International Electronic Journal of Algebra
Volume 29 (2021) 223-238
DOI: 10.24330/ieja.852237

SOME PROPERTIES OF THE REPRESENTATION CATEGORY OF TWISTED DRINFELD DOUBLES OF FINITE GROUPS

Deepak Naidu

Received: 21 June 2020; Accepted 7 August 2020
Communicated by Sait Halıcıoğlu

Abstract. A criterion for a simple object of the representation category $\mathrm{Rep}(D^\omega(G))$ of the twisted Drinfeld double $D^\omega(G)$ to be a generator is given, where G is a finite group and ω is a 3-cocycle on G. A description of the adjoint category of $\mathrm{Rep}(D^\omega(G))$ is also given.

Mathematics Subject Classification (2020): 18M20
Keywords: Drinfeld double, quantum double

1. Introduction

Modular tensor categories arise in several diverse areas such as quantum group theory, vertex operator algebras, and rational conformal field theory. Let G be a finite group, let $D(G)$ denote the Drinfeld double of G, a quasi-triangular semisimple Hopf algebra, and let $\mathrm{Rep}(D(G))$ denote the category of finite-dimensional complex representations of $D(G)$. The category $\mathrm{Rep}(D(G))$ is a modular tensor category [1], and it is perhaps the most accessible constructions of a modular tensor category. As such, it is desirable to have a thorough understanding of this category. In this paper, we make a contribution towards this goal. The category $\mathrm{Rep}(D(G))$ is equivalent to the G-equivariantization of Vec_G, and it is also equivalent to the center $\mathcal{Z}(\mathrm{Vec}_G)$ of the tensor category Vec_G of finite-dimensional G-graded complex vector spaces.

In the papers [3,4], R. Dijkgraaf, V. Pasquier, and P. Roche introduce a quasi-triangular semisimple quasi-Hopf algebra $D^\omega(G)$, often called the twisted Drinfeld double of G, where ω is a 3-cocycle on G. When $\omega = 1$ this quasi-Hopf algebra coincides with the Drinfeld double $D(G)$ considered above. The category $\mathrm{Rep}(D^\omega(G))$ of finite-dimensional complex representations of $D^\omega(G)$ is a modular tensor category. Analogous to the $\omega = 1$ case, the category $\mathrm{Rep}(D^\omega(G))$ is equivalent to the G-equivariantization of Vec_G^ω, and it is also equivalent to the center $\mathcal{Z}(\mathrm{Vec}_G^\omega)$ of the tensor category Vec_G^ω of finite-dimensional G-graded complex vector spaces with associativity constraint defined using ω. Every braided group-theoretical fusion category is equivalent to a full fusion subcategory of some $\mathrm{Rep}(D^\omega(G))$, and all such subcategories were parametrized in the paper [12].

This paper contains two main results, stated below. The first gives a criterion for a simple object of $\mathrm{Rep}(D^\omega(G))$ to be a generator, and the second gives a description of the adjoint category of $\mathrm{Rep}(D^\omega(G))$.

Theorem. *Let G be a finite group, let ω be a normalized 3-cocycle on G, and let (a,χ) be a simple object of $\mathrm{Rep}(D^\omega(G))$. Then (a,χ) is a generator of $\mathrm{Rep}(D^\omega(G))$ if and only if the following two conditions hold.*

(a) *The normal closure of a in G is equal to G.*
(b) *For all $b \in Z(G)$ and $\chi' \in \mathrm{Irr}_{\beta_b}(G)$, if $\chi(b)\chi'(a) = \deg\chi \deg\chi'$ (equivalently, (a,χ) and (b,χ') centralize each other), then $b = e$ and $\chi' = 1$.*

Theorem. *Let G be a finite group, and let ω be a normalized 3-cocycle on G. Then*

$$\mathrm{Rep}(D^\omega(G))_{\mathrm{pt}} = \mathcal{S}(Z_\omega(G), [G,G], B)$$

and

$$\mathrm{Rep}(D^\omega(G))_{\mathrm{ad}} = \mathcal{S}([G,G], Z_\omega(G), (B^{\mathrm{op}})^{-1})$$

where $B : Z_\omega(G) \times [G,G] \to \mathbb{C}^\times$ is the G-invariant ω-bicharacter defined in Lemma 4.4.

Organization:

In Section 2, we recall basic facts about the modular tensor category $\mathrm{Rep}(D^\omega(G))$. In Section 3, we prove the first theorem above, and in Section 4, we prove the second theorem.

Convention and notation:

Throughout this paper we work over the field $\mathbb{C}$ of complex numbers. The multiplicative group of nonzero complex numbers is denoted $\mathbb{C}^\times$. Let G be a finite group. The identity element of G is denoted e, and the center of G is denoted $Z(G)$. For any character χ of G, the degree of χ is denoted $\deg\chi$, the complex conjugate of χ is denoted $\overline{\chi}$, and the kernel of χ is denoted $\mathrm{Ker}\,\chi$. Let μ be a 2-cocycle on G with coefficients in $\mathbb{C}^\times$. The set of irreducible μ-characters of G is denoted $\mathrm{Irr}_\mu(G)$. When $\mu = 1$, we write $\mathrm{Irr}(G)$ instead of $\mathrm{Irr}_1(G)$. Finally, the coboundary operator on the space of cochains of G with coefficients in $\mathbb{C}^\times$ is denoted d.

2. Drinfeld doubles of finite groups

Let G be a finite group. As stated earlier, the category $\mathrm{Rep}(D(G))$ of finite-dimensional representations of the Drinfeld double $D(G)$ is a modular tensor category [1]. The simple objects of $\mathrm{Rep}(D(G))$ are in bijection with the set of pairs (a, χ), where a is a representative of a conjugacy class of G, and χ is an irreducible character of the centralizer $C_G(a)$ of a in G. The S-matrix and the T-matrix of $\mathrm{Rep}(D(G))$ are square matrices indexed by the simple objects of $\mathrm{Rep}(D(G))$, and are given by the following formulas [1,2].

$$S_{(a,\chi),(b,\chi')} = \frac{1}{|C_G(a)||C_G(b)|} \sum_{g \in G(a,b)} \overline{\chi}(gbg^{-1})\overline{\chi'}(g^{-1}ag),$$

$$T_{(a,\chi),(b,\chi')} = \delta_{a,b}\delta_{\chi,\chi'} \frac{\chi(a)}{\deg \chi},$$

where $G(a,b)$ denotes the set $\{g \in G \mid agbg^{-1} = gbg^{-1}a\}$.

Let $\omega : G \times G \times G \to \mathbb{C}^\times$ be a normalized 3-cocycle. Then

$$\omega(b,c,d)\omega(a,bc,d)\omega(a,b,c) = \omega(ab,c,d)\omega(a,b,cd)$$

for all $a, b, c, d \in G$, and $\omega(a, b, c) = 1$ if a, b, or c is the identity element. Replacing ω by a cohomologous 3-cocycle, if necessary, we may assume that the values of ω are roots of unity.

For each $a \in G$, define a function $\beta_a : G \times G \to \mathbb{C}^\times$ by

$$\beta_a(x,y) = \frac{\omega(a,x,y)\omega(x,y,y^{-1}x^{-1}axy)}{\omega(x,x^{-1}ax,y)}. \tag{1}$$

The 3-cocycle condition on ω ensures that the relation

$$\beta_{x^{-1}ax}(y,z)\beta_a(x,yz) = \beta_a(xy,z)\beta_a(x,y)$$

holds for all $a, x, y, z \in G$. Therefore, for any $a \in G$, the restriction of β_a to the centralizer $C_G(a)$ of a in G is a normalized 2-cocycle, that is,

$$\beta_a(y,z)\beta_a(x,yz) = \beta_a(xy,z)\beta_a(x,y)$$

for all $x, y, z \in C_G(a)$, and $\beta_a(x, y) = 1$ if x or y is the identity element.

For each $a \in G$, define a function $\gamma_a : G \times G \to \mathbb{C}^\times$ by

$$\gamma_a(x,y) = \frac{\omega(x,y,a)\omega(a,a^{-1}xa,a^{-1}ya)}{\omega(x,a,a^{-1}ya)}.$$

Direct calculations using the 3-cocycle condition of ω show that

$$\frac{\beta_a(x,y)\beta_b(x,y)}{\beta_{ab}(x,y)} = \frac{\gamma_{xy}(a,b)}{\gamma_x(a,b)\gamma_y(x^{-1}ax,x^{-1}bx)}$$

for all $a, b, x, y \in G$. For all $a \in G$, the functions β_a and γ_a are equal when restricted to $C_G(a)$. Therefore, we have

$$\frac{\beta_a(x,y)\beta_b(x,y)}{\beta_{ab}(x,y)} = \frac{\beta_{xy}(a,b)}{\beta_x(a,b)\beta_y(a,b)} \tag{2}$$

for all $a, b \in Z(G)$, and $x, y \in G$.

As stated earlier, the category $\mathrm{Rep}(D^\omega(G))$ of finite-dimensional representations of the twisted Drinfeld double $D^\omega(G)$ is a modular tensor category. The simple objects of $\mathrm{Rep}(D^\omega(G))$ are in bijection with the set of pairs (a, χ), where a is a representative of a conjugacy class of G, and χ is an irreducible β_a-character of the centralizer $C_G(a)$ of a in G. The S-matrix and the T-matrix of $\mathrm{Rep}(D^\omega(G))$ are square matrices indexed by the simple objects of $\mathrm{Rep}(D(G))$, and are given by the following formulas [2].

$$S_{(a,\chi),(b,\chi')} = \sum_{\substack{g \in C\ell_G(a) \\ g' \in C\ell_G(b) \cap C_G(g)}} \overline{\left(\frac{\beta_a(x,g')\beta_a(xg',x^{-1})\beta_b(y,g)\beta_b(yg,y^{-1})}{\beta_a(x,x^{-1})\beta_b(y,y^{-1})} \right)} \overline{\chi}(xg'x^{-1})\overline{\chi}'(ygy^{-1})$$

$$T_{(a,\chi),(b,\chi')} = \delta_{a,b}\delta_{\chi,\chi'}\frac{\chi(a)}{\deg \chi},$$

where $g = x^{-1}ax$, $g' = y^{-1}by$, and $C\ell_G(a)$ denotes the conjugacy class of a in G.

3. Tensor generators

In this section, we give a criterion for a simple object (a, χ) of $\mathrm{Rep}(D^\omega(G))$ to be a tensor generator, that is, the full fusion subcategory given by the intersection of all fusion subcategories of $\mathrm{Rep}(D^\omega(G))$ that contain (a, χ) is $\mathrm{Rep}(D^\omega(G))$.

Let $\mathcal{C}$ be a modular tensor category with braiding c. Two objects $X, Y \in \mathcal{C}$ *centralize* each other if

$$c_{Y,X} \circ c_{X,Y} = \mathrm{id}_{X \otimes Y}\,.$$

Let $\mathcal{D}$ be a full (not necessarily tensor) subcategory of $\mathcal{C}$. In the paper [10], M. Müger defined the *centralizer* of $\mathcal{D}$ in $\mathcal{C}$ as the full subcategory of $\mathcal{C}$, denoted $\mathcal{D}'$, consisting of all objects in $\mathcal{C}$ that centralize every object in $\mathcal{D}$. That is,

$$\mathrm{Obj}(\mathcal{D}') = \{X \in \mathrm{Obj}(\mathcal{C}) \mid c_{Y,X} \circ c_{X,Y} = \mathrm{id}_{X \otimes Y} \text{ for all } Y \in \mathrm{Obj}(\mathcal{D})\}\,.$$

It was shown in [10] that $\mathcal{D}'$ is a fusion subcategory, and that if $\mathcal{D}$ is a fusion subcategory, then $\mathcal{D}'' = \mathcal{D}$; we refer to this result as the *double centralizer theorem.*

We recall the following result from [11].

Proposition 3.1. *Let G be a finite group, and let (a,χ) and (b,χ') be simple objects of* $\mathrm{Rep}(D(G))$*. Then (a,χ) and (b,χ') centralize each other if and only if the following two conditions hold.*

(a) *The conjugacy classes of a and b commute elementwise.*

(b) *For all $g \in G$, $\chi(gbg^{-1})\chi'(g^{-1}ag) = \deg\chi \deg\chi'$.*

Below, we record a special case of the result above.

Proposition 3.2. *Let G be a finite group, and let (a,χ) and (b,χ') be simple objects of* $\mathrm{Rep}(D(G))$*, where b lies in the center of G, so that $\chi' \in \mathrm{Irr}(G)$. Then (a,χ) and (b,χ') centralize each other if and only if the following holds.*

(i) $\chi(b)\chi'(a) = \deg\chi \deg\chi'$.

If $b = e$ or $\chi = 1$, then the condition above is equivalent to the condition

(i′) $a \in \mathrm{Ker}\,\chi'$.

Proof. If $b = e$ or $\chi = 1$, then the equality in condition (i) is equivalent to the equality $\chi'(a) = \deg\chi'$, which is equivalent to condition (i′).

Suppose that (a,χ) and (b,χ') centralize each other. Putting $g = e$ in condition (b) of Proposition 3.1, we get $\chi(b)\chi'(a) = \deg\chi \deg\chi'$, which is condition (i).

Conversely, suppose that condition (i) holds. Since b lies in the center of G, we know that $C_G(b) = G$ and χ' is a character of G. Condition (a) of Proposition 3.1 clearly holds. For all $g \in G$, we have $\chi(gbg^{-1})\chi'(g^{-1}ag) = \chi(b)\chi'(a)$, since b is in the center of G and χ' is a class function on G. By supposition, $\chi(b)\chi'(a) = \deg\chi \deg\chi'$, and so condition (b) of Proposition 3.1 holds. Hence (a,χ) and (b,χ') centralize each other. □

Theorem 3.3. *Let G be a finite group, and let (a,χ) be a simple object of* $\mathrm{Rep}(D(G))$*. Then (a,χ) is a generator of* $\mathrm{Rep}(D(G))$ *if and only if the following two conditions hold.*

(a) *The normal closure of a in G is equal to G.*

(b) *For all $b \in Z(G)$ and $\chi' \in \mathrm{Irr}(G)$, if $\chi(b)\chi'(a) = \deg\chi \deg\chi'$ (equivalently, (a,χ) and (b,χ') centralize each other), then $b = e$ and $\chi' = 1$.*

Proof. By the double centralizer theorem, the simple object (a,χ) is a generator of $\mathrm{Rep}(D(G))$ if and only if the only simple object that centralizes (a,χ) is the trivial simple object $(e,1)$.

Suppose that conditions (a) and (b) in the statement of the theorem hold, and let (b,χ') be a simple object of $\mathrm{Rep}(D(G))$ that centralizes (a,χ). By Proposition 3.1, the conjugacy classes of a and b commute elementwise; combining this fact with condition (a), we deduce that b lies in the center of G, and so χ' lies in $\mathrm{Irr}(G)$. By Proposition 3.2, we must have $\chi(b)\chi'(a) = \deg\chi\deg\chi'$. Applying condition (b), we get $b = e$ and $\chi' = 1$, and it follows that (a,χ) is a generator of $\mathrm{Rep}(D(G))$.

Conversely, suppose that (a,χ) is a generator of $\mathrm{Rep}(D(G))$. Then the only simple object that centralizes (a,χ) is the trivial simple object $(e,1)$. Let $b \in Z(G)$, let $\chi' \in \mathrm{Irr}(G)$, and suppose that $\chi(b)\chi'(a) = \deg\chi\deg\chi'$. Then the simple objects (a,χ) and (b,χ') centralize each other, by Proposition 3.2. Since the only simple object that centralizes (a,χ) is the trivial simple object $(e,1)$, it follows that $b = e$ and $\chi' = 1$, showing that condition (b) holds.

To see that condition (a) holds, let H denote the normal closure of a in G, and suppose that $H \neq G$. By Proposition 3.2, for all $\chi' \in \mathrm{Irr}(G)$, the simple objects (a,χ) and (e,χ') centralize each other if and only if $a \in \mathrm{Ker}\,\chi'$, equivalently, $H \leq \mathrm{Ker}\,\chi'$. Since H is proper in G, the action of G on the coset space G/H is not trivial, and so the corresponding representation contains a nontrivial irreducible constituent; let χ' denote the character of this constituent. Since H is normal in G, it acts trivially on G/H, and so $H \leq \mathrm{Ker}\,\chi'$. It follows that (a,χ) and (e,χ') centralize each other, a contradiction. Hence $H = G$, showing that condition (a) holds. □

Corollary 3.4. *Let G be a finite group with trivial center, and let (a,χ) be a simple object of $\mathrm{Rep}(D(G))$. Then (a,χ) is a generator of $\mathrm{Rep}(D(G))$ if and only if the normal closure of a in G is equal to G.*

Proof. Suppose that the normal closure of a in G is equal to G. To see that condition (b) of Theorem 3.3 holds, let $\chi' \in \mathrm{Irr}(G)$, and suppose that the simple objects (a,χ) and (e,χ') centralize each other. By Proposition 3.2, the element a belongs to $\mathrm{Ker}\,\chi'$. Since $\mathrm{Ker}\,\chi'$ is a normal subgroup of G, the supposition forces $\mathrm{Ker}\,\chi' = G$, equivalently, $\chi' = 1$, and so condition (b) of Theorem 3.3 holds. Hence (a,χ) is a generator of $\mathrm{Rep}(D(G))$.

The converse clearly holds, by Theorem 3.3 □

Next, we address the twisted case. Of course, the untwisted case above is a special case of the twisted case below, but we find it instructive to treat the untwisted case separately, as in [11] and [12]. We recall the following result from [11].

Proposition 3.5. *Let G be a finite group, let ω be a normalized 3-cocycle on G, and let (a,χ) and (b,χ') be simple objects of $\mathrm{Rep}(D^{\omega}(G))$. Then (a,χ) and (b,χ') centralize each other if and only if the following two conditions hold.*

(a) *The conjugacy classes of a and b commute elementwise.*

(b) *For all* $x, y \in G$,
$\frac{\beta_a(x,y^{-1}by)\beta_a(xy^{-1}by,x^{-1})\beta_b(y,x^{-1}ax)\beta_b(yx^{-1}ax,y^{-1})}{\beta_a(x,x^{-1})\beta_b(y,y^{-1})}\chi(xy^{-1}byx^{-1})\chi'(yx^{-1}axy^{-1})$
$= \deg\chi \deg\chi'$.

We will need the following result from [12].

Lemma 3.6. *Let* G *be a finite group, let* ω *be a normalized 3-cocycle on* G*, and let* $a, b, x \in G$*. If* $ab = ba$*, then*

$$\frac{\beta_a(x,x^{-1})}{\beta_a(x,b)\beta_a(xb,x^{-1})} = \frac{\beta_b(x^{-1},x)}{\beta_b(x^{-1},a)\beta_b(x^{-1}a,x)}.$$

Lemma 3.7. *Let* G *be a finite group, and let* μ *be a normalized 2-cocycle on* G*.*

(a) *For all* $a, x, y \in G$,

$$\frac{\mu(y,x^{-1}ax)\mu(yx^{-1}ax,y^{-1})\mu(xy^{-1},yx^{-1}axy^{-1})}{\mu(y,y^{-1})\mu(a,xy^{-1})} = \frac{\mu(x,x^{-1}ax)}{\mu(a,x)}.$$

(b) *For all* $a, x \in G$,

$$\frac{\mu(x,x^{-1})}{\mu(x^{-1},a)\mu(x^{-1}a,x)} = \frac{\mu(x,x^{-1}a)}{\mu(x^{-1}a,x)} = \frac{\mu(x,x^{-1}ax)}{\mu(a,x)}.$$

Proof. That μ is a normalized 2-cocyle means that

$$\mu(y,z)\mu(x,yz) = \mu(xy,z)\mu(x,y)$$

for all $x, y, z \in G$, and $\mu(x, y) = 1$ if x or y is the identity element. Applying the 2-cocycle condition of μ to the triple $(xy^{-1}, yx^{-1}ax, y^{-1})$ gives $\mu(yx^{-1}ax,y^{-1})\mu(xy^{-1},yx^{-1}axy^{-1}) = \mu(ax,y^{-1})\mu(xy^{-1},yx^{-1}ax)$. Making this substitution in the expression

$$\frac{\mu(y,x^{-1}ax)\mu(yx^{-1}ax,y^{-1})\mu(xy^{-1},yx^{-1}axy^{-1})}{\mu(y,y^{-1})\mu(a,xy^{-1})}$$

yields

$$\frac{\mu(y,x^{-1}ax)\mu(ax,y^{-1})\mu(xy^{-1},yx^{-1}ax)}{\mu(y,y^{-1})\mu(a,xy^{-1})}.$$

Applying the 2-cocycle condition of μ to the triple (a, x, y^{-1}) gives $\frac{\mu(ax,y^{-1})}{\mu(a,xy^{-1})} = \frac{\mu(x,y^{-1})}{\mu(a,x)}$. Making this substitution in the expression above yields

$$\frac{\mu(y,x^{-1}ax)\mu(xy^{-1},yx^{-1}ax)\mu(x,y^{-1})}{\mu(y,y^{-1})\mu(a,x)}.$$

Applying the 2-cocycle condition of μ to the triple $(y, y^{-1}, yx^{-1}ax)$ gives $\frac{\mu(y,x^{-1}ax)}{\mu(y,y^{-1})} = \frac{1}{\mu(y^{-1},yx^{-1}ax)}$. Making this substitution in the expression above yields

$$\frac{\mu(xy^{-1},yx^{-1}ax)\mu(x,y^{-1})}{\mu(a,x)\mu(y^{-1},yx^{-1}ax)}.$$

Applying the 2-cocycle condition of μ to the triple $(x, y^{-1}, yx^{-1}ax)$ gives $\frac{\mu(xy^{-1}, yx^{-1}ax)\mu(x,y^{-1})}{\mu(y^{-1}, yx^{-1}ax)} = \mu(x, x^{-1}ax)$. Making this substitution in the expression above yields

$$\frac{\mu(x, x^{-1}ax)}{\mu(a, x)},$$

establishing (a).

Applying the 2-cocycle condition of μ to the triple (x, x^{-1}, a) gives $\frac{\mu(x,x^{-1})}{\mu(x^{-1},a)} = \mu(x, x^{-1}a)$. Making this substitution in the expression

$$\frac{\mu(x, x^{-1})}{\mu(x^{-1}, a)\mu(x^{-1}a, x)}$$

yields

$$\frac{\mu(x, x^{-1}a)}{\mu(x^{-1}a, x)},$$

Applying the 2-cocycle condition of μ to the triple $(x, x^{-1}a, x)$ gives $\frac{\mu(x,x^{-1}a)}{\mu(x^{-1}a,x)} = \frac{\mu(x,x^{-1}ax)}{\mu(a,x)}$, and so the expression above is equal to

$$\frac{\mu(x, x^{-1}ax)}{\mu(a, x)},$$

establishing (b). □

Proposition 3.8. *Let G be a finite group, let ω be a normalized 3-cocycle on G, and let (a, χ) and (b, χ') be simple objects of $\operatorname{Rep}(D^\omega(G))$, where b lies in the center of G, so that $\chi' \in \operatorname{Irr}_{\beta_b}(G)$. Then (a, χ) and (b, χ') centralize each other if and only if the following holds.*

(i) $\chi(b)\chi'(a) = \deg \chi \deg \chi'$.

If $b = e$, then the condition above is equivalent to the condition

(i′) $a \in \operatorname{Ker} \chi'$.

Proof. If $b = e$, then χ' is an ordinary character, and the equality in condition (i) is equivalent to the equality $\chi'(a) = \deg \chi'$, which is equivalent to condition (i′).

Suppose that (a, χ) and (b, χ') centralize each other. Putting $x = y = e$ in condition (b) of Proposition 3.5, we get $\chi(b)\chi'(a) = \deg \chi \deg \chi'$, which is condition (i).

Conversely, suppose that condition (i) holds. Since b lies in the center of G, we know that $C_G(b) = G$, β_b is a 2-cocycle on G, and χ' is a β_b-character of G. Condition (a) of Proposition 3.5 clearly holds. It remains to show that condition

(b) of Proposition 3.5 holds. Let $x, y \in G$. Since b lies in the center of G, the left-hand side of condition (b) of Proposition 3.5 reduces to

$$\frac{\beta_a(x,b)\beta_a(xb,x^{-1})\beta_b(y,x^{-1}ax)\beta_b(yx^{-1}ax,y^{-1})}{\beta_a(x,x^{-1})\beta_b(y,y^{-1})}\chi(b)\chi'(yx^{-1}axy^{-1}). \tag{3}$$

Let $\rho : G \to \mathrm{GL}(V)$ be a projective β_b-representation of G whose character is χ', and let $z \in G$. Then

$$\rho(a)\rho(z) = \beta_b(a,z)\rho(az) = \beta_b(a,z)\rho(z(z^{-1}az)) = \frac{\beta_b(a,z)}{\beta_b(z,z^{-1}az)}\rho(z)\rho(z^{-1}az),$$

and so

$$\rho(z^{-1}az) = \frac{\beta_b(z,z^{-1}az)}{\beta_b(a,z)}[\rho(z)]^{-1}\rho(a)\rho(z).$$

Taking the trace of both sides, we get

$$\chi'(z^{-1}az) = \frac{\beta_b(z,z^{-1}az)}{\beta_b(a,z)}\chi'(a).$$

Putting $z = xy^{-1}$ in the equation above, we get

$$\chi'(yx^{-1}axy^{-1}) = \frac{\beta_b(xy^{-1},yx^{-1}axy^{-1})}{\beta_b(a,xy^{-1})}\chi'(a).$$

Substituting the expression above in (3), we get

$$\frac{\beta_a(x,b)\beta_a(xb,x^{-1})\beta_b(y,x^{-1}ax)\beta_b(yx^{-1}ax,y^{-1})\beta_b(xy^{-1},yx^{-1}axy^{-1})}{\beta_a(x,x^{-1})\beta_b(y,y^{-1})\beta_b(a,xy^{-1})}\chi(b)\chi'(a).$$

Using Lemma 3.6, we see that the expression above is equal to

$$\frac{\beta_b(x^{-1},a)\beta_b(x^{-1}a,x)\beta_b(y,x^{-1}ax)\beta_b(yx^{-1}ax,y^{-1})\beta_b(xy^{-1},yx^{-1}axy^{-1})}{\beta_b(x^{-1},x)\beta_b(y,y^{-1})\beta_b(a,xy^{-1})}\chi(b)\chi'(a).$$

Applying Lemma 3.7 with $\mu = \beta_b$, and noting that $\beta_b(x^{-1},x) = \beta_b(x,x^{-1})$, we see that the expression above reduces to $\chi(b)\chi'(a)$. By supposition, $\chi(b)\chi'(a) = \deg\chi\deg\chi'$, and so condition (b) of Proposition 3.5 holds. Hence (a,χ) and (b,χ') centralize each other. □

Theorem 3.9. *Let G be a finite group, let ω be a normalized 3-cocycle on G, and let (a,χ) be a simple object of $\mathrm{Rep}(D^\omega(G))$. Then (a,χ) is a generator of $\mathrm{Rep}(D^\omega(G))$ if and only if the following two conditions hold.*

(a) *The normal closure of a in G is equal to G.*
(b) *For all $b \in Z(G)$ and $\chi' \in \mathrm{Irr}_{\beta_b}(G)$, if $\chi(b)\chi'(a) = \deg\chi\deg\chi'$ (equivalently, (a,χ) and (b,χ') centralize each other), then $b = e$ and $\chi' = 1$.*

Proof. The proof to be given is almost identical to the one given for the untwisted case. By the double centralizer theorem, the simple object (a,χ) is a generator of $\mathrm{Rep}(D^\omega(G))$ if and only if the only simple object that centralizes (a,χ) is the trivial simple object $(e,1)$.

Suppose that conditions (a) and (b) in the statement of the theorem hold, and let (b,χ') be a simple object of $\mathrm{Rep}(D^\omega(G))$ that centralizes (a,χ). By Proposition 3.5, the conjugacy classes of a and b commute elementwise; combining this fact with condition (a), we deduce that b lies in the center of G, and so χ' lies in $\mathrm{Irr}_{\beta_b}(G)$. By Proposition 3.8, we must have $\chi(b)\chi'(a) = \deg\chi\deg\chi'$. Applying condition (b), we get $b = e$ and $\chi' = 1$, and it follows that (a,χ) is a generator of $\mathrm{Rep}(D^\omega(G))$.

Conversely, suppose that (a,χ) is a generator of $\mathrm{Rep}(D^\omega(G))$. Then the only simple object that centralizes (a,χ) is the trivial simple object $(e,1)$. Let $b \in Z(G)$, let $\chi' \in \mathrm{Irr}_{\beta_b}(G)$, and suppose that $\chi(b)\chi'(a) = \deg\chi\deg\chi'$. Then the simple objects (a,χ) and (b,χ') centralize each other, by Proposition 3.8. Since the only simple object that centralizes (a,χ) is the trivial simple object $(e,1)$, it follows that $b = e$ and $\chi' = 1$, showing that condition (b) holds.

To see that condition (a) holds, let H denote the normal closure of a in G, and suppose that $H \neq G$. By Proposition 3.8, for all $\chi' \in \mathrm{Irr}(G)$, the simple objects (a,χ) and (e,χ') centralize each other if and only if $a \in \mathrm{Ker}\,\chi'$, equivalently, $H \le \mathrm{Ker}\,\chi'$. Since H is proper in G, the action of G on the coset space G/H is not trivial, and so the corresponding representation contains a nontrivial irreducible constituent; let χ' denote the character of this constituent. Since H is normal in G, it acts trivially on G/H, and so $H \le \mathrm{Ker}\,\chi'$. It follows that (a,χ) and (e,χ') centralize each other, a contradiction. Hence $H = G$, showing that condition (a) holds. □

Corollary 3.10. *Let G be a finite group with trivial center, let ω be a normalized 3-cocycle on G, and let (a,χ) be a simple object of $\mathrm{Rep}(D^\omega(G))$. Then (a,χ) is a generator of $\mathrm{Rep}(D^\omega(G))$ if and only if the normal closure of a in G is equal to G.*

Example 3.11. Let G be a finite group, and let ω be a normalized 3-cocycle on G. If G has trivial center, and a is an element of G whose normal closure is G, then, by Corollary 3.10, for every irreducible β_a-character of $C_G(a)$, the simple object (a,χ) is a generator of $\mathrm{Rep}(D^\omega(G))$. We give three related examples below.

(a) Take $G = S_n$, the symmetric group on n letters, with $n \ge 3$. Then S_n has trivial center, and the normal closure of the transposition $\sigma = (12)$ is S_n. Therefore, for every irreducible β_σ-character of $C_{S_n}(\sigma)$, the simple object (σ,χ) is a generator of $\mathrm{Rep}(D^\omega(S_n))$.

(b) Let $n \geq 3$ be an odd integer, and take $G = \mathrm{Dih}_n$, the dihedral group of order $2n$ generated by the elements a and b subject to the relations $a^n = e$, $b^2 = e$, and $ba = a^{-1}b$. Then Dih_n has trivial center, and the normal closure of the element b is Dih_n. Therefore, for every irreducible β_b-character of $C_{\mathrm{Dih}_n}(b) = \{e, b\}$, the simple object (b, χ) is a generator of $\mathrm{Rep}(D^\omega(\mathrm{Dih}_n))$. Note that, since the Schur multiplier of a cyclic group is trivial, the 2-cocycle β_b is cohomologically trivial, and so χ may be identified with an ordinary character.

(c) Suppose that G is nonabelian and simple. Then G has trivial center, and the normal closure of every nontrivial element a is G. Therefore, for every nontrivial element a, and for every irreducible β_a-character of $C_G(a)$, the simple object (a, χ) is a generator of $\mathrm{Rep}(D^\omega(G))$.

Example 3.12. Let p be an odd prime, and consider the special linear group $\mathrm{SL}(2, p)$ consisting of all 2×2 matrices of determinant 1 whose entries belong to the field of p elements. This group has order $p^3 - p^2$. The matrices

$$X = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \qquad \text{and} \qquad Y = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

generate the group $\mathrm{SL}(2, p)$. The center of $\mathrm{SL}(2, p)$ is a subgroup of order 2, consisting of the matrices $\pm \left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right)$.

The character table of $\mathrm{SL}(2, p)$ was first obtained by F. G. Frobenius. Later, I. Schur [13] and independently H. Jordan [9] obtained the characters of the special linear groups over arbitrary finite fields [8]. We use the exposition given in [5]. The group $\mathrm{SL}(2, p)$ has exactly $p + 4$ distinct irreducible characters. For the purpose of this example, we will only need a portion of the character table of $\mathrm{SL}(2, p)$. Set $\epsilon = (-1)^{(p-1)/2}$. The table below gives the values of the irreducible characters evaluated at the identity matrix I and at the matrix Y, omitting identical columns.

I	1	p	$p+1$	$p-1$	$\frac{p+1}{2}$	$\frac{p+1}{2}$	$\frac{p-1}{2}$	$\frac{p-1}{2}$
Y	1	0	1	-1	$\frac{1+\sqrt{\epsilon p}}{2}$	$\frac{1-\sqrt{\epsilon p}}{2}$	$\frac{-1+\sqrt{\epsilon p}}{2}$	$\frac{-1-\sqrt{\epsilon p}}{2}$

It is easily verified that $X = Y^{-1}(XY^{-1}X^{-1})Y^{-1}$, and since the matrices X and Y generate $\mathrm{SL}(2, p)$, it follows that the normal closure of Y is $\mathrm{SL}(2, p)$.

The conjugacy class of Y contains $(p^2 - 1)/2$ elements, and so the centralizer of Y in $\mathrm{SL}(2, p)$ has order $2p$. The matrix Y has order p, so the matrix $-Y$ has order $2p$, and it follows that the centralizer of Y in $\mathrm{SL}(2, p)$ is a cyclic group with generator $-Y$.

Let ζ be a primitive $2p$-th root of unity. For each $1 \leq i \leq 2p$, let $\chi_i : \langle -Y \rangle \to \mathbb{C}^\times$ denote the group homomorphism that sends $-Y$ to ζ^i. Then the χ_i constitute all of the irreducible characters of the centralizer of Y in $\mathrm{SL}(2,p)$. We have

$$\chi_i(-I) = \chi_i((-Y)^p)) = \zeta^{pi} = (-1)^i.$$

Suppose that i is odd. We will show that the simple object (Y, χ_i) is a generator of $\mathrm{Rep}(D(\mathrm{SL}(2,p)))$. We have shown above that condition (a) of Theorem 3.3. To see that condition (b) of Theorem 3.3 holds, let χ' be an irreducible character of $\mathrm{SL}(2,p)$. Suppose that the simple objects (Y, χ_i) and (I, χ') centralize each other. Then $Y \in \operatorname{Ker} \chi'$, by Proposition 3.2. Since the normal closure of Y is $\mathrm{SL}(2,p)$, it follows that $\chi' = 1$. We have

$$\chi_i(-I)\chi'(Y) = -\chi'(Y)$$

and

$$\deg \chi_i \deg \chi' = \deg \chi'.$$

Inspecting the partial character table given above, we see that $-\chi'(Y) \neq \deg \chi'$, and so the simple objects (Y, χ_i) and $(-I, \chi')$ do not centralize each other, by Proposition 3.2. It follows that condition (b) of Theorem 3.3 also holds, proving that (Y, χ_i) is a generator of $\mathrm{Rep}(D(\mathrm{SL}(2,p)))$.

Note that if i is even, then the simple objects (Y, χ_i) and $(-I, 1)$ centralize each other, and so (Y, χ_i) is not a generator of $\mathrm{Rep}(D(\mathrm{SL}(2,p)))$, by Theorem 3.3.

4. Adjoint category

In this section, we describe the adjoint category of $\mathrm{Rep}(D^\omega(G))$. The case where $\omega = 1$ was addressed in the paper [12]. For a fusion category $\mathcal{C}$, *adjoint category* of $\mathcal{C}$, denoted $\mathcal{C}_{\mathrm{ad}}$, is the full fusion subcategory of $\mathcal{C}$ generated by all subobjects of $X \otimes X^*$, where X runs through simple objects of $\mathcal{C}$. For example, for a finite group G, we have $\mathrm{Rep}(G)_{\mathrm{ad}} \cong \mathrm{Rep}(G/Z(G))$.

Lemma 4.1. *Let G be a finite group, and let ω be a normalized 3-cocycle on G. The set*

$$Z_\omega(G) = \{a \in Z(G) \mid \beta_a \text{ is cohomologically trivial}\}$$

is a subgroup of $Z(G)$.

Proof. Since $\beta_e = 1$, the identity element e lies in $Z_\omega(G)$. Let $a, b \in Z_\omega(G)$. Define a function $\tau_{a,b} : G \to \mathbb{C}^\times$ by $\tau_{a,b}(x) = \beta_x(a,b)$. It follows from (2) that

$$\beta_{ab} = \beta_a \cdot \beta_b \cdot d\tau_{a,b},$$

showing that β_{ab} and $\beta_a\beta_b$ are cohomologous, where d denotes the coboundary operator. Since β_a and β_b are cohomologically trivial, the same is true for β_{ab}, and so $ab \in Z_\omega(G)$. □

The following definition is taken from [12].

Definition 4.2. Let G be a finite group, let ω be a normalized 3-cocycle on G, let K and H be normal subgroups of G that commute elementwise, and let $B : K \times H \to \mathbb{C}^\times$ be a function. We say that B is an ω-*bicharacter* on $K \times H$ if

(a) $B(x, uv) = \beta_x^{-1}(u, v)B(x, u)B(x, v)$, and
(b) $B(xy, u) = \beta_u(x, y)B(x, u)B(y, u)$

for all $x, y \in K$ and $u, v \in H$. We say that B is G-*invariant* if

$$B(g^{-1}xg, u) = \frac{\beta_x(g, u)\beta_x(gu, g^{-1})}{\beta_x(g, g^{-1})} B(x, gug^{-1})$$

for all $g \in G$, $x \in K$, and $u \in H$.

We refer the reader to [12] for an explanation of the G-invariance property and the apparent lack of symmetry. It was shown in [12] that the fusion subcategories of $\mathrm{Rep}(D^\omega(G))$ are parametrized by triples (K, H, B), where K and H are normal subgroups of G that commute elementwise, and B is an ω-bicharacter on $K \times H$. Given such a triple (K, H, B), denote by $\mathcal{S}(K, H, B)$ the full abelian subcategory generated by simple objects (a, χ) such that $a \in K$ and $\chi(h) = B(a, h) \deg \chi$ for all $h \in H$. It was shown in [12] that $\mathcal{S}(K, H, B)$ is, in fact, a fusion subcategory of $\mathrm{Rep}(D^\omega(G))$, and

$$\mathcal{S}(K, H, B)' = \mathcal{S}(H, K, (B^{\mathrm{op}})^{-1}), \tag{4}$$

where $(B^{\mathrm{op}})^{-1} : H \times K \to \mathbb{C}^\times$ is defined by $(B^{\mathrm{op}})^{-1}(h, k) = B(k, h)^{-1}$.

Lemma 4.3. *Let G be a finite group, and let ω be a normalized 3-cocycle on G.*

(a) *For all $a, g, x, y \in G$,*

$$\beta_a(gxg^{-1}, gyg^{-1}) \cdot \frac{\beta_a(g, g^{-1})}{\beta_a(g, xy)\beta_a(gxy, g^{-1})} = \beta_{g^{-1}ag}(x, y) \cdot \frac{\beta_a(g, g^{-1})}{\beta_a(g, x)\beta_a(gx, g^{-1})} \cdot \frac{\beta_a(g, g^{-1})}{\beta_a(g, y)\beta_a(gy, g^{-1})}.$$

(b) *For all $a, g, x, y \in G$, if a lies in $Z_\omega(G)$, then*

$$\beta_a(gxg^{-1}, gyg^{-1}) = \beta_a(x, y).$$

Proof. Part (a) was proved in [12]. To see (b), let $g, x, y \in G$, and let $a \in Z_\omega(G)$. Applying Lemma 3.7 with $\mu = \beta_a$ to the equality in (a), we get

$$\beta_a(gxg^{-1}, gyg^{-1}) \cdot \frac{\beta_a(g^{-1}, gxy)}{\beta_a(gxy, g^{-1})} = \beta_a(x, y) \cdot \frac{\beta_a(g^{-1}, gx)}{\beta_a(gx, g^{-1})} \cdot \frac{\beta_a(g^{-1}, gy)}{\beta_a(gy, g^{-1})}.$$

Since β_a is cohomologically trivial, it is symmetric, and so the equation above reduces to

$$\beta_a(gxg^{-1}, gyg^{-1}) = \beta_a(x, y),$$

proving (b). □

Lemma 4.4. *Let G be a finite group, let ω be a normalized 3-cocycle on G, let K be a subgroup of $Z_\omega(G)$, and let H be a subgroup of the commutator subgroup $[G, G]$ of G. For each $a \in K$, choose a function $\sigma_a : G \to \mathbb{C}^\times$ such that $d\sigma_a = \beta_a$. The function $B : K \times H \to \mathbb{C}^\times$ defined by $B(a, x) = \sigma_a(x)$ does not depend on the choice of the σ_a, and it is a G-invariant ω-bicharacter on $K \times H$.*

Proof. Since the restriction of any homomorphism $G \to \mathbb{C}^\times$ to the subgroup H is trivial, we deduce that for any two functions $f_1 : G \to \mathbb{C}^\times$ and $f_2 : G \to \mathbb{C}^\times$, if $df_1 = df_2$, then f_1 and f_2 are equal when restricted to H. It follows that the function B does not depend on the choice of the σ_a.

The condition $d\sigma_a = \beta_a$ is equivalent to the first condition in the definition of ω-character. To see that the second condition in the definition of ω-character holds, let $a, b \in K$. Define a function $\tau_{a,b} : G \to \mathbb{C}^\times$ by $\tau_{a,b}(x) = \beta_x(a, b)$. As seen in the proof of Lemma 4.1,

$$\beta_{ab} = \beta_a \cdot \beta_b \cdot d\tau_{a,b} = d(\sigma_a \cdot \sigma_b \cdot \tau_{a,b}).$$

Since we also have $\beta_{ab} = d\sigma_{ab}$, we deduce that the functions σ_{ab} and $\sigma_a \cdot \sigma_b \cdot \tau_{a,b}$ are equal when restricted to H, that is, for all $x \in H$,

$$\sigma_{ab}(x) = \sigma_a(x)\sigma_b(x)\tau_{a,b}(x),$$

equivalently,

$$B(ab, x) = \beta_x(a, b)B(a, x)B(b, x),$$

which is the second condition in the definition of ω-character.

To see that B is G-invariant, let $g \in G$, let $a \in K$, and let $x \in H$. Applying the definition of B and Lemma 3.7 with $\mu = \beta_a$ to the expression

$$\frac{\beta_a(g, x)\beta_a(gx, g^{-1})}{\beta_a(g, g^{-1})} B(a, gxg^{-1}),$$

we get

$$\frac{\beta_a(g^{-1}, gx)}{\beta_a(gx, g^{-1})}\sigma_a(gxg^{-1}).$$

Since β_a is cohomologically trivial, it is symmetric, and so the expression above reduces to $\sigma_a(gxg^{-1})$. By Lemma 4.3, $(\beta_a)^g = \beta_a$, and so $d(\sigma_a)^g = (\beta_a)^g = \beta_a = d\sigma_a$, where the superscript denotes the conjugation action. Therefore, the functions $(\sigma_a)^g$ and σ_a are equal when restricted to H, and so $\sigma_a(gxg^{-1}) = \sigma_a(x) = B(a, b)$, proving that B is G-invariant. □

A fusion category $\mathcal{C}$ is said to be *pseudounitary* if its categorical dimension coincides with its Frobenius-Perron dimension [6]. In this case, $\mathcal{C}$ admits a canonical spherical structure with respect to which categorical dimensions of objects coincide with their Frobenius-Perron dimensions [6]. The category $\mathrm{Rep}(D^\omega(G))$ is pseudounitary. In the paper [7], S. Gelaki and D. Nikshych showed that, for a pseudounitary modular category $\mathcal{C}$, the adjoint subcategory $\mathcal{C}_{ad}$ and the full maximal pointed subcategory $\mathcal{C}_{\mathrm{pt}}$ are centralizers of each other, that is,

$$\mathcal{C}_{ad} = (\mathcal{C}_{\mathrm{pt}})'. \tag{5}$$

Theorem 4.5. *Let G be a finite group, and let ω be a normalized 3-cocycle on G. Then*

$$\mathrm{Rep}(D^\omega(G))_{\mathrm{pt}} = \mathcal{S}(Z_\omega(G), [G, G], B)$$

and

$$\mathrm{Rep}(D^\omega(G))_{\mathrm{ad}} = \mathcal{S}([G, G], Z_\omega(G), (B^{\mathrm{op}})^{-1})$$

where $B : Z_\omega(G) \times [G, G] \to \mathbb{C}^\times$ is the G-invariant ω-bicharacter defined in Lemma 4.4.

Proof. The dimension of a simple object (a, χ) of $\mathrm{Rep}(D^\omega(G))$ is $\frac{|G|}{|C_G(a)|}\deg\chi$, which is equal to 1 if and only if a lies in the center of G and $\deg\chi = 1$. The latter condition implies that β_a is cohomologically trivial. It follows that $\mathrm{Rep}(D^\omega(G))_{\mathrm{pt}} = \mathcal{S}(Z_\omega(G), [G, G], B)$. Applying (4) and (5) to the previous equation, we get
$\mathrm{Rep}(D^\omega(G))_{\mathrm{ad}} = \mathcal{S}([G, G], Z_\omega(G), (B^{\mathrm{op}})^{-1})$. □

References

[1] B. Bakalov and A. Kirillov Jr., Lectures on Tensor Categories and Modular Functors, University Lecture Series, 21, American Mathematical Society, Providence, RI, 2001.

[2] A. Coste, T. Gannon and P. Ruelle, *Finite group modular data*, Nuclear Phys. B, 581(3) (2000), 679-717.

[3] R. Dijkgraaf, V. Pasquier and P. Roche, *Quasi-quantum groups related to orbifold models*, Modern quantum field theory (Bombay, 1990), World Sci. Publ., River Edge, NJ, (1991), 375-383.

[4] R. Dijkgraaf, V. Pasquier and P. Roche, *Quasi-Hopf algebras, group cohomology, and orbifold models*, Integrable systems and quantum groups (Pavia, 1990), World Sci. Publ., River Edge, NJ, (1992), 75-98.

[5] L. Dornhoff, Group Representation Theory, Part A, M. Dekker (1971).

[6] P. Etingof, D. Nikshych and V. Ostrik, *On fusion categories*, Ann. of Math. (2), 162 (2005), 581-642.

[7] S. Gelaki and D. Nikshych, *Nilpotent fusion categories*, Adv. Math., 217 (2008), 1053-1071.

[8] J. E. Humphreys, *Representation of* $\mathrm{SL}(2,p)$, Amer. Math. Monthly, 82 (1975), no. 1, 21-39.

[9] H. Jordan, *Group characters of various types of linear groups*, Amer. J. Math., 29 (1907), 387-405.

[10] M. Müger, *On the structure of modular categories*, Proc. London Math. Soc., 87(2) (2003), 291-308.

[11] D. Naidu and D. Nikshych, *Lagrangian subcategories and braided tensor equivalences of twisted quantum doubles of finite groups*, Comm. Math. Phys., 279 (2008), 845-872.

[12] D. Naidu, D. Nikshych and S. Witherspoon, *Fusion subcategories of representation categories of twisted quantum doubles of finite groups*, Int. Math. Res. Not. IMRN, 22 (2009), 4183-4219.

[13] I. Schur, *Untersuchungen über die Darstellungen der endlichen Gruppen durch gebrochene lineare Substitutionen*, J. Reine Angew. Math., 132 (1907) 85-137.

Deepak Naidu
Department of Mathematical Sciences
Northern Illinois University
DeKalb, Illinois 60115, USA
email: dnaidu@math.niu.edu