim follows from (1) and (3). □

In view of Theorem 2.2, we have the following corollary.

Corollary 2.3. *Let $p \geq 3$ be a prime positive integer, and let $R = Z_p \times Z_p$. Then*

(1) $ZD(R) = \Gamma(R) = K_{p-1,p-1}$.

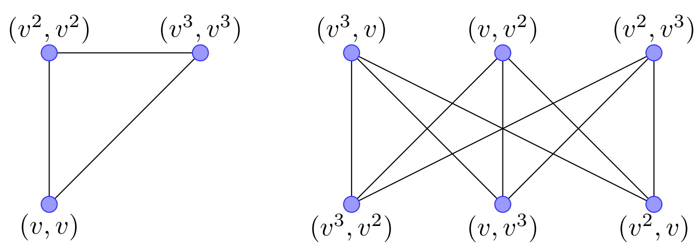
(2) *If* $4 \nmid p-1$, *then* $UD(R)$ *is the disjoint union of* $(p-1)/2$ $K_{p-1,p-1}$.

(3) *If* $4 \mid p-1$, *then* $UD(R)$ *is the disjoint union of two* K_{p-1}*'s and* $(p-3)/2$ $K_{p-1,p-1}$*'s.*

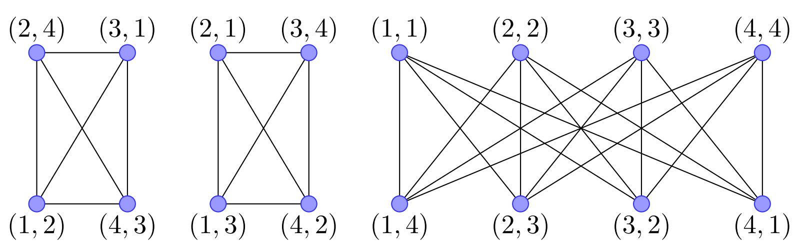
(4) *If* $4 \nmid p-1$, *then* $TD(R)$ *is the disjoint union of* $(p+1)/2$ $K_{p-1,p-1}$*'s.*

(5) *If* $4 \mid p-1$, *then* $TD(R)$ *is the disjoint union of two* K_{p-1}*'s and* $(p-1)/2$ $K_{p-1,p-1}$*'s.*

Example 2.4. Let $A = \frac{Z_2[X]}{(X^2+X+1)}$. Then A is a finite field with 4 elements. Let $v = X + (X^2 + X + 1) \in A$. Since $(A^*, .)$ is a cyclic group and $A^* =< v >$, we have $A = \{0, v, v^2, v^3 = 1 + (X^2 + X + 1)\}$. Let $R = A \times A$. Then the $UD(R)$ is the disjoint union of one K_3 and one $K_{3,3}$ by Theorem 2.1(2). The following is the graph of $UD(R)$.



Example 2.5. Let $A = Z_5$ and $R = A \times A$. Then $UD(R)$ is the disjoint union of two K_4 and one $K_{4,4}$ by Corollary 2.3(3). The following is the graph of $UD(R)$.



3. Unit dot product graph of $R = Z_n \times Z_n$

Let $n > 1$ and write $n = p_1^{k_1} \cdots p_m^{k_m}$, where the p_i's are distinct prime positive integers. Then $U(Z_n) = \{1 \leq a < n \mid \text{a is an integer and } gcd(a, n) = 1\}$. It is known that $U(Z_n)$ is a group under multiplication modulo n and $|U(Z_n)| = \phi(n) = (p_1-1)p_1^{k_1-1}(p_2-1)p_2^{k_2-1}\cdots(p_m-1)p_m^{k_m-1}$.

The following lemma is needed.

Lemma 3.1. *Let n be a positive integer and write $n = p_1^{k_1}p_2^{k_2}\cdots p_r^{k_r}$, where the p_i's are distinct prime positive integers. Then*

(1) *If $4 \mid n$, then $a^2 \not\equiv n-1 \pmod{n}$ for each $a \in U(Z_n)$.*

(2) *If $4 \nmid n$, then $x^2 \equiv n-1 \pmod{n}$ has a solution in $U(Z_n)$ if and only if $4 \mid (p_i - 1)$ for each odd prime factor p_i of n. Furthermore, if $x^2 \equiv n-1$ $(mod\ n)$ has a solution in $U(Z_n)$, then it has exactly 2^{r-1} distinct solutions in $U(Z_n)$ if n is even and it has exactly 2^r distinct solutions in $U(Z_n)$ if n is odd.*

Proof. (1) Suppose that $4 \mid n$. Then $n \geq 4$. Since $4 \nmid (n-2)$, $n-1 \not\equiv 1$ $(mod\ 4)$ and thus $a^2 \not\equiv n-1 \pmod{n}$ for each $a \in U(Z_n)$ by [19, Theorem 5.1].

(2) Suppose that $4 \nmid n$. Then $a^2 \equiv n-1$ $(mod\ n)$ for some $a \in U(Z_n)$ if and only if $a^2 \equiv n-1$ $(mod\ p_i)$ for each odd prime factor p_i of n by [19, Theorem 5.1]. Thus $a^2 \equiv n-1$ $(mod\ n)$ for some $a \in U(Z_n)$ if and only if $(a\ mod\ p_i)^2 \equiv p_i - 1$ $(mod\ p_i)$ for each odd prime factor p_i of n. Since $U(Z_{p_i}) = Z_{p_i}^* = \{1, \ldots, p_i - 1\}$ for each prime factor p_i of n, we have $|U(Z_{p_i})| = p_i - 1$. For each $x \in U(Z_{p_i})$, $1 \leq i \leq r$, let $|x|$ denotes the order of x in $U(Z_{p_i})$. Let p_i, $1 \leq i \leq r$, be an odd prime factor of n. Since $|p_i - 1| = 2$ in $U(Z_{p_i})$, $b^2 = p_i - 1$ in $U(Z_{p_i})$ for some $b \in U(Z_{p_i})$ if and only if $|b| = 4$ in $U(Z_{p_i})$. Since $|U(Z_{p_i})| = p_i - 1$, we conclude that $b^2 = p_i - 1$ in $U(Z_{P_i})$ for some $b \in U(Z_{p_i})$ if and only if $4 \mid (p_i - 1)$. Thus $x^2 \equiv n-1$ $(mod\ n)$ has a solution in $U(Z_n)$ if and only if $4 \mid (p_i - 1)$ for each odd prime p_i factor of n. Suppose that $x^2 \equiv n-1$ $(mod\ n)$ has a solution in $U(Z_n)$. We consider two cases: **Case 1**. Suppose that n is an even integer. Then there are exactly $r-1$ distinct odd prime factors of n. Since $4 \nmid n$, $x^2 \equiv n-1$ $(mod\ n)$ has exactly 2^{r-1} distinct solutions in $U(Z_n)$ by [19, Theorem 5.2]. **Case 2**. Suppose that n is an odd integer. Then there are exactly r distinct odd prime factors of n. Thus $x^2 \equiv n-1$ $(mod\ n)$ has exactly 2^r distinct solutions in $U(Z_n)$ by [19, Theorem 5.2]. □

Let $A = Z_n$, where n is not prime. Then $TD(A \times A)$ is connected by [15, Theorem 2.3]. In the following result, we show that $UD(A \times A)$ is disconnected, and we give a complete description of the structure of $UD(A \times A)$.

Theorem 3.2. *Let $n \geq 3$ be an integer, $R = Z_n \times Z_n$ and $\phi(n) = m$. Write $n = p_1^{k_1}p_2^{k_2}\cdots p_r^{k_r}$, where the p_i's are distinct prime positive integers, $1 \leq i \leq r$. Then*

(1) *If* $4 \mid n$*, then* $UD(R)$ *is the disjoint union of* $m/2$ $K_{m,m}$*'s.*

(2) *If* $4 \nmid n$ *and* $4 \nmid (p_i - 1)$ *for at least one of the* p_i*'s in the prime factorization of* n*, then* $UD(R)$ *is the disjoint union of* $m/2$ $K_{m,m}$*'s.*

(3) *If* $4 \nmid n$ *and* $4 \mid (p_i - 1)$ *for all the odd* p_i*'s in the prime factorization of* n*, then we consider two cases:*

Case I. *If* n *is even, then* $UD(R)$ *is the disjoint union of* $(m/2) - 2^{r-2}$ $K_{m,m}$*'s and* 2^{r-1} K_m*'s.*

Case II. *If* n *is odd, then* $UD(R)$ *is the disjoint union of* $(m/2) - 2^{r-1}$ $K_{m,m}$*'s and* 2^r K_m*'s.*

Proof. Let $A = Z_n$. Then $R = A \times A$. Note that $UD(R)$ has exactly m^2 vertices. Let $v_1, v_2 \in U(R)$. Since R is a vector space over A, $v_1 = u(1, a) \in R$ and $v_2 = v(1, b) \in R$ for some $u, v, a, b \in U(A)$. Hence v_1 is adjacent to v_2 if and only if $v_1 \cdot v_2 = uv + uvab = 0$ in A if and only if $b = -a^{-1}$ in A. Thus for each $a \in U(A)$, let $X_a = \{u(1, a) \mid u \in U(A)\}$ and $Y_a = \{u(1, -a^{-1}) \mid u \in U(A)\}$. It is clear that $|X_a| = |Y_a| = m$.

(1) Since $4 \mid n$, $a^2 \not\equiv n - 1 \pmod{n}$ for each $a \in U(Z_n)$ by Lemma 3.1(1). Hence $X_a \cap Y_a = \emptyset$; so every two distinct vertices in X_a are not adjacent, and every two distinct vertices in Y_a are not adjacent. By construction of X_a and Y_a, it is clear that every vertex in X_a is adjacent to every vertex in Y_a. Thus the vertices in $X_a \cup Y_a$ form the graph $K_{m,m}$ that is a complete bi-partite subgraph of $TD(R)$. By construction, there are exactly $m/2$ disjoint complete bi-partite $K_{m,m}$ subgraphs of $TD(R)$. Hence $UD(R)$ is the disjoint union of $m/2$ $K_{m,m}$'s.

(2) Write $n = p_1^{k_1} p_2^{k_2} \cdots p_r^{k_r}$, where the p_i's are distinct prime positive integers. Since $4 \nmid n$ and $4 \nmid (p_i - 1)$ for at least one of the p_i's, $a^2 \not\equiv n - 1 \pmod{n}$ for each $a \in U(Z_n)$ by Lemma 3.1. Thus by the same argument as in (1), $UD(R)$ is the disjoint union of $m/2$ $K_{m,m}$'s.

(3) Write $n = p_1^{k_1} p_2^{k_2} \cdots p_r^{k_r}$, where the p_i's are distinct prime positive integers. Suppose that $4 \nmid n$ and $4 \mid p_i - 1$ for all the odd p_i's in the prime factorization of n. Let $B = \{b \in U(Z_n) \mid b^2 = n - 1 \text{ in } U(Z_n)\}$ and $C = \{c \in U(Z_n) \mid c^2 \neq n - 1 \text{ in } U(Z_n)\}$. We consider two cases: **Case I**. Suppose that n is even. Then $|B| = 2^{r-1}$ by Lemma 3.1(2) and hence $|C| = m - 2^{r-1}$. For each $a \in B$, we have $X_a = Y_a$ and hence the dot product of every two distinct vertices in X_a is zero. Thus the vertices in X_a form the graph K_m that is a complete subgraph of $TD(R)$. Hence $UD(Z_n)$ has exactly 2^{r-1} disjoint K_m's. For each $a \in C$, we have $X_a \cap Y_a = \emptyset$; so every two distinct vertices in X_a are not adjacent, and every two distinct vertices in Y_a are not adjacent. By construction, it is clear that every vertex in X_a is adjacent to every

vertex in Y_a. Thus the vertices in $X_a \cup Y_a$ form the graph $K_{m,m}$ that is a complete bi-partite subgraph of $TD(R)$. Thus $UD(Z_n)$ has exactly $\frac{m-2^{r-1}}{2} = \frac{m}{2} - 2^{r-2}$ disjoint $K_{m,m}$'s. **Case II**. Suppose that n is odd. Then $|B| = 2^r$ by Lemma 3.1(2) and hence $|C| = m - 2^r$. For each $a \in B$, we have $X_a = Y_a$ and hence the dot product of every two distinct vertices in X_a is zero. Thus the vertices in X_a form the graph K_m that is a complete subgraph of $TD(R)$. Hence $UD(Z_n)$ has exactly 2^r disjoint K_m's. For each $a \in C$, we have $X_a \cap Y_a = \emptyset$, every two distinct vertices in X_a are not adjacent, and every two distinct vertices in Y_a are not adjacent. By construction, it is clear that every vertex in X_a is adjacent to every vertex in Y_a. Thus the vertices in $X_a \cup Y_a$ form the graph $K_{m,m}$ that is a complete bi-partite subgraph of $TD(R)$. Thus $UD(Z_n)$ has exactly $\frac{m-2^r}{2} = \frac{m}{2} - 2^{r-1}$ disjoint $K_{m,m}$'s. □

Recall that a graph G is called *completely disconnected* if every two vertices of G are not connected by an edge in G.

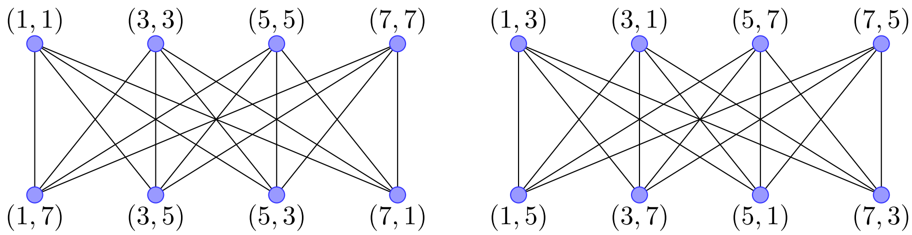
Theorem 3.3. *Let $n \geq 4$ be an even integer, and let $R = Z_n \times Z_n \times \cdots \times Z_n$ (k times), where k is an odd positive integer. Then $UD(R)$ is completely disconnected.*

Proof. Let $x = (x_1, \ldots, x_k), y = (y_1, \ldots, y_k) \in U(R)$. Then $x_i, y_i \in U(Z_n)$ for every i, $1 \leq i \leq k$. Since n is an even integer, x_i and y_i are odd integers for every i, $1 \leq i \leq k$. Hence, since k is an odd integer, $x \cdot y = x_1y_1 + \cdots + x_ky_k$ is an odd integer, and thus $x \cdot y = x_1y_1 + \cdots + x_ky_k \neq 0$ in Z_n, since n is even. Thus $UD(R)$ is completely disconnected. □

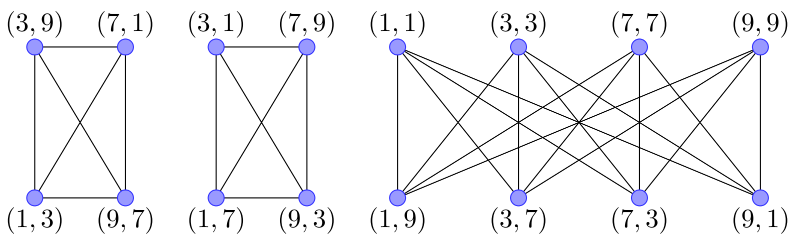
Theorem 3.4. *Let $n \geq 4$ be an even integer, and let $R = Z_n \times Z_n$. Then the vertex $(n/2, n/2)$ in $ZD(R)$ is adjacent to every vertex in $UD(R)$.*

Proof. It is clear that $(\frac{n}{2}, \frac{n}{2})$ is a vertex of $ZD(R)$. Let $u \in U(Z_n)$. Since n is even, u is an odd integer. Thus $u - 1 = 2m$ for some integer m. Hence $\frac{n}{2}(u-1) = \frac{n}{2}(2m) = mn = 0 \in Z_n$. Thus $\frac{n}{2}u = \frac{n}{2}$ in Z_n. Now let $(a, b) \in U(R)$. Then $a, b \in U(Z_n)$ are odd integers. Hence $(a, b)(\frac{n}{2}, \frac{n}{2}) = \frac{n}{2} + \frac{n}{2} = n = 0 \in Z_n$. Thus the vertex $(n/2, n/2)$ in $ZD(R)$ is adjacent to every vertex in $UD(R)$. □

Example 3.5. Let $A = Z_8$ and $R = A \times A$. Then $UD(R)$ is the disjoint union of two $K_{4,4}$ by Theorem 3.2(1). The following is the graph of $UD(R)$.



Example 3.6. Let $A = Z_{10}$ and $R = A \times A$. Then $UD(R)$ is the disjoint union of two K_4 and one $K_{4,4}$ by Theorem 3.2(3, Case I). The following is the graph of $UD(R)$.



4. Subgraphs of the zero-divisor dot product graph of $Z_n \times Z_n$

For an integer $n \geq 2$, let $R_1 = \{(u_1, z_1) \mid u_1 \in U(Z_n) \text{ and } z_1 \in Z(Z_n)\}$ and $R_2 = \{(z_2, u_2) \mid u_2 \in U(Z_n) \text{ and } z_2 \in Z(Z_n)\}$. It is clear that $R_1 \subset Z(Z_n \times Z_n)$ and $R_2 \subset Z(Z_n \times Z_n)$. In this section, we study the induced subgraph $ZD(R_1 \cup R_2)$ of $ZD(Z_n \times Z_n)$ with vertices $R_1 \cup R_2$.

Theorem 4.1. *Let $n \geq 2$, $R = Z_n \times Z_n$, and $\phi(n) = m$. Then*

(1) *If n is prime, then $ZD(R_1 \cup R_2) = ZD(Z_n \times Z_n) = \Gamma(R) = K_{n-1,n-1}$.*

(2) *If n is not prime, then $ZD(R_1 \cup R_2)$ is the disjoint union of of $(n - m)$ $K_{m,m}$'s.*

Proof. (1) Suppose that n is prime. Then it is clear that $R_1 \cup R_2 = Z(Z_n \times Z_n)$. If $n = 2$, then it is trivial to see that $ZD(R_1 \cup R_2) = ZD(Z_n \times Z_n) = \Gamma(R) = K_{1,1}$. If $n \geq 3$, then the claim is clear by Corollary 2.3(1).

(2) Let $A = Z_n$. Suppose that n is not prime. It is clear that every two vertices in R_i are not adjacent for every $i \in \{1, 2\}$. Let $v_1 \in R_1$ and $v_2 \in R_2$. Then $v_1 = u(1, a) \in R_1$ and $v_2 = v(b, 1) \in R_2$ for some $u, v \in U(A)$ and some $a, b \in Z(A)$. Then v_1 is adjacent to v_2 if and only if $v_1 \cdot v_2 = uvb + uva = 0$ in A if and only if $b = -a$ in A. Hence for each $a \in Z(A)$, let $X_a = \{u(1, a) \mid u \in U(A)\}$ and $Y_a = \{u(-a, 1) \mid u \in U(A)\}$. It is clear that $|X_a| = |Y_a| = m$. For each

$a \in Z(A)$, $X_a \cap Y_a = \emptyset$, every two distinct vertices in X_a are not adjacent, and every two distinct vertices in Y_a are not adjacent. By construction, it is clear that every vertex in X_a is adjacent to every vertex in Y_a. Thus the vertices in $X_a \cup Y_a$ form the graph $K_{m,m}$ that is a complete bi-partite subgraph of $ZD(R)$. Since $|R_1| = |R_2| = m(n-m)$ and $R_1 \cap R_2 = \emptyset$, we have $|R_1 \cup R_2| = 2m(n-m)$. Thus $ZD(R_1 \cup R_2)$ is the disjoint union of of $(n-m)$ $K_{m,m}$'s. □

5. Equivalence dot product graph

Let $A = Z_n$ and $R = A \times A$. Define a relation $\sim$ on $U(R)$ such that $x \sim y$, where $x, y \in U(R)$, if $x = (c,c)y$ for some $(c,c) \in U(R)$. It is clear that $\sim$ is an equivalence relation on $U(R)$. If S is an equivalence class of $U(R)$, then there is an $a \in U(A)$ such that $S = \overline{(1,a)} = \{u(1,a) \mid u \in U(Z_n)\}$. Let $E(U(R))$ be the set of all distinct equivalence classes of $U(R)$. We define the *equivalence unit dot product graph* of $U(R)$ to be the (undirected) graph $EUD(R)$ with vertices $E(U(R))$, and two distinct vertices X and Y are adjacent if and only if $a \cdot b = 0 \in A$ for every $a \in X$ and every $b \in Y$ (where $a \cdot b$ denote the normal dot product of a and b). We have the following results.

Theorem 5.1. *Let $n \geq 1$, $m = 2^n - 1$ and $R = GF(2^n) \times GF(2^n)$. Then $EUD(R)$ is the disjoint union of one K_1 and $(2^{(n-1)} - 1)$ $K_{1,1}$'s.*

Proof. Let $A = GF(2^n)$. For each $a \in U(A)$, let X_a and Y_a be as in the proof of Theorem 2.1. Then $X_a, Y_a \in E(U(R))$. Since $|X| = m$ for each $X \in E(U(R))$, we conclude that each K_m of $UD(R)$ is a K_1 of $EUD(R)$ and each $K_{m,m}$ of $UD(R)$ is a $K_{1,1}$ of $EUD(R)$. Hence the claim follows by the proof of Theorem 2.1. □

Theorem 5.2. *Let $p \geq 3$ be a positive prime integer, $n \geq 1$, $m = p^n - 1$, and let $R = GF(p^n) \times GF(p^n)$. Then*

(1) *If $4 \nmid m$, then $EUD(R)$ is the disjoint union of $m/2$ $K_{1,1}$'s.*
(2) *If $4 \mid m$, then $EUD(R)$ is the disjoint union of two K_1's and $(m-2)/2$ $K_{1,1}$'s.*

Proof. Let $A = GF(p^n)$. For each $a \in U(A)$, let X_a and Y_a be as in the proof of Theorem 2.2. Then $X_a, Y_a \in E(U(R))$. Since $|X| = m$ for each $X \in E(U(R))$, we conclude that each K_m of $UD(R)$ is a K_1 of $EUD(R)$ and each $K_{m,m}$ of $UD(R)$ is a $K_{1,1}$ of $EUD(R)$. Hence the claim follows by the proof of Theorem 2.2. □

Theorem 5.3. *Let $n \geq 3$ be an integer, $R = Z_n \times Z_n$ and $\phi(n) = m$. Write $n = p_1^{k_1} p_2^{k_2} \cdots p_r^{k_r}$, where the p_i's are distinct prime positive integers, $1 \leq i \leq r$. Then*

(1) *If $4 \mid n$, then $EUD(R)$ is the disjoint union of $m/2$ $K_{1,1}$'s.*

(2) *If $4 \nmid n$ and $4 \nmid (p_i - 1)$ for at least one of the p_i's in the prime factorization of n, then $EUD(R)$ is the disjoint union of $m/2$ $K_{1,1}$'s.*

(3) *If $4 \nmid n$ and $4 \mid (p_i - 1)$ for all the odd p_i's in the prime factorization of n, then we consider two cases:*

Case I. *If n is even, then $EUD(R)$ is the disjoint union of $(m/2) - 2^{r-2}$ $K_{1,1}$'s and 2^{r-1} K_1's.*

Case II. *If n is odd, then $EUD(R)$ is the disjoint union of $(m/2) - 2^{r-1}$ $K_{1,1}$'s and 2^r K_1's.*

Proof. Let $A = Z_n$. For each $a \in U(A)$, let X_a and Y_a be as in the proof of Theorem 3.2. Then $X_a, Y_a \in E(U(R))$. Since $|X| = m$ for each $X \in E(U(R))$, we conclude that each K_m of $UD(R)$ is a K_1 of $EUD(R)$ and each $K_{m,m}$ of $UD(R)$ is a $K_{1,1}$ of $EUD(R)$. Hence the claim follows by the proof of Theorem 3.2. □

Let $R_1 = \{(u_1, z_1) \mid u_1 \in U(Z_n) \text{ and } z_1 \in Z(Z_n)\}$ and $R_2 = \{(z_2, u_2) \mid u_2 \in U(Z_n) \text{ and } z_2 \in Z(Z_n)\}$, see Section 4. We define a relation $\sim$ on $R_1 \cup R_2$ such that $x \sim y$, where $x, y \in R_1 \cup R_2$, if $x = (c, c)y$ for some $(c, c) \in U(Z_n \times Z_n)$. It is clear that $\sim$ is an equivalence relation on $R_1 \cup R_2$. By construction of R_1 and R_2, it is clear that if $x \sim y$ for some $x, y \in R_1 \cup R_2$, then $x, y \in R_1$ or $x, y \in R_2$. Hence if S is an equivalence class of $R_1 \cup R_2$, then there is an $a \in Z(Z_n)$ such that either $S = (\overline{(1, a)} = \{u(1, a) \mid u \in U(Z_n)\}$ or $S = \overline{(a, 1)} = \{u(a, 1) \mid u \in U(Z_n)\}$. Let $E(R_1 \cup R_2)$ be the set of all distinct equivalence classes of $R_1 \cup R_2$. We define the *equivalence zero-divisor dot product graph* $R_1 \cup R_2$ to be the (undirected) graph $EZD(R_1 \cup R_2)$ with vertices $E(R_1 \cup R_2)$, and two distinct vertices X and Y are adjacent if and only if $a \cdot b = 0 \in A$ for every $a \in X$ and every $b \in Y$ (where $a \cdot b$ denote the normal dot product of a and b). We have the following result.

Theorem 5.4. *Let $n \geq 2, R = Z_n \times Z_n$, and $\phi(n) = m$. Then*

(1) *If n is prime, then $EZD(R_1 \cup R_2) = K_{1,1}$.*

(2) *If n is not prime, then $EZD(R_1 \cup R_2)$ is the disjoint union of of $(n - m)$ $K_{1,1}$'s.*

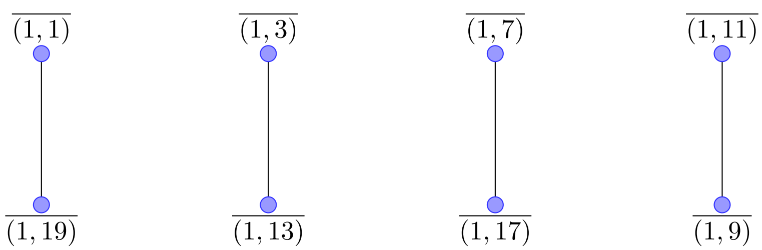
Proof. (1) If n is prime, then $E = \{\overline{(1, 0)}, \overline{(0, 1)}\}$. Thus $EZD(R_1 \cup R_2) = K_{1,1}$.

(2) Suppose that n is not prime, and let $A = Z_n$. For each $a \in Z(A)$, let X_a and Y_a be as in the proof of Theorem 4.1. Then $X_a, Y_a \in E(R_1 \cup R_2)$. Since $|X| = m$

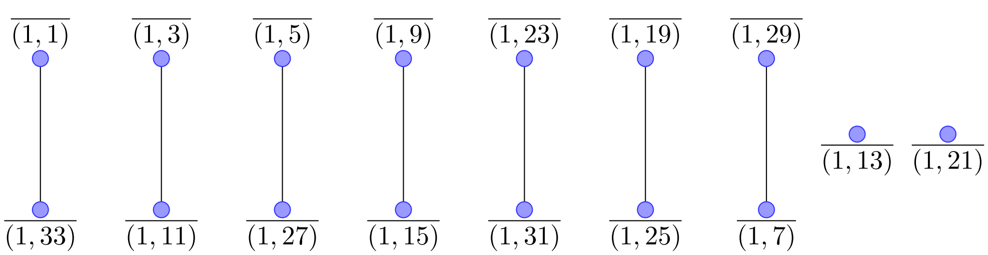
for each $X \in E(R_1 \cup R_2)$, we conclude that each $K_{m,m}$ of $ZD(R_1 \cup R_2)$ is a $K_{1,1}$ of $EZD(R_1 \cup R_2)$. Hence the claim follows by the proof of Theorem 4.1. $\square$

Remark 5.5.
(1) Let $A = Z_n$ and $R = Z_n \times Z_n$. Since for each $X \in E(U(R))$ there exists an $a \in U(A)$ such that $X = \overline{(1,a)} = \{u(1,a) \mid u \in U(A)\}$, note that we can recover the graph $UD(R)$ from the graph $EUD(R)$. However, drawing $EUD(R)$ is much simpler than drawing $UD(R)$.
(2) Since for each $X \in E(R_1 \cup R_2)$ there exists an $a \in Z(Z_n)$ such that either $X = \overline{(1,a)} = \{u(1,a) \mid u \in U(Z_n)\}$ or $X = \overline{(a,1)} = \{u(a,1) \mid u \in U(Z_n)\}$, note that we can recover the graph $ZD(R_1 \cup R_2)$ from the graph $EZD(R_1 \cup R_2)$. However, drawing $EZD(R_1 \cup R_2)$ is much simpler than drawing $ZD(R_1 \cup R_2)$.

Example 5.6. Let $A = Z_{20}$ and $R = A \times A$. Then $EUD(R)$ is the disjoint union of 4 $K_{1,1}$ by Theorem 5.3(1), and thus $UD(R)$ is the disjoint union of 4 $K_{8,8}$. The following is the graph of $EUD(R)$.



Example 5.7. Let $A = Z_{34}$ and $R = A \times A$. Then $EUD(R)$ is the disjoint union of 7 $K_{1,1}$'s and 2 K_1's by Theorem 5.3(3, Case I), and thus $UD(R)$ is the disjoint union of 7 $K_{16,16}$ and 2 K_8. The following is the graph of $EUD(R)$.



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