ON \((m,n)\)-CLOSED IDEALS IN AMALGAMATED ALGEBRA

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Abstract. Let \(R\) be a commutative ring with \(1 \neq 0\) and let \(m\) and \(n\) be integers with \(1 \leq n < m\). A proper ideal \(I\) of \(R\) is called an \((m,n)\)-closed ideal of \(R\) if whenever \(a^m \in I\) for some \(a \in R\) implies \(a^n \in I\). Let \(f: A \to B\) be a ring homomorphism and let \(J\) be an ideal of \(B\). This paper investigates the concept of \((m,n)\)-closed ideals in the amalgamation of \(A\) with \(B\) along \(J\) with respect \(f\) denoted by \(A \triangleright◁ f J\). Namely, Section 2 investigates this notion to some extensions of ideals of \(A\) to \(A \triangleright◁ f J\). Section 3 features the main result, which examines when each proper ideal of \(A \triangleright◁ f J\) is an \((m,n)\)-closed ideal. This allows us to give necessary and sufficient conditions for the amalgamation to inherit the radical ideal property with applications on the transfer of von Neumann regular, \(\pi\)-regular and semisimple properties.

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1. Introduction

We assume throughout the whole paper that all rings are commutative with \(1 \neq 0\). The notion of \((m,n)\)-closed ideal was introduced and defined by Anderson and Badawi in [2], as follow: Let \(R\) be a ring and \(m\) and \(n\) be two positive integers with \(1 \leq n < m\). A proper ideal \(I\) of \(R\) is called an \((m,n)\)-closed ideal of \(R\) if whenever \(a^m \in I\) for some \(a \in R\) implies \(a^n \in I\). Also, an ideal \(I\) of \(R\) is a semi-\(n\)-absorbing ideal of \(R\) if and only if \(I\) is an \((n+1,n)\)-closed ideal of \(R\). Recall that an ideal \(I\) of \(R\) is a radical ideal if and only if \(I\) is a \((2,1)\)-closed ideal. Among other things, they gave the basic properties of semi-\(n\)-absorbing ideals and \((m,n)\)-closed ideals and they also determined when every proper ideal of \(R\) is \((m,n)\)-closed for integers \(1 \leq n < m\). Further, they gave several examples illustrating their results. Recall that a proper ideal \(I\) of \(R\) is called an \(n\)-absorbing ideal of \(R\) as in [1] if \(a_1, a_2, \ldots, a_{n+1} \in R\) and \(a_1 a_2 \cdots a_{n+1} \in I\), then there are \(n\) of the \(a_i\)’s whose product is in \(I\). Notice that an \(n\)-absorbing ideal is an \((m,n)\)-closed for every
positive integer \( m \). In [4], the authors studied the notions of \( n \)-absorbing ideals, strongly \( n \)-absorbing ideals and \((m, n)\)-closed ideals in the trivial ring extension. Let \( A \) be a commutative ring and \( E \) be an \( A \)-module. The trivial ring extension of \( A \) by \( E \) (also called the idealization of \( E \) over \( A \)) is the ring \( R := A \rtimes E \) whose underlying group is \( A \times E \) with multiplication given by \((a, e)(a', e') = (aa', ae' + a'e)\).

Trivial ring extensions have been studied extensively. Considerable work, part of which is summarized in Glaz's book [12] and Huckaba's book [13], has been concerned with trivial ring extension; these extensions have been useful for solving many open problems and conjectures in both commutative and non-commutative ring theory. See for instance [3, 10, 12, 13]. We recall that if \( I \) is a proper ideal of \( A \), then \( I \rtimes E \) is an ideal of \( A \rtimes E \). And if \( F \) is a submodule of \( E \) such that \( IE \subset F \), then \( I \rtimes F \) is an ideal of \( A \rtimes E \). The ideals of \( A \rtimes E \) are not all of the form \( I \rtimes E \) or \( I \rtimes F \), but if \( A \) is an integral domain, and \( E \) an \( A \)-module divisible, the ideals of \( A \rtimes E \) are of the form \( I \rtimes E \) or \( 0 \rtimes F \), where \( I \) is an ideal of \( A \) and \( F \) a submodule of \( E \).

Let \((A, B)\) be a pair of rings, \( f : A \to B \) be a ring homomorphism and \( J \) be an ideal of \( B \). In this setting, we can consider the following subring of \( A \times B \):

\[ A \rhd_f J := \{(a, f(a) + j) \mid a \in A, j \in J\} \]

called the amalgamation of \( A \) and \( B \) along \( J \) with respect to \( f \), introduced and studied by D'Anna, Finocchiaro and Fontana in [9,10]. In particular, they have studied amalgamations in the frame of pullbacks which allowed them to establish numerous (prime) ideal and ring-theoretic basic properties for this new construction. This construction is a generalization of the amalgamated duplication of a ring along an ideal (introduced and studied by D'Anna and Fontana in [7,8]). The interest of amalgamation resides, partly, in its ability to cover several basic constructions in commutative algebra, including pullbacks and trivial ring extensions (also called Nagata's idealizations) (cf. [16, page 2]). Moreover, other classical constructions (such as the \( A + XB[X] \), \( A + XB[[X]] \) and the \( D + M \) constructions) can be studied as particular cases of the amalgamation ([9, Examples 2.5 and 2.6]) and other classical constructions, such as the CPI extensions (in the sense of Boisen and Sheldon [5]) are strictly related to it ([9, Example 2.7 and Remark 2.8]). In [9], the authors studied the basic properties of this construction (e.g., characterizations for \( A \rhd_f J \) to be a Noetherian ring, an integral domain, a reduced ring) and they characterized those distinguished pullbacks that can be expressed as an amalgamation. Moreover, in [10], they pursued the investigation on the structure of the rings of the form \( A \rhd_f J \), with particular attention to the prime spectrum, to the chain properties...
and to the Krull dimension. For more details on amalgamation rings, we refer the reader to [11], [14], [15].

In this paper, we study the notion of \((m, n)\)-closed ideals in the amalgamation of \(A\) with \(B\) along \(J\) with respect \(f\) denoted by \(A \bowtie^f J\). For any ring \(R\), we denote by \(\text{Nil}(R)\) (resp., \(\text{dim}(R)\)), the set of nilpotent elements of \(R\) (resp., the Krull dimension of \(R\)).

2. On some \((m, n)\)-closed ideals of amalgamation \(A \bowtie^f J\)

To avoid unnecessary repetition, let us fix notation for the rest of the paper. Let \((A, B)\) be a pair of rings, \(f : A \to B\) be a ring homomorphism and \(J\) be an ideal of \(B\). All along this paper, \(A \bowtie^f J\) will denote the amalgamation of \(A\) and \(B\) along \(J\) with respect to \(f\). Let \(I\) be an ideal of \(A\) and \(K\) be an ideal of \(f(A) + J\). Notice that \(I \bowtie^f J := \{(i, f(i) + j)/i \in I, j \in J\}\) and \(\overline{K}^f := \{(a, f(a) + j)/a \in A, j \in J, f(a) + j \in K\}\) are ideals of \(A \bowtie^f J\). Our first result gives a characterization about when the ideals \(I \bowtie^f J\) and \(\overline{K}^f\) are \((m, n)\)-closed ideals of \(A \bowtie^f J\), for all positive integers \(m\) and \(n\), with \(1 \leq n < m\).

Proposition 2.1. Under the above notations, the following statements hold:

1. \(I \bowtie^f J\) is an \((m, n)\)-closed ideal of \(A \bowtie^f J\) if and only if \(I\) is an \((m, n)\)-closed ideal of \(A\).

2. \(\overline{K}^f\) is an \((m, n)\)-closed ideal of \(A \bowtie^f J\) if and only if \(K\) is an \((m, n)\)-closed ideal of \(f(A) + J\).

Proof. (1) Assume that \(I \bowtie^f J\) is an \((m, n)\)-closed ideal of \(A \bowtie^f J\) for \(m\) and \(n\) two positive integers with \(1 \leq n < m\). Let \(a^m \in I\), with \(a \in A\). Clearly \((a, f(a))^m \in I \bowtie^f J\). Since \(I \bowtie^f J\) is an \((m, n)\)-closed ideal of \(A \bowtie^f J\), we have \((a, f(a))^m \in I \bowtie^f J\) and so \(a^n \in I\). Hence, \(I\) is an \((m, n)\)-closed ideal of \(A\). Conversely, assume that \(I\) is an \((m, n)\)-closed ideal of \(A\). Let \(x^m = (a, f(a) + j)^m \in I \bowtie^f J\) with \(x = (a, f(a) + j) \in A \bowtie^f J\). Clearly, \(a^m \in I\). Since \(I\) is an \((m, n)\)-closed ideal of \(A\), we have \(a^n \in I\). One can easily check that \(x^n = (a^n, (f(a) + j)^n) \in I \bowtie^f J\), as desired.

(2) Suppose that \(\overline{K}^f\) is an \((m, n)\)-closed ideal of \(A \bowtie^f J\). We claim that \(K\) is an \((m, n)\)-closed ideal of \(f(A) + J\). Indeed, let \((f(a) + j)^m \in K\) with \((f(a) + j) \in f(A) + J\). Then \((a^m, (f(a) + j)^m) \in \overline{K}^f\). Since \(\overline{K}^f\) is an \((m, n)\)-closed ideal, \((a, f(a) + j)^n \in \overline{K}^f\). Therefore, \((f(a) + j)^n \in K\). Hence, \(K\) is an \((m, n)\)-closed ideal of \(f(A) + J\). Conversely, assume that \(K\) is an \((m, n)\)-closed ideal of \(f(A) + J\). Let \((a, f(a) + j)^m \in \overline{K}^f\) with \((a, f(a) + j) \in A \bowtie^f J\).
Obviously, \( f(a) + j \in f(A) + J \) and \((f(a) + j)^m \in K\) which is an \((m, n)\)-closed ideal. So, \((a^n, (f(a) + j)^n) \in \overline{K}^f\). Hence, \(\overline{K}^f\) is an \((m, n)\)-closed ideal of \(A \bowtie^f J\), as desired.

The following corollary is an immediate consequence of Proposition 2.1.

**Corollary 2.2.** Under the above notations, the following statements hold:

1. \(I \bowtie^f J\) is a radical ideal of \(A \bowtie^f J\) if and only if \(I\) is a radical ideal of \(A\).
2. \(I \bowtie^f J\) is a semi-\(n\)-absorbing ideal of \(A \bowtie^f J\) if and only if \(I\) is a semi-\(n\)-absorbing ideal of \(A\).
3. \(\overline{K}^f\) is a semi-\(n\)-absorbing ideal of \(A \bowtie^f J\) if and only if \(K\) is a semi-\(n\)-absorbing ideal of \(f(A) + J\).
4. \(\overline{K}^f\) is a radical ideal of \(A \bowtie^f J\) if and only if \(K\) is a radical ideal of \(f(A) + J\).

Let \(I\) be a proper ideal of \(A\). The (amalgamated) duplication of \(A\) along \(I\) is a special amalgamation given by

\[
A \bowtie^f I := A \bowtie^f \text{id}_A I = \{(a, a + i) \mid a \in A, i \in I\}.
\]

The next corollary is an immediate consequence of Proposition 2.1 and Corollary 3.5 on the transfer of \((m, n)\)-closed ideal property to duplications.

**Corollary 2.3.** Let \(A\) be a ring and \(I\) be an ideal of \(A\). Consider \(K\) an ideal of \(A\). Then the following statements hold:

(1) \(K \bowtie I\) is an \((m, n)\)-closed ideal of \(A \bowtie I\) if and only if \(I\) is an \((m, n)\)-closed ideal of \(A\).
(2) \(K \bowtie I\) is a semi-\(n\)-absorbing ideal of \(A \bowtie I\) if and only if \(I\) is a semi-\(n\)-absorbing ideal of \(A\).
(3) \(K \bowtie I\) is a radical ideal of \(A \bowtie I\) if and only if \(I\) is a radical ideal of \(A\).

Let \(I\) (resp., \(K\)) be an ideal of \(A\) (resp., \(f(A) + J\)). Observe that

\[
\overline{I \times K}^f := \{(a, f(a) + j) \mid j \in J, a \in I, f(a) + j \in K\}
\]

is an ideal of \(A \bowtie^f J\). The following proposition establishes a partial result about when the ideal \(\overline{I \times K}^f\) is an \((m, n)\)-closed ideal of \(A \bowtie^f J\).

**Proposition 2.4.** Let \(m_1, n_1, m_2\) and \(n_2\) be positive integers such that \(m_1 < n_1\) and \(m_2 < n_2\). Under the above notations: If \(I\) is an \((m_1, n_1)\)-closed ideal of \(A\) and \(K\) is an \((m_2, n_2)\)-closed ideal of \(f(A) + J\), then \(\overline{I \times K}^f\) is an \((m, n)\)-closed ideal of \(A \bowtie^f J\) for all positive \(m \leq \min(m_1, m_2)\) and \(n \geq \max(n_1, n_2)\).
Proof. Assume that $I$ is an $(m_1, n_1)$-closed ideal of $A$ and $K$ is an $(m_2, n_2)$-closed ideal of $f(A) + J$. Notice that $I \times K$ is an $(m, n)$-closed ideal of $A \times (f(A) + J)$ for all positive $m \leq \min(m_1, m_2)$ and $n \geq \max(n_1, n_2)$. Since $A \bowtie J \subset A \times (f(A) + J)$, by [2, Corollary 2.11(1)], it follows that $I \times K$ is an $(m, n)$-closed ideal of $A \bowtie J$ for all positive $m \leq \min(m_1, m_2)$ and $n \geq \max(n_1, n_2)$, as desired. \qed

As a direct consequence of Proposition 2.4, we obtain the following corollary:

**Corollary 2.5.**

1. If $I$ is a semi-$n$-absorbing ideal of $A$ and $K$ is a semi-$n$-absorbing ideal of $f(A) + J$, then $I \times K$ is a semi-$n$-absorbing ideal of $A \bowtie J$.

2. If $I$ is a radical ideal of $A$ and $K$ is a radical ideal of $f(A) + J$, then $I \times K$ is a radical ideal of $A \bowtie J$.

Let $f : A \rightarrow B$ be a ring homomorphism and $J$ be an ideal of $B$. Consider an ideal $I$ (resp., $H$) of $A$ (resp., $f(A) + J$) such that $f(I)J \subseteq H \subseteq J$. Observe that $I \bowtie H := \{(i, f(i) + h) / i \in I, h \in H\}$ is an ideal of $A \bowtie J$.

**Remark 2.6.** Under the above notations. Let $m$ and $n$ be two integers with $1 \leq n < m$. If $I \bowtie H$ is an $(m, n)$-closed ideal of $A \bowtie J$, then by using similar argument as statement (1) of Proposition 2.1, it follows that $I$ is an $(m, n)$-closed ideal of $A$. The converse is not true, in general, as shown by the next example which exhibits an ideal $I$ that is a $(2, 1)$-closed ideal of $A$ such that $f(I)J \subseteq H \subset J$ but $I \bowtie H$ is not a $(2, 1)$-closed ideal of $A \bowtie J$.

**Example 2.7.** Let $A := \mathbb{Z}$ be the ring of integers, $B := \mathbb{Q}[[X]]$ be the ring of formal power series over $\mathbb{Q}$ in an indeterminate $X$, $f : \mathbb{Z} \hookrightarrow \mathbb{Q}[[X]]$ be the natural embedding and $J := XQ[[X]]$. Let $H = \{XP(X) : P \in f(A) + J = \mathbb{Z} + XQ[[X]]\}$. Obviously, $H$ is an ideal of $f(A) + J = \mathbb{Z} + XQ[[X]]$. Note that $I := 0$ is a prime ideal of $A$ and so is an $(m, 1)$-closed ideal of $A$ for all positive integer $m$. Since $f(I)J = 0 \subset H \subset J$, $0 \bowtie H$ is an ideal of $\mathbb{Z} \bowtie XQ[[X]]$. We claim that $0 \bowtie H$ is not a $(2, 1)$-closed ideal of $\mathbb{Z} \bowtie XQ[[X]]$. Indeed, $0 \bowtie H = \{(0, XP(X)) : P(X) \in \mathbb{Z} + XQ[[X]]\}$, we have $(0, \sqrt{2}X)^2 = (0, 2X^2) \subset 0 \bowtie H$ but $(0, \sqrt{2}X) \notin 0 \bowtie H$. Hence, $0 \bowtie H$ is not a $(2, 1)$-closed ideal of $\mathbb{Z} \bowtie XQ[[X]]$.

Now, we examine about when the ideal $I \bowtie H$ is an $(m, n)$-closed ideal of $A \bowtie J$. 


Proposition 2.8. Let $f : A \to B$ be a ring homomorphism and $J$ be an ideal of $B$. Let $H$ be an ideal of $f(A) + J$ such that $f(I)J \subseteq H \subseteq J$. Then the following statements hold:

1. If $I \asymp f H$ is an $(m, n)$-closed ideal of $A \asymp f J$, then $I$ is an $(m, n)$-closed ideal of $A$, for all positive integers $m$ and $n$, with $1 \leq n < m$.

2. Assume that $I$ is an $(m, 1)$-closed ideal of $A$ and $x^n \in H$ for every $x \in J$ with $n \geq 1$ a positive integer. Then $I \asymp f H$ is an $(m, n)$-closed ideal of $A \asymp f J$.

Proof. (1) From Remark 2.6.

(2) Assume that $I$ is an $(m, 1)$-closed ideal of $A$. We claim that $I \asymp f H$ is an $(m, n)$-closed ideal of $A \asymp f J$. Let $x^m = (a, f(a) + j)^m \in I \asymp f H$ for some $x = (a, f(a) + j) \in A \asymp f J$. Then $a^m \in I$. Since $I$ is an $(m, 1)$-closed ideal of $A$, we have $a \in I$ and so $a^i \in I$ for every positive integer $1 \leq i \leq m$. Let $h = \sum_{i=1}^{n-1} \binom{n}{i} f(a^i)j^{n-i} + j^n$. Observe that for every positive integer $i \leq n - 1$, we have $f(a^i)j^{n-i} \in H$ (as $(a, f(a) + j)^m \in I \asymp f H$). Since $j^n \in H$ for every $j \in J$, by using the Binomial theorem, it follows that $x^n = (a^n, f(a^n) + h) \in I \asymp f H$. Hence, $I \asymp f H$ is an $(m, n)$-closed ideal of $A \asymp f J$, as desired. \hfill \Box

The following corollary is a consequence of Proposition 2.8.

Corollary 2.9. Let $f : A \to B$ be a homomorphism of rings and $J$ be an ideal of $B$. Assume that $I$ is a $(m, 1)$-closed ideal of $A$ and $x^n \in (f(I)B)J$ for every $x \in J$, with $n \geq 1$ a positive integer. Then the extension ideal of $I$ to $A \asymp f J$, denoted by $I^c := I \asymp f (f(I)B)J$ is an $(m, n)$-closed ideal of $A \asymp f J$, for all positive integers $m$ and $n$, with $1 \leq n < m$.

Proof. Applying Proposition 2.8 with $H := (f(I)B)J$, it follows that $I^c$ is an $(m, n)$-closed ideal of $A \asymp f J$, as desired. \hfill \Box

Next, we show how one may use Proposition 2.8 to construct original example of $(m, n)$-closed ideals of the form $I \asymp f H$ of amalgamation $A \asymp f J$.

Example 2.10. Let $A$ be a ring, $E$ be an $A$-module and $B := A \otimes E$ be the trivial ring extension of $A$ by $E$. Let $I$ be an ideal of $A$ and $F$ be a submodule of $E$ such that $IE \subseteq F$, and $J := I \otimes E$ be an ideal of $B$. Consider the ring homomorphism $f : A \hookrightarrow B$ defined by $f(a) = (a, 0)$. Notice that $H := I \otimes F$ is an ideal of $f(A) + J = A \otimes 0 + I \otimes E = (A + I) \otimes E = A \otimes E$ and
Let $n$ be a positive integer and let $(i, e) \in J$. Clearly, $(i, e)^n = (i^n, m^{n-1}e) \in H$. By Proposition 2.8, we conclude that $I$ is an $(m,1)$-closed ideal of $A$ if and only if $I \trianglelefteq J$ is an $(m, n)$-closed ideal of $A \trianglelefteq J$ for every positive integer $1 \leq n < m$. (For instance, if $I$ is a prime ideal of $A$, then $I \trianglelefteq J$ is an $(m,1)$-closed ideal of $A \trianglelefteq J$ for every positive integer $m \geq 1$. Therefore, $I \trianglelefteq J$ is a radical ideal of $A \trianglelefteq J$.)

Now, we give a characterization of $(m,n)$-closed ideals of the form $I \trianglelefteq J$ in the case $(A, M)$ is local and $J$ an ideal of $B$ such that $f(M)J = 0$.

**Theorem 2.11.** Let $A$ be a local ring with maximal ideal $M$, $f : A \to B$ be a ring homomorphism and $J$ be an ideal of $B$ such that $f(M)J = 0$. Let $I$ be a proper ideal of $A$. Consider an ideal $H$ of $f(A) + J$ such that $H \subset J$ and $x^n \in H$ for every $x \in J$. Then the following statements are equivalent:

1. $I \trianglelefteq J$ is an $(m,n)$-closed ideal of $A \trianglelefteq J$ for every positive integer $1 \leq n < m$.
2. $I$ is an $(m,n)$-closed ideal of $A$ for every positive integer $1 \leq n < m$.

**Proof.** Notice that $I \trianglelefteq J$ is an ideal of $A \trianglelefteq J$ since $f(I)J = 0 \subset H$.

(1) $\Rightarrow$ (2) Assume that $I \trianglelefteq J$ is an $(m,n)$-closed ideal of $A \trianglelefteq J$ for every positive integer $1 \leq n < m$. Then by Remark 2.6, it follows that $I$ is an $(m,n)$-closed ideal of $A$ for every positive integer $1 \leq n < m$.

(2) $\Rightarrow$ (1) Let $x = (a, f(a) + j) \in A \trianglelefteq J$ such that $x^n \in I \trianglelefteq J$. Then $a^n$ is an element of $I$ which is an $(m,n)$-closed ideal of $A$. Therefore, $a^n \in I$. Let $1 \leq l \leq n - 1$. Two cases are then possible:

Case 1: $a^l \in M$. Then $(a^l)\gamma^{n-l} = 0$ and so by using the Binomial theorem, it follows that $(a, f(a) + j)^n = (a^n, f(a^n) + j^n) \in I \trianglelefteq J$.

Case 2: $a^l \notin M$. Then $a^n$ is invertible in $A$ and so $a^n \in I$ is invertible, which is a contradiction since $I$ is a proper ideal of $A$. So, $x^n = (a^n, (f(a) + j)^n) \in I \trianglelefteq J$. Hence, in all cases, $I \trianglelefteq J$ is an $(m,n)$-closed ideal of $A \trianglelefteq J$.

Before giving an explicit example of Theorem 2.11, we establish the following lemma which will be useful.

**Lemma 2.12.** Let $(A, M)$ be local ring such that $M^n = 0$, where $n$ is a positive integer. Then every ideal of $A$ is an $n$-absorbing ideal of $A$. In particular, every proper ideal of $A$ is an $(m,n)$-ideal of $A$, for every positive integer $1 \leq n < m$. 

3. When every proper ideal of amalgamation of $A \triangleleft J$ is an $(m,n)$-closed ideal of $A$

Proof. Let $I$ be a proper ideal of $A$. Let $a_1, \ldots, a_{n+1} \in A$ such that $\prod_{i=1}^{n+1} a_i \in I$. Two cases are then possible:

Case 1: There exists $j \in \{1, \ldots, n+1\}$ such that $a_j \notin M$. Then $a_j$ is invertible and so it follows that $\prod_{i=1, i \neq j}^{n+1} a_i \in I$, as desired.

Case 2: For every $j \in \{1, \ldots, n+1\}$, $a_j \in M$. So, for every $j \in \{1, \ldots, n+1\}$, $a_j$ is not invertible in $A$. Therefore, $\prod_{i=1}^{n} a_i = 0 \in I$. Thus in all cases, $I$ is an $n$-absorbing ideal of $A$. In particular, $I$ is an $(m,n)$-closed ideal of $A$. □

Next, we show how one may use Theorem 2.11 and Lemma 2.12 to construct original examples of $(m,n)$-closed ideals of the form $I \triangleright J \ H$.

Example 2.13. Let $(A, M)$ be a local ring with a maximal ideal $M$ such that $M^n = 0$, and $E$ be an $\frac{A}{M}$-vector space. Consider the ring homomorphism $f : A \rightarrow B := A \times E$ defined by $f(a) = (a, 0)$, for every $a \in A$. Let $J := M \times E$ be an ideal of $B$ and $H = I \times E$ be an ideal of $f(A) + J$, where $I \subseteq M$ is an ideal $A$ with $m$ and $n$ two positive integers such that $2 \leq n \leq m$. One can easily check that $J^n = 0$ and for every ideal $I$ of $A$, we get $f(I)J = (I \times E)(M \times E) \subseteq H$, and $J^n \subseteq H$. Hence, by Theorem 2.11, the ideal $I \triangleright J$ is an $(m,n)$-closed ideal of $A \triangleright J$ since $I$ is an $n$-absorbing ideal of $A$ (by Lemma 2.12).

We denote by $\text{Char}(R)$, the characteristic of a ring $R$. We close this section by giving a characterization of $(m,n)$-closed ideals of the form $I \triangleright J$ under the condition “$\text{Char}(f(A) + J) = n$”.

Proposition 2.14. Let $f : A \rightarrow B$ be a ring homomorphism and $J$ be an ideal of $B$. Assume that $\text{char}(f(A) + J) = n$. Let $H$ be an ideal of $f(A) + J$ such that $f(I)J \subset H \subset J$ and for every $j \in J$, $j^n \in H$. Then $I \triangleright J$ is an $(m,n)$-closed ideal of $A \triangleright J$ if and only if $I$ is an $(m,n)$-closed ideal of $A$, for all positive integer $m \geq n$.

Proof. Assume that $I$ is an $(m,n)$-closed ideal of $A$, for all positive integer $m \geq n$.

Let $(a, f(a) + j)^m \in I \triangleright J$ for every $(a, f(a) + j) \in A \triangleright J$. From assumption, it follows that $a^n \in I$. Using the fact that $\text{char}(f(A) + J) = n$ and $j^n \in H$, then $(a, f(a) + j)^n = (a^n, f(a^n) + j^n) \in I \triangleright J$. Hence, $I \triangleright J$ is an $(m,n)$-closed ideal of $A \triangleright J$. The converse is trivial via Remark 2.6. □

3. When every proper ideal of amalgamation $A \triangleright J$ is an $(m,n)$-closed ideal?

The following result gives a characterization about when every proper ideal of the amalgamation $A \triangleright J$ is an $(m,n)$-closed ideal, for some integers $1 \leq n < m$.
**Theorem 3.1.** Assume that $f^{-1}(J)$ is a radical ideal of $A$. Then every proper ideal of $A \triangleright J$ is an $(m, n)$-closed ideal of $A \triangleright J$ if and only if the following statements hold:

(i) Every proper ideal of $A$ is an $(m, n)$-closed ideal of $A$.

(ii) Every proper ideal of $f(A) + J$ is an $(m, n)$-closed ideal of $f(A) + J$.

The proof of this theorem requires the following lemmas.

**Lemma 3.2.** [2, Theorem 2.14] Let $R$ be a commutative ring and $m$ and $n$ integers with $1 \leq n < m$. Then the following statements are equivalent.

1. Every proper ideal of $R$ is an $(m, n)$-closed ideal of $R$.
2. $\dim(R) = 0$ and $w^n = 0$ for every $w \in \text{Nil}(R)$.

**Lemma 3.3.** [6, Lemma 2.10] Let $f : A \to B$ be a ring homomorphism and $J$ be an ideal of $B$. Then:

\[
\text{Nil}(A \triangleright J) := \{(a, f(a) + j)/a \in \text{Nil}(A), j \in \text{Nil}(B) \cap J\}.
\]

**Lemma 3.4.** [10, Proposition 4.1] Let $f : A \to B$ be a ring homomorphism and $J$ be an ideal of $B$. Then: $\dim(A \triangleright J) = \text{Max}(\dim(A), \dim(f(A) + J))$.

**Proof of Theorem 3.1:** Assume that every proper ideal of $A \triangleright J$ is an $(m, n)$-closed ideal of $A \triangleright J$.

(i) By Lemmas 3.2 and 3.4, it follows that $\dim(A \triangleright J) = \text{Max}(\dim(A), \dim(f(A) + J)) = 0$. So, $\dim(A) = 0$. Next, let $a \in \text{Nil}(A)$. Then $(a, f(a)) \in \text{Nil}(A \triangleright J)$. Using the fact that every proper ideal of $A \triangleright J$ is an $(m, n)$-closed ideal of $A \triangleright J$, then by Lemma 3.2, it follows that $(a, f(a))^n = (0, 0)$. Therefore, $a^n = 0$. Hence, every proper ideal of $A$ is an $(m, n)$-closed ideal of $A$.

(ii) With similar argument as (i) above, it follows that $\dim(f(A) + J) = 0$. Let $f(a) + j \in \text{Nil}(f(A) + J)$. Clearly, $f(a)^k \in J$ for some integer $k \geq 1$. So, $a^k \in f^{-1}(J)$ which is radical. Therefore, $a \in f^{-1}(J)$. Consequently, $f(a) + j \in J$.

By Lemma 3.3, $(0, f(a) + j) \in \text{Nil}(A \triangleright J)$. Hence, $(0, f(a) + j)^n = (0, 0)$ and so $(f(a) + j)^n = 0$. From Lemma 3.2, it follows that every proper ideal of $f(A) + J$ is an $(m, n)$-closed ideal of $f(A) + J$. Conversely, assume that every proper ideal of $A$ (resp., $f(A) + J$) is an $(m, n)$-closed ideal of $A$ (resp., $f(A) + J$). We claim that every proper ideal of $A \triangleright J$ is an $(m, n)$-closed ideal of $A \triangleright J$. Indeed, by Lemma 3.4, $\dim(A \triangleright J) = \text{Max}(\dim(A), \dim(f(A) + J)) = 0$ since $\dim(A) = \dim(f(A) + J) = 0$. It remains to show that for all $(a, f(a) + j) \in \text{Nil}(A \triangleright J)$, $(a, f(a) + j)^n = 0$. Let $(a, f(a) + j) \in \text{Nil}(A \triangleright J)$. Then $a \in \text{Nil}(A)$ and $f(a) + j \in \text{Nil}(f(A) + J)$. By Lemma 3.2, it follows that $a^n = 0$ and $(f(a) + j)^n = 0$. 

\[
\text{Nil}(A \triangleright J) := \{(a, f(a) + j)/a \in \text{Nil}(A), j \in \text{Nil}(B) \cap J\}.
\]

\[
\text{Nil}(A \triangleright J) := \{(a, f(a) + j)/a \in \text{Nil}(A), j \in \text{Nil}(B) \cap J\}.
\]
Therefore, \((a, f(a) + j)^n = 0\). Hence, every proper ideal of \(A \bowtie^f J\) is an \((m, n)\)-closed ideal of \(A \bowtie^f J\), as desired. \(\square\)

The following corollary is an immediate consequence of Theorem 3.1.

**Corollary 3.5.** Let \(f : A \rightarrow B\) be a ring homomorphism and \(J\) be an ideal of \(B\). Then the following statements hold:

1. Every proper ideal of \(A \bowtie^f J\) is a radical ideal of \(A \bowtie^f J\) if and only if the following statements hold:
   \begin{itemize}
   \item[(i)] Every proper ideal of \(A\) is a radical ideal of \(A\).
   \item[(ii)] Every proper ideal of \(f(A) + J\) is a radical ideal of \(f(A) + J\).
   \end{itemize}

2. Assume that \(f^{-1}(J)\) is a radical ideal of \(A\). Then every proper ideal of \(A \bowtie^f J\) is a semi-\(n\)-absorbing ideal of \(A \bowtie^f J\) if and only if the following statements hold:
   \begin{itemize}
   \item[(i)] Every proper ideal of \(A\) is a semi-\(n\)-absorbing ideal of \(A\).
   \item[(ii)] Every proper ideal of \(f(A) + J\) is a semi-\(n\)-absorbing ideal of \(f(A) + J\).
   \end{itemize}

**Remark 3.6.** Observe that the assumption “\(f^{-1}(J)\) is a radical ideal of \(A\)” is omitted in Corollary 3.5(1). Indeed, if every proper ideal of \(A \bowtie^f J\) is a radical ideal of \(A \bowtie^f J\), then \(f^{-1}(J) \times \{0\}\) is a radical ideal of \(A \bowtie^f J\) and so \(f^{-1}(J) \simeq f^{-1}(J) \times \{0\}\) is also a radical ideal of \(A\). Conversely if the statements (i) and (ii) hold, then \(f^{-1}(J)\) is radical ideal of \(A\).

Theorem 3.1 and Corollary 3.5 cover the special case of duplications, as recorded below.

**Corollary 3.7.** Let \(A\) be a ring and \(I\) be an ideal of \(A\). Then the following statements hold:

1. Assume that \(I\) is a radical ideal of \(A\). Then every proper ideal of \(A \bowtie I\) is an \((m, n)\)-closed ideal of \(A \bowtie I\) if and only every proper ideal of \(A\) is an \((m, n)\)-closed ideal of \(A\).

2. Every proper ideal of \(A \bowtie I\) is a radical ideal of \(A \bowtie I\) if and only every proper ideal of \(A\) is a radical ideal of \(A\).

3. Assume that \(I\) is a radical ideal of \(A\). Then every proper ideal of \(A \bowtie I\) is a semi-\(n\)-absorbing ideal of \(A \bowtie I\) if and only every proper ideal of \(A\) is a semi-\(n\)-absorbing ideal of \(A\).

Theorem 3.1 recovers a known result for trivial ring extensions which is [4, Theorem 6.12].
Corollary 3.8. Let $A$ be an integral domain with quotient field $K$, $E$ be a $K$-vector space, and $B := A \otimes E$ be the trivial ring extension of $A$ by $E$. Then the following statements are equivalent:

1. Every proper ideal of $A$ is an $(m,n)$-closed ideal of $A$ for some integers $1 \leq n < m$.
2. Every proper ideal of $B$ is an $(m,n)$-closed ideal of $B$ for some integers $1 \leq n < m$.

Proof. Consider the injective ring homomorphism $f : A \hookrightarrow B$ defined by $f(a) = (a,0)$, for every $a \in A$, $J := 0 \otimes E$ is an ideal of $B$. Clearly, $f^{-1}(J) = 0$. Therefore, from [9, Proposition 5.1 (3)], $f(A) + J = A \otimes 0 + 0 \otimes E = A \otimes E = B \simeq A \otimes f^0 J$. Since $A$ is an integral domain, $f^{-1}(J) = 0$ is a radical ideal of $A$. Hence, by Theorem 3.1, we have the desired result. □

As a consequence of Theorem 3.1, we give a complete characterization of those $D + M$ rings such that every proper ideal is $(m,n)$-closed.

Corollary 3.9. Let $M$ be a maximal ideal of an integral domain $T$, and $D$ be a subring of $T$ such that $D \cap M = \{0\}$. Then every proper ideal of $D + M$ is an $(m,n)$-closed ideal of $D + M$ if and only if every proper ideal of $D$ is an $(m,n)$-closed ideal of $D$ for some integers $1 \leq n < m$.

Proof. Let $f : D \rightarrow T$ be the natural embedding and $J := M$ is the maximal ideal of $T$. Since $f^{-1}(J) = D \cap M = \{0\}$ (which is radical ideal of $D$, as 0 is prime ideal of $D$), it is easy to see that $D + M \simeq D \otimes f^0 M$. Therefore, by Theorem 3.1, it follows that every proper ideal of $D + M$ is an $(m,n)$-closed ideal of $D + M$ if and only if every proper ideal of $D \otimes f^0 M$ is a $(m,n)$-closed ideal if and only if every proper ideal of $D$ is an $(m,n)$-closed ideal of $D$, for some integers $1 \leq n < m$. □

As an application Theorem 3.1, we give a new characterization for the amalgamation $A \otimes f^0 J$ to be von Neumann regular. Recall that a ring $R$ is von Neumann regular if and only if every proper ideal of $R$ is a radical ideal [2, Theorem 2.13 (2)]. A combination of this fact and Corollary 3.5(1) establishes the transfer of von Neumann regular property to the amalgamation $A \otimes f^0 J$.

Corollary 3.10. Let $f : A \rightarrow B$ be a ring homomorphism and $J$ be an ideal of $B$. Then $A \otimes f^0 J$ is von Neumann regular if and only if $A$ and $f(A) + J$ are von Neumann regular.
The next result is an application of Corollary 3.10 on the transfer of $\pi$-regular property to the amalgamation. Recall that a ring $R$ is $\pi$-regular if and only if $R/\text{Nil}(R)$ is von Neumann regular.

**Corollary 3.11.** Let $f : A \to B$ be a ring homomorphism and $J$ be an ideal of $B$. Then $A \bowtie f J$ is $\pi$-regular if and only if $A$ and $f(A) + J$ are $\pi$-regular.

**Proof.** Set $\overline{A} = A/\text{Nil}(A)$, $\overline{B} = B/\text{Nil}(B)$, $p : B \to \overline{B}$ the canonical projection and $\overline{J} = p(J)$ and let $\overline{f} : \overline{A} \to \overline{B}$, defined by $\overline{f}(\overline{x}) = \overline{f(x)}$. Observe that $\overline{f}$ is well defined. From [6, Proof of Theorem 2.9], we get $A \bowtie \overline{f} J/\text{Nil}(A \bowtie \overline{f} J) \simeq \overline{A} \bowtie \overline{J}$. Consequently, $A \bowtie f J$ is a $\pi$-regular ring if and only if $\overline{A} \bowtie \overline{f} \overline{J}$ is von Neumann regular if and only if $\overline{A}$ and $\overline{f(A)} + \overline{J}$ are von Neumann regular (by Corollary 3.10). Hence, the conclusion is straightforward. \hfill $\square$

As another application of Theorem 3.1, we get necessary and sufficient conditions for an amalgamation to be semisimple.

**Corollary 3.12.** Let $f : A \to B$ be a ring homomorphism and $J$ be an ideal of $B$. Then $A \bowtie f J$ is semisimple if and only if $A$ and $f(A) + J$ are semisimple.

**Proof.** It is well known that semisimple rings collapse with von Neumann regular rings that are Noetherian. A combination of this fact with Corollary 3.10 and [9, Proposition 5.6] (on the transfer of the Noetherian property) leads to the conclusion. \hfill $\square$

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